2 Thermal physics

2.1 Thermodynamics in FRW

Recap: Recall for thermodynamics in a finite volume V start with the first law;

$$dE = TdS - pdV + \mu dN \tag{204}$$

where we have included a conserved total particle number N. The quantities E, S depend on T, V and N in general. Note that if there is no conserved number or $\mu = 0$ then they depend only on T and V.

We now address how to think about thermodynamics of a homogeneous system in a (potentially infinite) FRW universe.

Thermodynamics of matter in FRW:

Consider a thermodynamic system in an FRW universe that respects the cosmological symmetry. Then there can be no net flow of energy, entropy or particles out of or into a comoving region.

We consider the thermodynamics associated to some component of matter in the universe, assuming it is in equilibrium. There may or may not be a heat bath. If we think of all the matter in the universe, clearly there is nothing else. On the other hand, it we think about charged particles, there is the photon radiation which may act as a heat bath. Thus in the former we should think in a microcanonical setting, in the latter we may think in a canonical one.

We take V_R to be the (physical) volume of a comoving region R, and consider changes of E_R, S_R, N_R associated to that region. We only consider N_R to be constant – either there is not thermal bath, or it is not one that exchanges particles.

The entropy of the region may change if we perform a non-adiabatic process. Also V_R may change. We think of V_R defined by the spatial coordinates x^i of $ds^2 = -dt^2 + a(t)^2 h_{ij}(x) dx^i dx^j$ so we write $V_R = a^3 c_R$ where c_R is the 'coordinate volume' of the region ie. $c_R \int_R d^3x \sqrt{\det h_{ij}}$. We restrict to changes dV_R due to changes in the scale factor, rather than changes in the region. Hence $dV_R/V_R = 3da/a$.

In equilibrium all quantities can be taken to depend on T, V_R and N_R if there is a conserved particle number. We may define intensive quantities, the number density n using the conserved particle number, and energy and entropy density ρ, s using the comoving volume V_R, as,

$$N = nV_R \ , \quad E = \rho V_R \ , \quad S = sV_R \tag{205}$$

Then either there is no conserved N or if there is, N is constant, so either way we have;

$$d\left(\rho V_R\right) = Td\left(sV_R\right) - pdV_R \tag{206}$$

then rearranging and using $V_R \propto a^3$;

$$d\rho - Tds \ = \ \frac{dV_R}{V_R}\left(Ts - \rho - p\right) = 3\frac{da}{a}\left(Ts - \rho - p\right) \tag{207}$$

Our thermodynamic variables E, T, p, μ depend on S, V and (if conserved N). However, we have converted to all intensive variables, ρ, T, p, μ, so these can only depend on $s = S/V$ and (if conserved) $n = N/V$. In particular this means they cannot directly depend on a, only s and (if conserved n).

No conserved N: In the situation that there is no conserved N, or equivalently that $\mu = 0$, then their independence on a implies from above that the coefficient of da/a must vanish;

$$Ts \ = \ \rho + p \tag{208}$$

Consider a perfect fluid with this property; take EOS $p = w\rho$, then,

$$s \ = \ \frac{\rho + p}{T} = (1 + w)\frac{\rho}{T} \tag{209}$$

Conserved N: For a conserved N then the functions depend on s and n, and using $n \propto \frac{1}{a^3}$ we may eliminate a to obtain;

$$d\rho - Tds \ = \ -\frac{dn}{n}\left(Ts - \rho - p\right) \tag{210}$$

which describes the thermodynamic variations and their dependence on s, n.

Variations in time: Now suppose we consider a matter system such that it is non-interacting with anything else, and we consider the variations $d\rho$, ds, da due to the passage of time. Rearranging we obtain,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = T\left(\dot{s} + 3\frac{\dot{a}}{a}s\right) \tag{211}$$

but from stress energy conservation we see the RHS must vanish. Hence,

$$s \propto \frac{1}{a^3} \tag{212}$$

or equivalently S_R =constant. Thus in thermal equilibrium cosmological expansion is adiabatic. In a sense this is obvious – cosmological expansion can only be thought as maintaining equilibrium if it is sufficiently slow, and hence is adiabatic!

Behaviour of radiation for adiabatic expansion:

Consider the case that $\mu = 0$ or there is no conserved N and consider variation for such adiabatic expansion. Recall quantities then only depend on T. Consider a fluid with $p = w\rho$,

$$T = (1+w)\frac{\rho}{s} \propto (1+w)\rho a^3 \propto (1+w)(a^{-3(1+w)})a^3 \propto a^{-3w} \tag{213}$$

and hence if we know the T dependence, we can reexpress it in terms of a. For the case of radiation, $w = 1/3$ we obtain;

$$\rho \propto \frac{1}{a^4}, \quad s \propto \frac{1}{a^3}, \quad T \propto \frac{1}{a} \tag{214}$$

Note that we have not used the Einstein equation here, only conservation of the stress tensor – so this may just be one matter component that is non-interacting, it doesn't have to dominate the energy density.

Non-relativistic gas for adiabatic expansion:

In the case that we have a conserved N then we see that;

$$\frac{s}{n} = \text{const} \tag{215}$$

so the entropy per particle is constant for adiabatic expansion.

Consider the non-relativistic case with conserved particle number and chemical potential. The thermodynamic variables depend on s, n, or we can equivalently take T, n. Recall;

$$\rho(T,n) \;=\; \left(m + \frac{3}{2}kT\right) n \;, \quad P(T,n) = kTn \tag{216}$$

where m is the mass of a particle, and we are careful to include the rest mass energy density contribution above. Note then that $\rho \gg P$, ie this has EOS $w \simeq 0$. Now using this we see,

$$d\rho = \left(m + \frac{3}{2}kT\right) dn + \frac{3}{2}kndT \tag{217}$$

So putting this into our 1st law; $d\rho - Tds = \frac{dn}{n}\left(Ts - \rho - p\right)$ we obtain;

$$\left(m + \frac{3}{2}kT\right) dn + \frac{3}{2}k\,n\,dT - Tds \;=\; dn\left(\left(m + \frac{3}{2}kT\right) + kT + \frac{Ts}{n}\right) \tag{218}$$

so dividing by $-nT$,

$$\frac{ds}{n} - \frac{s}{n^2}dn \;=\; \frac{3k}{2}\frac{dT}{T} - k\frac{dn}{n} \quad\Longrightarrow\quad d\left(\frac{s}{n}\right) = d\left(k\log\left(\frac{T^{3/2}}{n}\right)\right) \tag{219}$$

One can solve this in general as;

$$\frac{s}{n} = C + k\log\left(\frac{T^{3/2}}{n}\right) \tag{220}$$

for a constant C.

