Imperial College London

MSc EXAMINATION May 2016

This paper is also taken for the relevant Examination for the Associateship

PARTICLE COSMOLOGY

For Students in Quantum Fields and Fundamental Forces

Friday, 13th May 2016: 14:00 to 17:00

Answer **THREE** *out of the following four questions.*
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature and choose units so that $\hbar = 1$ and $c = 1$.

You may find the following useful:

For the FRW metric,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

the Friedmann equation and conservation law are,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \qquad \dot{\rho} + 3H\left(\rho + P\right) = 0\,.$$

The real space number density for a non-relativistic particle with mass m, internal degrees of freedom g and chemical potential μ, at temperature T is,

$$n(T) = g\left(\frac{k_B m T}{2\pi}\right)^{\frac{3}{2}} e^{\frac{\mu - m}{kT}}\,.$$

In SI units the following constants have values;

$$\begin{aligned}
\hbar = \frac{h}{2\pi} &= 1.05 \times 10^{-34}\ \mathrm{kg\,m^2\,s^{-1}} \\
c &= 3.00 \times 10^{8}\ \mathrm{m\,s^{-1}} \\
G &= 6.67 \times 10^{-11}\ \mathrm{kg^{-1}\,m^3\,s^{-2}} \\
k_B &= 1.38 \times 10^{-23}\ \mathrm{kg\,m^2\,s^{-2}\,K^{-1}}
\end{aligned}$$

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1. (i) Consider an FRW spacetime with spatial curvature,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin\theta^2 d\phi^2\right)\right)$$

and a radially moving photon with affine parameter λ, so its position is $t = T(\lambda)$ and $r = R(\lambda)$, with θ, ϕ constant. It is emitted at $\lambda = \lambda_e$ with known energy in the comoving frame. Show that the redshift, Z, of the photon measured by a comoving observer at later time $T(\lambda)$ is,

$$1 + Z(\lambda) = \frac{\dot{T}(\lambda_e)}{\dot{T}(\lambda)} \tag{1.1}$$

where a dot represents differentiation w.r.t. λ, so $\dot{T} = dT/d\lambda$.

[5 marks]

(ii) Write a Lagrangian that may be varied to give the geodesic equations for T and R. Consider the null condition and the variation of T to show,

$$\dot{T}^2 = \frac{a(T)^2}{1-kR^2}\dot{R}^2\ , \quad -\ddot{T} = \frac{a'(T)}{a(T)}\dot{T}^2\ .$$

[5 marks]

(iii) Computing $dZ/d\lambda$ from equation (1.1) and using the other results above, show,

$$(1+Z)^2\left(\frac{dR}{dZ}\right)^2 = \frac{1-kR^2}{a'(T)^2}\ , \quad (1+Z)^2\left(\frac{dT}{dZ}\right)^2 = \frac{a(T)^2}{a'(T)^2}\ .$$

These are the evolution equations for photon position as a function of redshift for an FRW spacetime.

[3 marks]

(iv) Now consider the Lemaitre-Tolman-Bondi (LTB) model of cosmology. It is isotropic around the origin **but not homogeneous** and takes the form,

$$ds^2 = -dt^2 + \frac{\left(\partial_r A(t,r)\right)^2}{1-K(r)}dr^2 + A(t,r)^2\left(d\theta^2 + \sin\theta^2 d\phi^2\right)$$

where $A(t,r)$ is a function of both t and r, and $K(r)$ is only a function of the radial coordinate. These coordinates are comoving ie. an observer at constant r, θ, ϕ follows a timelike geodesic.

a) What are the functions $A(T,r)$ and $K(r)$ in order to recover the homogeneous FRW spacetime considered earlier in the question?

b) Repeat the calculations above for the LTB spacetime to derive evolution equations analogous to those in part iii) for the position of a radially travelling photon as a function of redshift measured by the comoving observers at constant spatial coordinate position.

[7 marks]

[Total 20 marks]

2. (i) Consider an FRW cosmology (including spatial curvature) with a cosmological constant and non-relativistic pressureless matter. Derive the form of the Friedmann equation,

$$H^2 = H_0^2 \left(\Omega_\Lambda + \Omega_{curv} \left(1 + Z\right)^2 + \Omega_M \left(1 + Z\right)^3 \right)$$

where Z is the redshift and you should define the parameters H_0, Ω_Λ, Ω_{curv} and Ω_M.

[6 marks]

(ii) Consider a spacetime with metric $ds^2 = -dt^2 + g_{ij}(t,x)dx^i dx^j$. The Boltzmann equation for a gas of free particles with phase space distribution $n(t, x^i, p_j)$ is,

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{p^t}\frac{\partial}{\partial x^i} + \frac{1}{2p^t} p^j p^k \partial_i g_{jk} \frac{\partial}{\partial p_i} \right) n = 0 \,.$$

Use this to show that for a homogeneous isotropic gas of free particles in a flat FRW spacetime the Boltzmann equation is,

$$\left(\left.\frac{\partial}{\partial t}\right|_p - Hp \left.\frac{\partial}{\partial p}\right|_t \right) n = 0$$

where $p = \sqrt{g^{ij} p_i p_j}$.

[5 marks]

(iii) Give the **general solution** of the Boltzmann equation for a homogeneous isotropic gas of free particles in flat FRW.

Confirm that for this general solution the number of particles in a comoving volume is conserved.

[4 marks]

(iv) Let us model our universe by a flat FRW cosmology. Suppose there is a time during the radiation era when there is a small fraction of non-relativistic dark matter particles of mass m in thermal equilibrium. They have no chemical potential and only one internal degree of freedom, so $g = 1$. (See front pages for the form of the thermal non-relativistic number density).

