Cosmology and Particle Physics **Toby Wiseman**

Example sheet 1

Qu. 1 Consider the 3-dimensional spatial geometry;

$$ds^2 = \tilde{g}_{ij} dx^i dx^j = S(r) dr^2 + r^2 d\Omega^2_{(2)}$$

where $d\Omega^2_{(2)}$ is the line element on a unit round 2-sphere and we take;

$$d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta d\phi^2$$

Firstly, show that the Christoffel symbol, $\tilde{\Gamma}$, of $\tilde{g}$ has non-vanishing components;

$$\begin{aligned}\tilde{\Gamma}^r{}_{rr} &= \frac{1}{2S}\frac{dS}{dr}\,,\ \tilde{\Gamma}^r{}_{\theta\theta} = -\frac{r}{S}\,,\ \tilde{\Gamma}^r{}_{\phi\phi} = -\frac{r}{S}\sin^2\theta \\ \tilde{\Gamma}^\theta{}_{r\theta} &= \frac{1}{r}\,,\ \tilde{\Gamma}^\theta{}_{\phi\phi} = -\cos\theta\sin\theta\,,\ \tilde{\Gamma}^\phi{}_{r\phi} = \frac{1}{r}\,,\ \tilde{\Gamma}^\phi{}_{\theta\phi} = \cot\theta\end{aligned}$$

Now show that the Ricci tensor has non-vanishing components;

$$\tilde{R}_r{}^r = \frac{1}{rS^2}\frac{dS}{dr}\,,\ \tilde{R}_\theta{}^\theta = \tilde{R}_\phi{}^\phi = \frac{1}{r^2} - \frac{1}{r^2 S} + \frac{1}{2rS^2}\frac{dS}{dr}$$

Hence determine that the Ricci scalar, $\tilde{R}$, of $\tilde{g}$ is;

$$\tilde{R} = \frac{2}{r^2}\left(1 - \frac{1}{S} + \frac{r}{S^2}\frac{dS}{dr}\right)$$

Use your result for the Ricci tensor to argue that if the geometry is *smooth* at $r = 0$ (so that $S(r) \simeq 1 + O(r^2)$) and *homogeneous*, then,

$$S(r) = \frac{1}{1 - Kr^2}$$

for K a constant.

In fact this metric is homogeneous and isotropic - see the last (optional) question on this sheet. Use your results to show that,

$$\tilde{R}_{ij} = 2K\tilde{g}_{ij}$$

for this metric.

Qu. 2 Consider FRW spacetime in coordinates;

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{ij}(x)dx^i dx^j$$

where $\tilde{g}_{ij}$ is the metric on a 3-space. Show that the Christoffel symbols for this metric are;

$$\begin{aligned} \Gamma^t{}_{ij} &= a\dot{a}\tilde{g}_{ij} \\ \Gamma^i{}_{tj} &= \frac{\dot{a}}{a}\delta_{ij} \\ \Gamma^i{}_{jk} &= \tilde{\Gamma}^i{}_{jk} \end{aligned}$$

where $\tilde{\Gamma}^i{}_{jk}$ is the Christoffel symbol of the 3-metric $\tilde{g}_{ij}$.

Assume that $\tilde{g}$ is the homogeneous isotropic space from question 1, so that,

$$\tilde{R}_{ij} = 2K\tilde{g}_{ij}$$

for constant K. Then use this to show that the non-zero components of the spacetime Ricci tensor are;

$$\begin{aligned} R_{tt} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= \left(2K + 2\dot{a}^2 + a\ddot{a}\right)\tilde{g}_{ij} \end{aligned}$$

Qu. 3 The stress tensor in FRW,

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{ij}(x)dx^i dx^j$$

takes the form,

$$T_{tt} = \rho(t)\,,\; T_{ti} = 0\,,\; T_{ij} = a^2(t)P(t)\tilde{g}_{ij}$$

for energy density, ρ, and pressure, P. Use the results from the last question for the FRW Ricci tensor to show that suitable linear combinations of the Einstein equations,

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

give the Friedmann equation;

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G_N}{3}\rho \tag{1}$$

and,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\rho + 3P\right) \tag{2}$$

Using the results for the Christoffel symbol for FRW (see previous question), show that stress energy conservation $\nabla^\mu T_{\mu\nu} = 0$ implies;

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + P\right) = 0 \tag{3}$$

Show that just using equations (1) and (3) we may derive the remaining Einstein equation (2). Thus we need not consider that equation as it is not independent, and instead just work with the Friedmann equation (1) and conservation equation (3).