Particle conservation gives $n \propto \frac{1}{a^3}$ so $\rho \propto \frac{1}{a^3}$ (since $\rho \simeq mn$). Consider an adiabatic expansion so then s/n =constant. Hence we obtain;

$$s \propto \frac{1}{a^3} \;, \quad n \propto T^{3/2} \tag{221}$$

and thus,

$$T \propto \frac{1}{a^2} \tag{222}$$

Naively we might think $kT \sim \frac{p^2}{2m}$ and we know momentum redshifts as $p \sim 1/a$ so $T \sim 1/a^2$.

2.2 Relativistic Statistical Mechanics

Let us now consider relativistic stat mech in Minkowski spacetime, in canonical Minkowski coordintes $x^\mu = (t, x^i)$. We will consider an ideal gas, formed by identical massless or massive point particles whose collisions are instantaneous.

Then for a gas made up of particles, each with 4-momentum $p^\mu_{(n)} = (E_{(n)}, p^i(n))$, with trajectory $x^\mu_{(n)}(t)$, so,

$$p^\mu_{(n)} = \frac{dx^\mu_{(n)}}{d\lambda_{(n)}} \tag{223}$$

for the particle's affine parameter $\lambda_{(n)}$. Note in the massive case, with mass m, we would have,

$$p^\mu_{(n)} = m\frac{dx^\mu_{(n)}}{d\tau_{(n)}} \tag{224}$$

and hence the affine parameter is related to the particles proper time (also an affine parameter) as $\lambda_{(n)} = \tau_{(n)}/m$.

We may express this momentum in terms of the Minkowski time t,

$$\begin{aligned} p^\mu_{(n)} &= \frac{dt}{d\lambda_{(n)}}\frac{dx^\mu_{(n)}}{dt} = p^0_{(n)}\frac{dx^\mu_{(n)}}{dt} \\ &= E_{(n)}\frac{dx^\mu_{(n)}}{dt} \end{aligned} \tag{225}$$

The stress tensor for this set of massless or massive particles is,

$$T^{\mu\nu} = \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right)\frac{1}{E_{(n)}}p^\mu_{(n)}p^\nu_{(n)} \tag{226}$$

For example, the energy is,

$$\begin{aligned} T^{tt} &= \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right)\frac{1}{E_{(n)}}p^t_{(n)}p^t_{(n)} \\ &= \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right)E_{(n)} \end{aligned} \tag{227}$$

as one would expect.

[**Breif aside:** Let us briefly motivate the form of this discrete stress tensor. This is easiest to understand in the massive case. Recall for dust the stress tensor is;

$$T^{\mu\nu} = \rho\, u^\mu u^\nu \tag{228}$$

where ρ is the rest mass density. Then a Lorentz-invariant discrete version of this is,

$$\begin{aligned} T^{\mu\nu}(x^\mu) &= \sum_n \int d\tau_{(n)}\, \delta^{(4)}\left(x^\alpha_{(n)}(\tau_{(n)}) - x^\alpha\right)\, m\, u^\mu_{(n)} u^\nu_{(n)} \\ &= \sum_n \int d\tau_{(n)}\, \delta^{(4)}\left(x^\alpha_{(n)}(\tau_{(n)}) - x^\alpha\right)\, \frac{1}{m}\, p^\mu_{(n)} p^\nu_{(n)} \end{aligned} \tag{229}$$

where $u^\mu_{(n)} = dx^\mu_{(n)}/d\tau_{(n)}$ is the 4-velocity of the particle, mass m, so that $p^\mu_{(n)} = m u^\mu_{(n)}$. Then we have $dt/d\tau_{(n)} = u^0_{(n)} = p^0_{(n)}/m = E_{(n)}/m$.

Note the important points; the continuous form $\rho\, u^\mu u^\nu$ for dust gives us the term $m\, u^\mu_{(n)} u^\nu_{(n)}$, and the delta function makes this discrete. Furthermore it is a spacetime delta function, and hence is Lorentz invariant.

However, we may simply integrate over the particle's timelike world line, using,

$$\begin{aligned} \int d\tau_{(n)}\, \delta^{(4)}\left(x^\alpha_{(n)}(\tau_{(n)}) - x^\alpha\right) &= \int d\tau_{(n)}\, \delta\left(t_{(n)}(\tau_{(n)}) - t\right) \delta^{(3)}\left(x^i_{(n)}(\tau_{(n)}) - x^i\right) \\ &= \int dt_{(n)} \frac{d\tau_{(n)}}{dt_{(n)}}\, \delta\left(t_{(n)}(\tau_{(n)}) - t\right) \delta^{(3)}\left(x^i_{(n)}(\tau_{(n)}) - x^i\right) \\ &= \int dt_{(n)} \delta\left(t_{(n)} - t\right) \frac{m}{E_{(n)}} \delta^{(3)}\left(x^i_{(n)}(t_{(n)}) - x^i\right) \\ &= \frac{m}{E_{(n)}} \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) \end{aligned} \tag{230}$$

Thus we find,

$$\begin{aligned} T^{\mu\nu}(x^\mu) &= \sum_n \int d\tau_{(n)}\, \delta^{(4)}\left(x^\alpha_{(n)}(\tau_{(n)}) - x^\alpha\right) \frac{1}{m}\, p^\mu_{(n)} p^\nu_{(n)} \\ &= \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) \frac{1}{E_{(n)}}\, p^\mu_{(n)} p^\nu_{(n)} \end{aligned} \tag{231}$$

as given above. Note that now the delta function is only spatial. This would not be Lorentz invariant were it not for the inverse energy factor. This is analogous to the invariant measure $d^3x/(2E)$ that we use in QFT. **End of aside.**]

We introduce a phase space density $n(t, x^i, p_j)$ where, $n(t, x^i, p_j)d^3xd^3p$ gives the number of particles in phase space volume d^3x, d^3p. Then we may write,

$$\int_V d^3x \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) \simeq \int_V d^3x \int_{-\infty}^{\infty} d^3p\, n\left(t, x^i, p_j\right) \tag{232}$$

where we have,

$$n\left(t, x^i, p_j\right) = \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) \delta^{(3)}\left(p_{(n)j}(t) - p_j\right) \tag{233}$$

Now consider a **large number** of particles We then approximate the phase space density as a **smooth function** rather than a distribution. Then we specialise to an equilibrium setting. Now time independence implies $n(t, x^i, p_j) = n(x^i.p_j)$ for our averaged phase space distribution (although not for the original distributional phase space density!).

We define the number density in real space n which is extracted from the phase space density as,

$$n(x^i) \simeq \int_{-\infty}^{\infty} d^3p\, n\left(x^i, p_j\right) \tag{234}$$

Now consider a **homogeneous isotropic** distribution which is parameterized by temperature T, chemical potential μ etc... In this case homogeneity implies no x^i dependence in the real space density or phase space density, so that $n(x^i, p_j) = n(p_j)$ only. Then,

$$n(T, \mu, \ldots) \;\simeq\; \int_{-\infty}^{\infty} d^3p\, n\left(p^i\,;\; T, \mu, \ldots\right) \tag{235}$$

Isotropy further implies;

$$n(T, \mu, \ldots) \;\simeq\; \int_0^{\infty} dp\, 4\pi p^2\, n\,(p\,;\; T, \mu, \ldots) \tag{236}$$

where $p^2 = \delta_{ij} p^i p^j$ and $4\pi p^2 n(p) dp$ gives the number density in the momentum shell dp.