Suppose the interactions keeping the gas in thermal equilibrium suddenly turn off at a temperature T_{freeze} in the radiation era and the particles subsequently travel freely without interacting. **Estimate** the density fraction Ω_{relic} today of the dark matter particles left over in terms of T_{freeze}, m, H_0 and the photon temperature T_{CMB} today.

[5 marks]

[Total 20 marks]

3. (i) Show that for a highly relativistic boson and fermion with g internal degrees of freedom the energy density at temperature T is,

$$\rho_{boson} = \frac{1}{2} g\, a\, T^4 \,, \qquad \rho_{fermion} = \frac{7}{16} g\, a\, T^4$$

where $a = \pi^2 k_B^4/15$ is the radiation constant (in units $\hbar = c = 1$). You may use the integral,

$$\int_0^\infty dx \frac{x^3}{e^x \pm 1} = \frac{15 \mp 1}{240} \pi^4 \,.$$

[5 marks]

(ii) Assume at some epoch in the radiation era there are g_B bosonic and g_F fermionic relativistic thermal degrees of freedom. Show that the expansion rate is given by,

$$H \simeq 0.15 \left(g_B + \frac{7}{8} g_F \right)^{\frac{1}{2}} \left(\frac{T}{10^{10}\,\mathrm{K}} \right)^2 \mathrm{s}^{-1} \,.$$

[5 marks]

(iii) Derive the Saha equation for the neutron and proton number densities, n_n and n_p, during nucleosynthesis,

$$\frac{n_n}{n_p} = e^{-\frac{Q}{k_B T}}$$

where $Q = m_n - m_p \simeq 2.3 \times 10^{-30}\,\mathrm{kg}$ (in SI units) is the mass difference between the neutron and proton.

(You may find the expression for number density of a non-relativistic species in the front pages useful).

[3 marks]

(iv) Use the Fermi interaction to estimate the **order of magnitude** of the rate Γ of this nucleosynthesis reaction. You may take the mass of the W bosons to be $\sim 10^{-25}\,\mathrm{kg}$.

Using part ii) estimate the temperature at which freezeout occurs. You should find approximately, $T_{freeze} \sim 10^{10} K$.

Assume $T_{freeze} = 10^{10} K$, then stating any assumptions estimate the ratio of Hydrogen to Helium in primordial gas clouds.

[7 marks]

[Total 20 marks]

4. (i) Consider the inflaton field ϕ with potential $V(\phi)$. For flat FRW the scalar field and Friedmann equations are,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\ , \qquad H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\ .$$

Starting from the slow-roll condition, $\dot{\phi}^2 \ll V(\phi)$, derive the two conditions,

$$|\ddot{\phi}| \ll |V'(\phi)|\ , \quad |V'(\phi)| \ll \sqrt{G}\,V(\phi)\ .$$

[4 marks]

(ii) Derive the following expression for the number of e-folds of inflation, N_{efolds}, assuming the inflaton slowly rolls from ϕ_{start} to ϕ_{finish},

$$N_{efolds} \;=\; -8\pi G\int_{\phi_{start}}^{\phi_{finish}} \frac{V}{V'}d\phi\ .$$

Consider the quadratic potential,

$$V(\phi) \;=\; m^2\phi^2$$

where m^2 is a positive constant. Estimate how many e-folds of inflation we should obtain for a starting position ϕ_{start}.

[5 marks]

(iii) Recall that we quantize fluctuations in the inflaton during slow roll as,

$$\delta\hat{\phi}(t,x) = \int d^3k_i\left(\delta\phi_{k_i}(t)e^{-ik_ix^i}\hat{a}_{k_i} + \delta\phi^\star_{k_i}(t)e^{+ik_ix^i}\hat{a}^\dagger_{k_i}\right)$$

where $[\hat{a}_{k_i}, \hat{a}_{q_j}] = [\hat{a}^\dagger_{k_i}, \hat{a}^\dagger_{q_j}] = 0$ and $[\hat{a}_{k_i}, \hat{a}^\dagger_{q_j}] = \delta^3(k_i - q_i)$ and we may approximate the mode functions to be those for flat deSitter,

$$\delta\phi_{k_i}(t) = \frac{1}{a(t)\sqrt{2k(2\pi)^3}}\, e^{+\frac{ik}{a(t)H}}\left(1 + \frac{ia(t)H}{k}\right)$$

for $k^2 = \delta^{ij}k_ik_j$. Compute the quantity $\Delta^2(t,k)$ defined by,

$$\langle 0|\delta\hat{\phi}(t,x)\,\delta\hat{\phi}(t,y)|0\rangle = \int \frac{d^3k_i}{k^3}\Delta^2(t,k)e^{-ik_i(x^i-y^i)}\ .$$

Show Δ^2 is constant in time on superhorizon scales.

[6 marks]

(iv) Consider the quadratic potential in part ii). Suppose the reheat temperature is such that the largest scales today come from 60 e-folds before the end of inflation. Recall the large scale CMB temperature fluctuations have a dimensionless power spectrum $\Delta^2_{(T)}$ which is estimated from Δ^2 by,

$$\Delta^2_{(T)} \;\simeq\; \frac{H^2}{\dot{\phi}^2}\Delta^2\ .$$

CMB measurements constrain this to be of order $\sqrt{\Delta^2_{(T)}} \sim 10^{-5}$. Estimate the parameter m^2 in the potential in order to fit this data.

[5 marks]

[Total 20 marks]

 End of examination paper