Imperial College London

MSci EXAMINATION May 2014

This paper is also taken for the relevant Examination for the Associateship

GENERAL RELATIVITY

For 4th-Year Physics Students

Monday 19th May 2014: 14:00 to 16:00

The paper consists of two *sections: A and B*
Section A contains one *question [40 marks total].*
Section B contains four *questions [30 marks each].*

Candidates are required to:
Answer **ALL** *parts of Section A and* **TWO QUESTIONS** *from Section B.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature.

You may find the following formulae useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right)$$

The covariant derivative of a vector field is,

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha$$

and for a covector field is,

$$\nabla_\mu w_\nu \equiv \partial_\mu w_\nu - \Gamma^{\alpha}{}_{\mu\nu} w_\alpha$$

For a Lagrangian of a curve $x^\mu(\lambda)$ of the form,

$$L = \int d\lambda \, \mathcal{L}(x^\mu, \frac{dx^\mu}{d\lambda})$$

the Euler-Lagrange equations are,

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial(\frac{dx^\mu}{d\lambda})} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

Section A

Answer all of section A.

SECTION A

1. This question concerns the covariant derivative.

(i) State how the components of a $(1,0)$ tensor v^μ and a $(0,1)$ tensor w_μ transform under a coordinate transformation $x \to x'$.

[8 marks]

(ii) Use your previous answer to show that $v^\mu w_\mu$ transforms as a scalar under a coordinate transformation $x \to x'$.

[8 marks]

(iii) Under a coordinate transformation the Christoffel symbol transforms as;

$$\Gamma'^{\mu'}{}_{\alpha'\beta'} = \Gamma^{\mu}{}_{\alpha\beta} \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial x^\alpha}{\partial x'^{\alpha'}} \frac{\partial x^\beta}{\partial x'^{\beta'}} - \left(\frac{\partial^2 x'^{\mu'}}{\partial x^\alpha \partial x^\beta} \right) \frac{\partial x^\alpha}{\partial x'^{\alpha'}} \frac{\partial x^\beta}{\partial x'^{\beta'}} \ .$$

Show that the Christoffel symbol does not transform as a tensor.

[2 marks]

(iv) Show that $\partial_\mu w_\nu$, the partial derivative of a covector field w_μ, does *not* transform as a tensor.

[8 marks]

(v) Starting from the relations,

$$\delta^\mu_\nu = \frac{\partial x^\mu}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x'^{\nu'}} \frac{\partial x'^{\nu'}}{\partial x^\nu}$$

take an appropriate partial derivative of this to derive,

$$\frac{\partial x'^{\alpha}}{\partial x^\alpha} \frac{\partial x'^{\beta'}}{\partial x^\beta} \frac{\partial^2 x^\mu}{\partial x'^{\alpha} \partial x'^{\beta'}} \quad = \quad -\frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial^2 x'^{\mu'}}{\partial x^\alpha \partial x^\beta} \ .$$

[6 marks]

(vi) Show that the covariant derivative of a covector field w_μ, defined as $\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^\alpha{}_{\mu\nu} w_\alpha$, does transform as a tensor.

[8 marks]

[Total 40 marks]

Section B

Answer 2 out of the 4 questions in the following section.

SECTION B

2. This question concerns the Newtonian spacetime, which we write using coordinates $x^\mu = (t, x^i)$ with $i = 1, 2, 3$ as,

$$ds^2 = \left(\eta_{\mu\nu} - 2\epsilon\Phi(x^i)\delta_{\mu\nu}\right) dx^\mu dx^\nu$$

where $\epsilon\Phi$ is the Newtonian potential, and we are interested in the Newtonian limit $\epsilon \to 0$. We take Φ to be static, and hence only a function of the x^i.

(i) State the stress tensor for a perfect fluid in a general spacetime in terms of its energy density ρ, pressure P and local 4-velocity u^μ (where $u^\mu u_\mu = -1$). What conservation equation does the stress tensor obey?

[6 marks]

(ii) In the limit $\epsilon \to 0$ the Ricci tensor to leading order $O(\epsilon)$ is;

$$\begin{aligned} R_{tt} &= \epsilon\, \delta_{ij}\partial_i\partial_j\Phi \\ R_{ti} &= 0 \\ R_{ij} &= \epsilon\, \delta_{ij}\ (\delta_{kl}\partial_k\partial_l\Phi)\ . \end{aligned}$$

Use this to compute the components of the stress tensor that satisfies the Einstein equations for this spacetime. Show that this is the stress tensor for a dust fluid (ie. fluid with zero pressure), and determine the 4-velocity and energy density of this dust in terms of the Newtonian potential $\epsilon\Phi$.

[12 marks]

(iii) By calculation, show that to leading order in ϵ,

$$\Gamma^i{}_{tt} = \epsilon\partial_i\Phi\ .$$

Using this, show that a non-accelerated particle with proper time τ that is *slowly moving* obeys (to leading order),

$$\frac{d^2x^i}{d\tau^2} = -\partial_i\,(\epsilon\Phi)\ .$$

[10 marks]

(iv) Use these answers to *briefly* explain how Newton's force law of gravity arises in General Relativity.

[2 marks]

[Total 30 marks]

3. This question concerns the Schwarzschild metric, which we write using coordinates $x^\mu = (t, r, \theta, \phi)$ as,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

for a mass M, with G the Newton constant.

(i) Consider a timelike geodesic $x^\mu(\tau) = (T(\tau), R(\tau), \Theta(\tau), \Phi(\tau))$ in the Schwarzschild metric where τ is proper time. Write a Lagrangian that we may vary to determine the geodesic. Deduce the Euler-Lagrange equations for Θ and Φ and show these are consistent with a geodesic that lies in the plane $\theta = \pi/2$. We now restrict our attention to such geodesics. Show then that,

$$R^2 \frac{d\Phi}{d\tau} = J$$

where J is a constant.