Then in our averaged large particle approximation we have;

$$T^{tt} \;=\; \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) E_{(n)} \simeq \int_0^{\infty} dp\, 4\pi p^2 n(p) E_p \tag{237}$$

where,

$$E_p^2 \;=\; p^2 + m^2 \tag{238}$$

Isotropy implies $T^{ti} = 0$ and $T^{ij} \propto \delta^{ij}$. Consider,

$$\begin{aligned}
\sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) p^i_{(n)} p^j_{(n)} \;&\simeq\; \int d^3p\, n(|p|) p^i p^j \\
&=\; \delta^{ij} \int dp\, 4\pi p^2\, n(p) f(p)
\end{aligned} \tag{239}$$

where we must deduce $f(p)$. Tracing we find;

$$\begin{aligned}
\sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) p^i_{(n)} p^j_{(n)} \delta_{ij} \;&=\; \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) |p_{(n)}|^2 \\
&=\; \int dp\, 4\pi p^2\, n(p)\, p^2
\end{aligned} \tag{240}$$

Hence $f(p) = p^2/3$. Then,

$$T^{ij} \;=\; \sum_n \delta^{(3)}\left(x^i_{(n)}(t) - x^i\right) \frac{1}{E_{(n)}} p^i_{(n)} p^j_{(n)} \simeq \delta^{ij} \int_0^{\infty} dp\, 4\pi p^2 n(p) \frac{p^2}{3E_p} \tag{241}$$

Thus we find the energy density and pressure;

$$
\begin{aligned}
T^{tt} &= \rho = \int_0^\infty dp\, 4\pi p^2 n(p) E_p \\
T^{ti} &= 0 \\
T^{ij} &= P\delta^{ij} = \delta^{ij} \int_0^\infty dp\, 4\pi p^2 n(p) \frac{p^2}{3E_p}
\end{aligned}
\tag{242}
$$

Relativistic case: $E_p = p$ and hence

$$P = \rho/3 \tag{243}$$

Note this is totally independent of the actual distribution $n(p)$. Thus *homogeneous isotropic* massless particles behave as an ideal fluid with $w = 1/3$, independent of whether they are in thermal equilibrium, or how they are interacting.

Non-relativistic case: we have $E_p \simeq m$ and $p \ll m$ so that,

$$P \ll \rho \tag{244}$$

This is our $w = 0$ equation of state, again independent of the precise form of $n(p)$, only requiring that $p \ll m$ behaviour dominates.

Thermal equilibrium

Consider the particles to have g internal degrees of freedom (eg. spins). Then at temperature T and chemical potential μ;

$$n\left(p; T, \mu\right) = \frac{g}{\left(2\pi\hbar\right)^3} \frac{1}{e^{\frac{E_p - \mu}{kT}} \pm 1} \tag{245}$$

with $+$ for fermions and $-$ for bosons.

Massless/relativitistic limit

Suppose we have $m = 0$ and $\mu = 0$,. Alternatively consider a massive field with potential, but at very high temperatures so that, $m, \mu \ll kT$. Then we can approximate $E_p \simeq p$, $\mu \simeq 0$. Then,

$$
\begin{aligned}
n(T) &= \int_0^\infty dp\, 4\pi p^2 n(p; T) = \frac{g}{\left(2\pi\hbar\right)^3} \int_0^\infty dp \frac{4\pi p^2}{e^{\frac{p}{kT}} \pm 1} \\
&= \frac{g\left(kT\right)^3}{2\pi^2\hbar^3} \int_0^\infty dx \frac{x^2}{e^x \pm 1}
\end{aligned}
\tag{246}
$$

Now we may evaluate;

$$\int_0^\infty dx \frac{x^2}{e^x \pm 1} = \frac{7 \mp 1}{4}\zeta(3) \qquad \text{ie. } \frac{3}{2}\zeta(3)\,,\; 2\zeta(3) \tag{247}$$

where $\zeta(3) = 1.202\ldots$ is the Riemann zeta function ($\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$). Now we define the radiation constant,

$$a = \frac{\pi^2 k^4}{15\hbar^3 c^3} \tag{248}$$

and then,

$$\begin{aligned} n_{fermion(-)}(T) &= \frac{3}{4} \times \frac{15\zeta(3)}{\pi^4} g \frac{aT^3}{k} \\ n_{boson(+)}(T) &= \frac{15\zeta(3)}{\pi^4} g \frac{aT^3}{k} \end{aligned} \tag{249}$$

We may evaluate the energy density as;

$$\begin{aligned} \rho(T) &= \int_0^\infty dp\, 4\pi p^2 E_p n(p;T) = \frac{g}{(2\pi\hbar)^3} \int_0^\infty dp \frac{4\pi p^3}{e^{\frac{p}{kT}} \pm 1} \\ &= \frac{g\,(kT)^4}{2\pi^2\hbar^3} \int_0^\infty dx \frac{x^3}{e^x \pm 1} \end{aligned} \tag{250}$$

We may evaluate;

$$\int_0^\infty dx \frac{x^3}{e^x \pm 1} = \frac{15 \mp 1}{240}\pi^4 \qquad \text{ie. } \frac{7}{8} \times \frac{1}{15}\pi^4\,,\; \frac{1}{15}\pi^4 \tag{251}$$

Then,

$$\begin{aligned} \rho_{fermion(-)}(T) &= \frac{7}{8} \times \frac{1}{2} g\, a\, T^4 \\ \rho_{boson(+)}(T) &= \frac{1}{2} g\, a\, T^4 \end{aligned} \tag{252}$$

Recall that $P = \rho/3$. This is the statistical mechanics description of our radiation/hot matter $w = 1/3$.

Note a fermion species contributes 4/3 to the number density and 7/8 to the energy density and pressure compared to a boson.

Recall from earlier that for matter without chemical potential, or $\mu \simeq 0$, then;

$$s = \frac{\rho + p}{T} = \frac{4}{3}\frac{\rho}{T} \tag{253}$$

Hence,

$$s_{boson} = \frac{2}{3}\, g\, a\, T^3 \,, \quad s_{fermion} = \frac{7}{8} \times \frac{2}{3}\, g\, a\, T^3 \tag{254}$$

Examples:
Consider photons so that $g = 2$. Then,

$$\rho_\gamma(T) = a\, T^4 \tag{255}$$

and hence the definition of the radiation constant a.