Qu. 4 Consider a perfect fluid with equation of state $P = w\rho$ for w =constant. Show that conservation of the fluid stress tensor implies that,

$$\rho = \rho_0 \left(\frac{a_0}{a} \right)^{3(1+w)} \tag{4}$$

for constants a_0, ρ_0. Show that the Friedmann equation implies that,

$$a = a_0 \left(\frac{3(1+w)}{2} H_0 t \right)^{\frac{2}{3(1+w)}} \tag{5}$$

Compute the age of the universe at $t = t_0$, when $a = a_0, \rho = \rho_0, H = H_0$. Confirm that for $w < -1/3$ so that $\rho + 3P < 0$ then this cosmology is accelerating.

Qu. 5 Consider geodesics in FRW in isotropic coordinates $x^\mu = (t, r, \theta, \phi)$ so that,

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right) \tag{6}$$

Take the geodesic curve to be parameterized as $x^\mu = (T(\lambda), R(\lambda), \Theta(\lambda), \Phi(\lambda))$, so that the parameter λ is affine with,

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \mu$$

with $\mu = -1, 0$ for timelike and null geodesics.

Vary the Lagrangian,

$$L = \int \lambda \mathcal{L} \,, \quad \mathcal{L} = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$

to obtain the Euler-Lagrange equations for Θ and Φ;

$$\begin{aligned} a^2 R^2 \sin^2 \Theta \frac{d\Phi}{d\lambda} &= \text{const} \\ \frac{d}{d\lambda} \left(a^2 R^2 \frac{d\Theta}{d\lambda} \right) &= a^2 R^2 \sin \Theta \cos \Theta \left(\frac{d\Phi}{d\lambda} \right)^2 \end{aligned}$$

Consider a ray passing through $r = 0$. Show that then $\Theta(\lambda)$ and $\Phi(\lambda)$ are constant ie. the geodesic's position on the 2-sphere is fixed.

Then use the R Euler Lagrange equation and the constraint $\mathcal{L} = \mu$ to deduce;

$$\begin{aligned} \frac{dR}{d\lambda} &= \frac{v}{a^2}\sqrt{1 - kr^2} \\ \frac{dT}{d\lambda} &= \frac{v}{a}\sqrt{1 - \frac{\mu a^2}{v^2}} \end{aligned}$$

for an integration constant v.

Show that for a massive particle (ie. $\mu = -1$), with 4-momentum $p^\mu = m dx^\mu / d\lambda$, then p^0 is the energy measured by comoving observers, and the spatial momentum measured is $|\mathbf{p}| = \sqrt{g_{ij} p^i p^j}$. Show that,

$$p^0 = \sqrt{m^2 + |\mathbf{p}|^2}\,, \quad |\mathbf{p}| = \frac{mv}{a}$$

For a massless particle (ie. $\mu = 0$), with 4-momentum $p^\mu = dx^\mu / d\lambda$, the energy measured by comoving observers is again p^0 and the momentum $|\mathbf{p}| = \sqrt{g_{ij} p^i p^j}$. Show that indeed,

$$p^0 = |\mathbf{p}| = \frac{v}{a}$$

Qu. 6 Show that the relation,

$$H^2 = H_0^2 \left(\Omega_\Lambda + \Omega_K (1+Z)^2 + \Omega_M (1+Z)^3 \right) \tag{7}$$

derives from the Friedmann and conservation equations for redshift Z. Note we have ignored radiation as we are thinking about late times in the universe when its energy density is tiny.

Use this to confirm that the age of the universe, T, for case where the universe is flat, so $\Omega_K = 0$, is,

$$T = \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \sinh^{-1} \sqrt{\frac{\Omega_\Lambda}{1-\Omega_\Lambda}}$$

[Hint; you may find that in order to integrate,

$$\int \frac{dy}{y\sqrt{1+y^{-3}}}$$

the substitution $z = y^{3/2}$ is useful.]

In the limits $\Omega_\Lambda = 0$ and $\Omega_\Lambda = 1$ show that this age agrees with that you have computed in question 4 for a matter ($w = 0$) and vacuum energy ($w = -1$) cosmology.

Qu. 7
Recall the derivation of the luminosity distance d_L, to confirm that,

$$d_L = a_0 R(1+Z)$$

where the source emits at radial coordinate $r = R$ at redshift Z, and the radiation is received at $Z = 0$, at $r = 0$ - here the coordinate r is the usual radial FRW coordinate as in the metric (6).

By considering the null ray and using the equation (7), show that the luminosity distance d_L in a universe with no curvature, so $\Omega_K = 0$ is,

$$d_L = \frac{1+Z}{H_0\sqrt{\Omega_\Lambda}}\left((1+Z)F\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}(1+Z)^3\right) - F\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}\right)\right)$$

where $F(x)$ is a certain Hypergeometric function which arises as;

$$\int \frac{dy}{y^{1/3}\sqrt{1+y^2}} = c + \frac{3}{2}y^{2/3}F\left(y^2\right)$$

with c a constant of integration.