[8 marks]

(ii) Further deduce the equations that govern T and R. Show that,

$$\left(1 - \frac{2GM}{r}\right)\frac{dT}{d\tau} = k$$

where k is a constant. Hence show the equation governing the radial motion in the plane $\theta = \pi/2$ looks like that of one dimensional motion for a unit mass particle in a potential $V(R)$ with constant energy E so,

$$E = \frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 + V(R) , \qquad V(R) = -\frac{GM}{R} + \frac{J^2}{2R^2} + \frac{\alpha J^2}{R^3}$$

where α is a constant depending on the mass M and Newton constant G that you should determine.

[8 marks]

(iii) Show that for a circular orbit, with constant radius $R = R_0$, then,

$$V''(R_0) = \frac{J^2}{R_0^4}\left(1 + \frac{6\alpha}{R_0}\right) .$$

[8 marks]

(iv) Compute the proper time τ_{ang} required for Φ to traverse an angle 2π. Show that for a circular orbit radius $R = R_0$ that is perturbed a little, so $R(\tau) \simeq R_0 + \delta R(\tau)$, the motion approximately performs simple harmonic oscillation with period,

$$\tau_{rad} = \frac{2\pi}{\sqrt{V''(R_0)}} .$$

Comment on the relation between τ_{ang} and τ_{rad}.

[6 marks]

[Total 30 marks]

4. (i) Consider a particle following a timelike curve $x^\mu(\tau)$ in a general spacetime, where τ is the particle's proper time and $v^\mu = dx^\mu/d\tau$ is its 4-velocity. By explicit calculation show that,

$$v^\mu \nabla_\mu v^\alpha = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\sigma\rho} \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} \ .$$

How is the 4-acceleration a^μ related to this expression?

[8 marks]

(ii) By carefully varying the action L_{free} for a free particle,

$$L_{free} = \int d\tau \mathcal{L}_{free} \ , \qquad \mathcal{L}_{free} = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

explicitly show that,

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}_{free}}{\partial \frac{dx^\alpha}{d\tau}} \right) - \frac{\partial \mathcal{L}_{free}}{\partial x^\alpha} = 2 g_{\alpha\beta} \left(\frac{d^2 x^\beta}{d\tau^2} + \Gamma^\beta{}_{\sigma\rho} \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} \right) \ .$$

[8 marks]

(iii) Suppose a particle's equation of motion is given by the variation of a different action, L, where,

$$L = \int d\tau \left(\mathcal{L}_{free} + \mathcal{L}_{interaction} \right)$$

with $\mathcal{L}_{free}$ as above and $\mathcal{L}_{interaction}$ determines the particle's interaction with some other field, so that the particle is accelerated and does not follow a geodesic. Use your previous results to deduce that the Euler-Lagrange equations of this action give rise to a particle motion with 4-acceleration given by,

$$a_\alpha \;\; = \;\; -\frac{1}{2} \left(\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}_{interaction}}{\partial \frac{dx^\alpha}{d\tau}} \right) - \frac{\partial \mathcal{L}_{interaction}}{\partial x^\alpha} \right)$$

[6 marks]

(iv) Consider now a particle coupled to a vector field $A_\mu(x)$ in a general spacetime so that its Lagrangian is modified to,

$$L = \int d\tau \left(g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + A_\mu(x) \frac{dx^\mu}{d\tau} \right) \ .$$

Show that the 4-acceleration of the particle is;

$$a^\mu = \frac{1}{2} F^{\mu\nu} v_\nu \ , \qquad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \ .$$

[8 marks]

[Total 30 marks]

5. (i) Show that the Christoffel symbol is related to partial derivatives of the metric as,

$$\partial_\alpha g_{\mu\nu} \quad = \quad g_{\mu\beta}\Gamma^\beta{}_{\alpha\nu} + g_{\nu\beta}\Gamma^\beta{}_{\alpha\mu} \ .$$

[6 marks]

(ii) The Lie derivative of a $(0, 2)$ tensor $A_{\mu\nu}$ with respect to a vector field w^μ is,

$$(Lie)(w, A)_{\mu\nu} = w^\alpha \partial_\alpha A_{\mu\nu} + A_{\mu\alpha}\partial_\nu w^\alpha + A_{\alpha\nu}\partial_\mu w^\alpha \ .$$

Suppose we consider the Lie derivative of the metric $g_{\mu\nu}$. Show that this can also be written in terms of the covariant derivative as,

$$(Lie)(w, g)_{\mu\nu} = \nabla_\mu w_\nu + \nabla_\nu w_\mu \ .$$

If this vanishes, we say w^μ is a *Killing vector field*.

[6 marks]

(iii) Consider a timelike particle with velocity $v^\mu = dx^\mu/d\tau$ for proper time τ. Suppose it follows a geodesic in a spacetime with a Killing vector field w^μ. Show that the quantity,

$$\phi = -w^\mu v_\mu$$

is constant along the particle's trajectory.

[8 marks]

(iv) Consider the spacetime with coordinates $x^\mu = (t, x^i)$

$$ds^2 = -N(x)dt^2 + g_{ij}(x)dx^i dx^j$$

where N and g_{ij} only depend on the spatial coordinates x^i and not time t. Show that there is a Killing vector w^μ for this spacetime and explicitly check that $\mathrm{Lie}(w, g) = 0$. In this spacetime how is the conserved quantity ϕ for a non-accelerated particle related to the energy of the particle as measured by observers sitting at constant spatial position?

[10 marks]

[Total 30 marks]