Consider electrons and antielectrons, so now $g = 2\times 2$ (particle + antiparticle, each with two spins). Then.

$$\rho_{e\bar{e}}(T) = \frac{7}{4} a\, T^4 \tag{256}$$

Non-relativitistic limit
Suppose $E_p - \mu \gg kT > 0$, hence one can only excite low momentum modes. We include the chemical potential here, but it may be that particle number is not conserved in which case one sets $\mu = 0$. Now,

$$\begin{aligned} n\,(p; T, \mu) &= \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{E_p - \mu}{kT}} \pm 1} \\ &\simeq \frac{g}{(2\pi\hbar)^3} e^{-\frac{E_p - \mu}{kT}} \end{aligned} \tag{257}$$

Now since,

$$E_p = m + \frac{p^2}{2m} + \ldots \tag{258}$$

and hence our condition implies $m - \mu \gg kT$. Then,

$$n \simeq \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu - m}{kT}} e^{-\frac{p^2}{2mkT}} \tag{259}$$

The number density may be evaluated;

$$n = \int_0^\infty dp\, 4\pi p^2 n(p) \simeq g \left(\frac{kmT}{2\pi\hbar^2} \right)^{\frac{3}{2}} e^{\frac{\mu - m}{kT}} \tag{260}$$

Now in the case that we have no chemical potential, so $\mu = 0$ then we see this implies $n \sim e^{\frac{-m}{kT}}$ and we have a strong exponential suppression of the particles since our analysis is only for temperatures where $kT \ll m$, ie. the non-relativistic regime.

In order to have an appreciable number density for non-relativistic particles as $T \to 0$ we require a conserved particle number, and hence chemical potential μ. We see then that in the $T \to 0$ limit we must have $\mu \to m$. At non-zero T we have μ determined by the equation above to maintain the conserved particle number $n \propto \frac{1}{a^3}$.

Continuing we may explicitly evaluate ρ and P integrals using the form of $n(p)$ above, and one finds;

$$\begin{aligned} \rho &= \int_0^\infty dp\, 4\pi p^2 n(p) E_p \simeq \left(m + \frac{3}{2} kT + \ldots \right) n \\ P &= \int_0^\infty dp\, 4\pi p^2 n(p) \frac{p^2}{3E_p} \simeq (kT + \ldots)\, n \end{aligned} \tag{261}$$

and hence we see,

$$P \ll \rho \tag{262}$$

This is our statistical mechanics description of cold matter $w = 0$. Thus by cold we really mean $m - \mu \gg kT$.

Photon radiation in our universe: Consider $\Omega_m \simeq 0.32$, $\Omega_B \simeq 0.05$, $\Omega_\gamma \ll 1$ and $h \simeq 0.67$. Then consider the relic CMB photons. These have a thermal distribution (although they are not in equilibrium) with a temperature today $T_{CMB} = 2.725K$. Thus we have,

$$\rho_\gamma = \frac{3H_o^2}{8\pi G} \Omega_\gamma = aT^4 \quad \Longrightarrow \quad \Omega_{gamma} \simeq 5.5 \times 10^{-5} \tag{263}$$

Note that the total radiation today is due to these photons together with relic neutrinos, which we will see later have a similar fraction $\Omega_{\nu\bar{\nu}}$. In fact as we will show later,

$$\begin{aligned}\Omega_r &= \Omega_\gamma + \Omega_{\nu\bar{\nu}} = \left(1 + 3 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{\frac{4}{3}}\right) \Omega_\gamma \\ &\simeq 1.68\, \Omega_\gamma \end{aligned} \quad (264)$$

Hence the total radiation fraction $\Omega_r \ll 1$ as we assumed earlier.

Note that while a negligible fraction of the energy density is in radiation today, in fact the photons vastly outnumber the baryons. We have,

$$\begin{aligned} n_\gamma &\simeq \frac{30\zeta(3)}{\pi^4} \frac{aT^3}{k} \sim 4 \times 10^8 m^{-3} \\ n_B &\simeq \frac{\rho_B}{m_{proton}} = \frac{3H_o^2}{8\pi G m_{proton}} \Omega_B \sim 0.25 m^{-3} \end{aligned} \quad (265)$$

and hence $n_\gamma \gg n_B$.

An important epoch was **radiation-matter equality**, the time when the energy density in radiation and matter were equal. Consider our ΛCDM model. Then,

$$\rho_{tot} = \rho_{crit} \left(\Omega_\Lambda + \Omega_m (1 + Z)^3 + \Omega_r (1 + Z)^4\right) \quad (266)$$

Then radiation-matter equality occurred at a redshift Z_{eq} when,

$$\Omega_m \simeq \Omega_r (1 + Z_{eq}) \qquad \Longrightarrow \qquad Z_{eq} \simeq 3500 \quad (267)$$

Note that since the redshift is large we can consistently ignore the cosmological constant contribution which will be totally subdominant.

As we discuss later in more detail, the photon temperature redshifts as $T \sim 1/a$. Hence the temperature at radiation-matter equality was,

$$T_{eq} \simeq T_{CMB} (1 + Z_{eq}) \sim 10^4 K \quad (268)$$

Application: photon-baryon mix Consider a mixture of photons and baryons in a adiabatically expanding universe. (This partly models our universe at temperatures $T \ll 10^{13} K$, although omits the leptons and neutrinos.)

The total entropy density $s_{tot} = s_\gamma + s_B$ and $s_{tot}a^3$ =constant. Then since baryon number is conserved we also have $n_B a^3$ =constant. This implies that the total entropy per baryon is constant. We define the dimensionless quantity,

$$\sigma = \frac{s_{tot}}{k\, n_B} = \text{const} \tag{269}$$

to measure this.

Now we use,

$$s_\gamma = \frac{4}{3}aT^3\ , \quad \frac{s_B}{n_B} = \text{const} + k \log\left(\frac{T^{3/2}}{Cn_B}\right) \tag{270}$$

for a constant C, and hence,

$$\sigma \quad = \quad \frac{4}{3}\frac{aT^3}{kn_B} + \log\left(\frac{T^{3/2}}{Cn_B}\right) \tag{271}$$

Recall that, $n_\gamma = \frac{30\zeta(3)aT^3}{\pi^4 k}$, then,

$$\text{const} = \sigma \quad = \quad \frac{4\pi^4}{3\times 30\zeta(3)}\left(\frac{n_\gamma}{n_B}\right) + \log\left(\frac{T^{3/2}}{Cn_B}\right) \tag{272}$$

This equation governs the dependence of n_B on n_γ and T.

Note that there are two interesting limits. Firstly $n_\gamma \gg n_B$ as in our universe. In this case, due to the logarithm, this implies,

$$\frac{n_\gamma}{n_B} \quad \simeq \quad \text{const} \tag{273}$$

and hence (since $n_B a^3$ =const and $n_\gamma \sim T^3$),

$$a^3 T^3 \quad \simeq \quad \text{const} \tag{274}$$

so we recover the behaviour as if there were only photons, namely $T \sim 1/a$.

Conversely if $n_\gamma \ll n_B$ (not the case in our universe) then the argument of the log, $T^{3/2}/n_B$, must be approximately constant, and we recover $T \sim 1/a^2$, the behaviour of baryons.

2.3 Chemical equilibrium and the Saha equation

Consider we have q particle species in equilibrium at fixed E, V. Consider their particle numbers N_i for $i = 1, \ldots, q$ to be conserved up to allowing one reaction between the particles. The question is then how do the particles want to partition their particle numbers? Due to this reaction one combination of N_i is no longer conserved, and hence there is one less chemical potential.

Suppose the reaction relates changes in particle numbers in the ratios;

$$dN_i = a_i dQ \; , \tag{275}$$

where a_i are integers. eg. for particles p_1, p_2, p_3 and a reaction process $p_1 + 2p_2 \to p_3$ we would have $a_1 = 1$, $a_2 = 2$, $a_3 = -1$. Then at equilibrium for fixed E, V reactions must lead to $dS = 0$. However $-TdS = \sum_i \mu_i dN_i$. Hence we learn for dN_i given by the reaction we must have;

$$0 = \sum_i \mu_i dN_i = dQ \sum_i \mu_i a_i \tag{276}$$

so we find;

$$\sum_i a_i \mu_i = 0 \tag{277}$$

which indeed eliminates one chemical potential.

For non-relativistic particles in equilibrium we have number densities with distributions (setting $\hbar = c = 1$);

$$n_i = g_i \left(\frac{m_i kT}{2\pi} \right)^{\frac{3}{2}} e^{\frac{\mu_i - m_i}{kT}} \tag{278}$$

Then the particular combination;

$$\prod_i (n_i)^{a_i} = \prod_i (g_i)^{a_i} \left(\frac{m_i kT}{2\pi} \right)^{\frac{3a_i}{2}} e^{-\frac{a_i m_i}{kT}} \tag{279}$$

is independent of the chemical potentials; this combination of number densities is then just determined by temperature and is the Saha equation.

2.4 Boltzmann equation for FRW

We may always find coordinates so that we can write the metric as,

$$ds^2 = -dt^2 + g_{ij}(t, x)dx^i dx^j \tag{280}$$

where g_{ij} is the metric on a (spatial - ie. Riemannian) manifold S. As before we assume we can describe a large number of identical particles by a phase space density $n(t, x^i, p_j)$. Formally the phase space is $T^\star S$. Note that the phase space momentum is a covector field, ie. has p_j with its index down stairs. The interpretation is the n gives the number of particles in a phase space volume $d^3x^i d^3p_j$. ie. the total number of particles N_V in a region of phase space V is,

$$N_V(t) = \int_V d^3x^i d^3p_j n(t, x^k, p_l) \tag{281}$$

The importance of momentum being a covector is that the phase space measure is trivial.

Now consider the case of **flat** FRW so,

$$ds^2 = -dt^2 + a(t)^2\, \delta_{ij} dx^i dx^j \tag{282}$$

and also restrict the phase space density to be homogeneous and isotopic. Homogeneity implies that n is independent of x^i. Isotropy implies that the dependence on p_i can only be isotropic, ie. we only have dependence on the magnitude of the momentum. Thus it is convenient to introduce the physical momentum p measured by comoving observers. We may then take p to be a coordinate on phase space, and homogeneity and isotropy implies $n = n(t, p)$.

Recovering the real space number density: We are interested in the real space number density $n(t)$. How is this derived from this phase space density $n(t, p)$? A comoving observer measures this physical momentum magnitude;

$$p = n_\mu p^\mu \tag{283}$$

where n_μ in the observers LIF is a unit magnitude spatial vector, pointing in the spatial direction of propagation ie. $n_t = 0$, $n_i \propto p_i$ and $\frac{1}{a^2}\delta^{ij} n_i n_j = 1$. Hence, $n_i = \frac{a(t)p_i}{\sqrt{\delta^{mn} p_m p_n}}$, and,

$$p = \frac{a(t)p_i}{\sqrt{\delta^{mn} p_m p_n}} p_j \frac{1}{a^2} \delta^{ij} = \frac{1}{a(t)} \sqrt{\delta^{ij} p_i p_j} \tag{284}$$

Note that the energy observed is then $E_p^2 = p^2 + m^2$. Suppose our matter was interacting to maintain a thermal equilibrium. Then $n(t,p)$ would be given by the Bose/Fermi distribution;

$$n(t,p) = \frac{1}{(2\pi\hbar)^3} \frac{1}{e^{\frac{E_p - \mu}{kT}} \pm 1} \ , \quad E_p^2 = p^2 + m^2 \tag{285}$$

Now let us derive the phase space measure. Suppose we parameterize the covector momentum as,

$$p_1 = \mu \cos\theta \cos\phi \ , \quad p_2 = \mu \cos\theta \sin\phi \ , \quad p_3 = \mu \sin\theta \tag{286}$$

so that $\mu^2 = \delta^{ij} p_i p_j$. Isotropy implies that $\mu = \mu(t,p)$ and there is no θ or ϕ dependence to n. n is a function of t, p or equivalently t, μ.

Consider the phase space measure $d^3 p_i$. Then,

$$\int d^3 p_j = \int_0^\infty d\mu \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\pi} d\phi \, \mu^2 \cos\theta = \int_0^\infty d\mu \mu^2 \int d\Omega = 4\pi \int_0^\infty d\mu \mu^2 \tag{287}$$

where $d\Omega$ is the angular measure. In terms of physical 3-momentum, $\mu = a(t)p$. Thus the phase space measure given by,

$$dN = n(t,p) \, d^3x^i d^3p_j = n(t,p) \, d^3x^i \mu^2 d\mu d\Omega = n(t,p) \, \left(a^3(t) \, d^3x^i\right) \left(p^2 dp d\Omega\right) \tag{288}$$

but now we recognise $a^3 d^3 x^i$ as the proper spatial volume measure. Hence we see the real space number density is derived from the our homogeneous isotropic phase space density in FRW as,

$$n(t) = \int_0^\infty dp \, 4\pi p^2 n(t,p) \tag{289}$$

Thus is just the usual expression as for flat space, where p is understood as being the physical momentum comoving observers see.

Obtaining the Boltzmann equation: The dynamics of the phase space density is determined by the Liouville theorem. This famously states that the the volume of a region of phase space is preserved under time evolution. In terms of the density it simply states;

$$L[n] \equiv \frac{d}{dt} n = 0 \tag{290}$$

where $L = d/dt$ is the Liouville operator, and must be understood as 'flow' derivative,

$$\frac{d}{dt} = \left.\frac{\partial}{\partial t}\right|_{x^i, p_j} + \dot{x}^i \left.\frac{\partial}{\partial x^i}\right|_{t, p_j} + \dot{p}_i \left.\frac{\partial}{\partial p_i}\right|_{t, x^i} \tag{291}$$

where $\dot{} = d/dt$.

We begin by treating the particles as non-interacting, and hence freely propagating along geodesics. Note, since they are non-interacting this is quite different to thermal equilibrium. We now want to derive the Boltzmann equation that determines the evolution of $n(t, p)$ in time. An important point; we argued previously that homogeneity implies for both massive and massless geodesics, the physical momentum observed by comoving observers as the geodesic passes then obeys;

$$a(t) p = \mathrm{const} \tag{292}$$

which implies that;

$$\frac{1}{p} \frac{\partial p}{\partial t} = -\frac{\dot{a}}{a} \tag{293}$$

Then applying the Liouville theorem using the coordinate p;

$$\begin{aligned} 0 = \frac{d}{dt} n(t, p) &= \left(\left.\frac{\partial}{\partial t}\right|_p + \frac{\partial p}{\partial t} \left.\frac{\partial}{\partial p}\right|_t \right) n(t, p) \\ &= \left(\left.\frac{\partial}{\partial t}\right|_p - p \frac{\dot{a}}{a} \left.\frac{\partial}{\partial p}\right|_t \right) n(t, p) \end{aligned} \tag{294}$$

Obviously we may solve this equation trivially as,

$$n(t, p) = n\left(a(t) p\right) \tag{295}$$

which since $a(t)p$ =const obviously implies $\frac{d}{dt}n(a(t)p) = 0$.

Example: Consider massless particles in FRW with a thermal distribution. Then consider the particle interactions immediately switch off at a time $t = t_i$, when the gas has temperature $T = T_i$. Thus,

$$n(t_i, p) = \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{p}{kT_i}} \pm 1} \tag{296}$$

Subsequently the evolution is governed by the Boltzmann equation, so, $n = n(a(t)p)$. Thus for $t > t_i$ the solution is,

$$n(t, p) = n(a(t)p) = \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{a(t)p}{ka(t_i)T_i}} \pm 1} \tag{297}$$

which obviously agrees with our boundary condition at $t = t_i$. We may write this as,

$$n(t, p) = \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{p}{kT_{eff}}} \pm 1} \ , \quad T_{eff}(Z) = \frac{a(t_i)}{a(t)} T_i = \frac{1}{1 + Z(t)} T_i \tag{298}$$

where $1 + Z(t) = a(t)/a(t_i)$ is the redshift of the past time t_i measured at the later time t.

An important application of this is that photons and neutrinos maintain an equilibrium Bose/Fermi distribution if they are initially in thermal equilibrium and then suddenly start **free streaming** - ie. stop interacting. The effective temperature governing their distribution goes as $T_{eff} \sim 1/a$.

2.5 Boltzmann equation with collision term

Now we consider an interacting gas. We describe this using our free Boltzmann equation and a 'collision term'. This collision term may have a simple form for local interactions, such as in perturbative QFT. In general we write,

$$L_{free}[n] = C[n] \tag{299}$$

In our FRW case we have,

$$\left(\frac{\partial}{\partial t}\bigg|_p - \frac{\dot{a}}{a} p \frac{\partial}{\partial p}\bigg|_t \right) n(t, p) = C\left[n(t, p)\right] \tag{300}$$

Consider the local density in real space;

$$n(t) = \int_0^\infty dp\, 4\pi p^2 n(t,p) \tag{301}$$

Then, integrating the Boltzmann equation;

$$\begin{aligned}\int_0^\infty dp 4\pi p^2 C\left[n\right] &= \int_0^\infty dp\, 4\pi p^2 \left(\left.\frac{\partial}{\partial t}\right|_p - \frac{\dot{a}}{a} p \left.\frac{\partial}{\partial p}\right|_t \right) n(t,p) \\ &= \left.\frac{\partial}{\partial t}\left(\int_0^\infty dp\, 4\pi p^2 n \right)\right|_p - 4\pi \frac{\dot{a}}{a} \int_0^\infty dp\, p^3 \left.\frac{\partial n}{\partial p}\right|_t \\ &= \frac{dn(t)}{dt} + 4\pi \frac{\dot{a}}{a} \int_0^\infty dp\, n \left.\frac{\partial (p^3)}{\partial p}\right|_t \end{aligned} \tag{302}$$

where we have integrated by parts and thrown away the boundary terms,

$$\left[p^3 n(t,p)\right]_0^\infty \tag{303}$$

which is justified by smoothness of $n(t,p)$ in p at $p \to 0$, and good high energy behaviour for $p \to \infty$. Then,

$$\begin{aligned}\int_0^\infty dp\, 4\pi p^2 C\left[n\right] &= \frac{dn(t)}{dt} + 3\frac{\dot{a}}{a} \int_0^\infty dp\, 4\pi p^2 n \\ &= \frac{dn(t)}{dt} + 3\frac{\dot{a}}{a} n(t) \end{aligned} \tag{304}$$

Hence we obtain,

$$\frac{1}{a^3(t)} \frac{d}{dt} \left(a^3(t) n(t) \right) = \int_0^\infty dp\, 4\pi p^2 C\left[n\right] \tag{305}$$

Note that $a^3(t)n(t)$ is proportional to the number of particles comoving volume. Defining the comoving number density as

$$N(t) = a^3(t) n(t) \tag{306}$$

then we have,

$$\frac{dN}{dt} = a(t)^3 \int_0^\infty dp\, 4\pi p^2 C\left[N\right] \tag{307}$$

so that in the absence of collisions we have the free Boltzmann equation and consequently $N =$constant.

2.6 Application 1: Decay into thermal products

Suppose we have a species X that decays into products Z_i which are in thermal equilibrium at a temperature T.

$$X \rightarrow Z_1 + Z_2 + \ldots \tag{308}$$

We may use the Boltzmann equation to study the out of equilibrium dynamics of X, in the approximation that the products Z_i are held in equilibrium by other interactions.

Let us denote the rate of decay of a single X particle as Γ, and denote the number density of X as $n(t)$ and the comoving number density as $N(t) = a^3 n$. Note that since the decay is into thermal products, this rate will be a function of temperature. We assume that the temperature of the products Z_i is $T = T(t)$. For example, typically $T \sim 1/a(t)$ for the expanding universe.

Then the Boltzmann equation is,

$$\begin{aligned} \frac{1}{a^3(t)} \frac{d}{dt} \left(a^3(t) n(t) \right) &= (\#created/vol) - (\#decay/vol) \\ &= (\#created/vol) - \Gamma(T) \times n(t) \end{aligned} \tag{309}$$

Now detailed balance tells use that in equilibrium the left hand side must vanish. Suppose at temperature T the equilibrium number density for X is $n_{eq}(T)$, and correspondingly the comoving density is $N_{eq}(T)$. Then,

$$\frac{1}{a^3(t)} \frac{d}{dt} \left(a^3(t) n(t) \right) = \Gamma(T) \left(n_{eq}(T) - n(t) \right) \tag{310}$$

and hence,

$$\frac{dN(t)}{dt} = \Gamma(T) \left(N_{eq}(T) - N(t) \right) \tag{311}$$

determines the time evolution of $N(t)$. For cosmology it is convenient to write the evolution in terms of a rather than t, using;

$$\frac{d}{dt} = \frac{da}{dt} \frac{d}{da} = H(a) \frac{d}{d \ln a} \tag{312}$$

Then we find,

$$\frac{d \ln N(a)}{d \ln a} = -\frac{\Gamma(T)}{H(a)} \left(1 - \frac{N_{eq}(T)}{N(a)} \right) \tag{313}$$

If we further assume that $a = 1/T$ (where say T is the photon temperature), we may write this as,

$$\frac{d\ln N}{d\ln T} = \frac{\Gamma}{H}\left(1 - \frac{N_{eq}}{N}\right) \tag{314}$$

where all quantities are functions of T.

Let us consider the behaviour of this equation. Suppose at high temperatures the particle X is close to thermal equilibrium. Let us take,

$$N = N_{eq}\,(1+f) \tag{315}$$

where initially $|f| \ll 1$. Then,

$$\ln N = \ln N_{eq} + f \tag{316}$$

so,

$$\frac{df}{d\ln T} + \frac{d\ln N_{eq}}{d\ln T} = \frac{\Gamma}{H}\left(1 - \frac{1}{1+f}\right) \simeq \frac{\Gamma}{H} f \tag{317}$$

and hence,

$$\frac{df}{d\ln T} = \frac{\Gamma}{H} f - \frac{d\ln N_{eq}}{d\ln T} \tag{318}$$

For decreasing temperature we may regard the first term on the right as a restoring term, and the second as a driving term. Suppose that $d\ln N_{eq}/d\ln T \sim O(1)$ and that $1 \ll \Gamma/H$. Then as T decreases, we will have,

$$f \simeq \frac{1}{\Gamma/H}\frac{d\ln N_{eq}}{d\ln T} \ll 1 \tag{319}$$

Consider examples of this driving term;

- Suppose X is highly relativistic - then $n_{eq} \sim T^3$ so N_{eq} ~constant. Thus $d\ln N_{eq}/d\ln T \simeq 0$.
- Suppose X is non relativistic with a potential so $n_{eq} \sim T^{3/2}$ so $N_{eq} \sim T^{-3/2}$. Thus $d\ln N_{eq}/d\ln T \simeq -3/2$.

Thus usually we have the case that $d \ln N_{eq}/d \ln T \sim O(1)$. (Note this is not always the case - eg. non-relativistic species with no potential $n_{eq} \sim T^{-3/2} e^{-m/kT}$.) Then the condition that equilibrium is preserved as the temperature drops is;

$$1 \ll \frac{\Gamma}{H} \tag{320}$$

This is a crucially important condition. Intuitively it says that if the decay (and hence creation) rate is much quicker than the expansion rate then the species is held in equilibrium. However, if the universe expands too fast, equilibrium cannot hold.

Suppose the converse is true, namely that $\Gamma/H \ll 1$. In this case we simply have our free Boltzmann equation,

$$\frac{d \ln N}{d \ln T} = 0 \qquad \Longrightarrow \qquad N = \text{const} \tag{321}$$

and hence $n \sim T^3 \sim 1/a^3$.

Suppose for some reason for $T > T_{freeze}$ we have $1 \ll \Gamma/H$, but then quickly changes for $T < T_{freeze}$ to $\Gamma/H \ll 1$. Then T_{freeze} is called the **freeze out** temperature. The the approximate solution to the boltzmann equation is (as we have seen before in our photon example above);

$$\begin{aligned} N &\simeq N_{eq}(T) & T &> T_{freeze} \\ N &\simeq N_{eq}(T_{freeze}) & T &< T_{freeze} \end{aligned} \tag{322}$$

For quick transitions this is a good approximation to a full solution of the Boltzmann equation. Note that rather than N decreasing as $N_{eq}(T)$, instead a **relic density** of X particles is left over.

The relic density of the matter today, t_0, is simply calculated. Suppose it is a massive particle, mass m. Then,

$$\begin{aligned} \rho_{relic} &= m\, n(t_0) = m \left(\frac{a^3(T_{freeze})}{a_0^3} \right) n_{eq}(T_{freeze}) \\ &= m \left(\frac{T_{CMB}}{T_{freeze}} \right)^3 n_{eq}(T_{freeze}) \end{aligned} \tag{323}$$

assuming that $a \sim 1/T_{photon}$ which is to a good approximation true. Then,

$$\Omega_{relic} = \frac{\rho_{relic}}{\rho_{crit}} = \frac{8\pi G}{3H_0^2}\rho_{relic} \tag{324}$$

Now recall that in in the radiation and matter eras $H \sim 1/t$. Hence for single particle decay, we might not expect that Γ/H decreases with temperature. However, let us consider decays involving more than one particle in the initial state.

2.7 Application 2: Decay via interaction with thermal species

A similar equation arises if we consider a process where X decays via interaction with a thermal species Y into thermal products Z_i, so,

$$X + Y \rightarrow Z_1 + Z_2 + \ldots \tag{325}$$

Now the rate of decay of an X particle, Γ, depends on the cross section of the process σ, the relative velocity v, and the number density of Y, $n_Y(T)$ as,

$$\Gamma = \langle \sigma v \rangle \, n_Y(T) \tag{326}$$

where the $\langle\rangle$ signify the expectation value in the thermal state, and given the distribution for X - thus it is a complicated expression. We may think of $\langle \sigma v \rangle$ as the 'interaction volume'.

Thus we have,

$$\frac{d \ln N}{d \ln T} = \frac{\Gamma}{H}\left(1 - \frac{N_{eq}}{N}\right) = \frac{\langle \sigma v \rangle n_Y}{H}\left(1 - \frac{N_{eq}}{N}\right) \tag{327}$$

Again the species will be driven to equilibrium for falling temperatures if $1 \ll \Gamma/H$. For $\Gamma/H \ll 1$ it will behave as a free 'relic'.

Example: If X is a non-relativistic particle, and Y is relativistic, then one often finds this quantity is roughly constant, so $\langle \sigma v \rangle \sim$const eg. think of a cannon ball.

Suppose particle Y has mass m_Y. Consider early times when $m_Y \ll kT$ so it behaves as a relativistic particle and then $n_Y \sim T^3$. Assume X is non-relativistic, with $\langle \sigma v \rangle \sim$ constant.

Then in the radiation era we have $H \sim 1/a^2 \sim T^2$. Hence we see at early times we expect $\Gamma/H \sim T^3/T^2 \sim T \gg 1$.

Assume this holds true until the temperature $kT \sim m_Y$ is reached. Then below this Y is non-relativistic. Suppose it has no potential so $n_Y \sim T^{3/2} e^{-m/kT}$. Then very quickly we will transition to, $\Gamma/H \sim e^{-m/kT}/T^{1/2} \ll 1$.

Example: neutron decay,

$$n + \bar{e} \rightarrow p + \bar{\nu} \tag{328}$$

We will discuss this later. In fact there $\Gamma/H \sim T^3$ early on so it is slightly more complicated.

2.8 Application 3: Decay of particle antiparticle

Consider a decay of a particle X and antiparticle $\bar{X}$ with conserved particle number $X - \bar{X}$. Consider decay into thermal products Z_i,

$$X + \bar{X} \quad \rightarrow \quad Z_1 + Z_2 + \ldots \tag{329}$$

Now assume a thermal distribution for X initially and there is no potential so $n_X = n_{\bar{X}}$ initially and for all times after. Thus we may write $n = n_X = n_{\bar{X}}$. Then we have,

$$\begin{aligned} \frac{1}{a^3(t)} \frac{d}{dt} \left(a^3(t) n_X(t) \right) &= (\#created/vol) - (\#decay/vol) \\ &= (\#created/vol) - \langle \sigma v \rangle n_X(t) n_{\bar{X}}(t) \\ &= (\#created/vol) - \langle \sigma v \rangle n(t)^2 \end{aligned} \tag{330}$$

and by detailed balance then,

$$\frac{1}{a^3(t)} \frac{d}{dt} \left(a^3(t) n_X(t) \right) = \langle \sigma v \rangle \left(n_{eq}^2 - n(t)^2 \right) \tag{331}$$

where $n_{eq}(T)$ is the equilibrium distribution for X and $\bar{X}$. Then,

$$\begin{aligned}\frac{d\ln N}{da} &= \frac{\langle\sigma v\rangle N}{H}\left(1-\frac{N_{eq}^2}{N^2}\right) \\ &= \frac{\Gamma}{H}\left(1-\frac{N_{eq}^2}{N^2}\right)\end{aligned} \tag{332}$$

with $\Gamma = \langle\sigma v\rangle n$.

2.9 Boltzmann equations and QFT

Our discussion of the behaviour of the real space number density has shown us that the quantity Γ/H controls when in a cooling universe a species breaks from equilibrium, and its approximate behaviour after. However we need to understand how to compute Γ directly from a QFT model. Let us return to the Boltzmann equation for the phase space density,

$$\frac{dn(t,p)}{dt} = C[n] \tag{333}$$

An interesting point to note is that elastic collisions (eg. Thompson scattering $e+\gamma \to e+\gamma$ with $E_\gamma \ll m_e$) has $C[n] = 0$. While there is scattering, the number of particles in a momentum shell dp is constant.

Let us reconsider a one body, decay as considered above, but with a specific QFT interaction. Consider again $X \to Z_i$ where the products are thermally distributed. Recall we can consider $C = C_{decay} + C_{creation}$ thinking of the forward and reverse processes separately. Then (see eg. Peskin and Schroder)

$$C_{decay} = \frac{n(t,p)}{2E}\prod_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)} \sum_{av\,spins\,and\,p_X} |M|^2 (2\pi)^4\delta^4(p_X - \sum_f p_f) \tag{334}$$

The factor $1/2E$ is required for Lorentz invariance, and then,

$$d\Gamma = C_{decay} 4\pi p^2 dp \tag{335}$$

gives $d\Gamma$ the rate of decay/volume of X due to a momentum shell p. Here M is the matrix element for the process $X \to Z_i$.

Comment on conventions: $\langle p|q\rangle = 2E_p(2\pi)^3\delta^3(p-q)$. And ${}_{out}\langle f|i\rangle_{in} = \langle f|S|i\rangle$. Then $S = 1+iT$ and $\langle f|iT|i\rangle = (2\pi)^4\delta^{(4)}(mom)iM$. Also $\hbar = c = 1$.

Example: Consider a scalar particle X, mass M, decaying into 2 identical scalar particles, mass m, in the final state. Suppose this is given in perturbation theory by a a tree level process with coupling λ. Suppose firstly the decay occurs in the zero temperature vacuum. In this case we simply have,

$$|M|^2 = \lambda^2 \tag{336}$$

with no momentum dependence. Furthermore,

$$\int \frac{d^3p_1}{(2E_1)}\int \frac{d^3p_2}{(2E_2)}\delta^4(p_X - p_1 - p_2) = \int \frac{d^3p_1}{(4E_1E_2)}\,\delta(E_X - E_1 - E_2)\big|_{p_X^i = p_1^i + p_2^i} \tag{337}$$

and using $E_1 = \sqrt{m^2 + |p_1|^2}$, and,

$$\int f(x)\delta(g(x))dx = \sum_{\{x|g(x)=0\}} \frac{f(x)}{|g'(x)|} \tag{338}$$

then,

$$\begin{aligned}\int \frac{d^3p_1}{2E_1}\delta(\bar{E} - \sqrt{m^2+|p_1|^2})\,(\ldots) &= \int_0^\infty \frac{|p_1|^2 d|p_1|\,d\Omega_1}{2E_1}\,\delta(\bar{E} - \sqrt{m^2+|p_1|^2})\,(\ldots)\\ &= \begin{cases} \int d\Omega_1 \frac{1}{2E_1}\left(|p_1|E_1\,(\ldots)\right)\Big|_{\bar{E}=E_1} & E > \bar{E}\\ 0 & E < \bar{E}\end{cases}\\ &= \begin{cases} \int d\Omega_1 \left(\frac{|p_1|}{2}\,(\ldots)\right)\Big|_{\bar{E}=E_1} & E > \bar{E}\\ 0 & E < \bar{E}\end{cases}\end{aligned} \tag{339}$$

Thus we can compute,

$$C_{decay} = \frac{n(t,p)}{2E}\sum_{av\,p_X}\frac{\lambda^2}{(2\pi)^2}\int d\Omega_1\left(\frac{|p_1|}{4E_2}\right)\Big|_{E_X=E_1+E_2, p_X^i=p_1^i+p_2^i} \tag{340}$$

This is still not trivial to compute. However, let us now assume that the particle X is very heavy and highly non-relativistic, and the products are much lighter, $m \ll M$. Then $E_X \simeq M$, and $p^i_X \sim 0$, and the products will be highly relativistic so $E_i \sim |p_i|$. In this case, the kinematics simply so that,

$$p^i_1 \simeq -p^i_2 \quad , \qquad E_1 \simeq E_2 \simeq |p_1| \simeq |p_2| \simeq \frac{M}{2} \tag{341}$$

Hence one obtains for $p \ll M$,

$$C_{decay} \;=\; \frac{n(t,p)}{2M}\frac{\lambda^2}{4\pi} \tag{342}$$

Thus we see the decay rate in this limit has no dependence on p except trivially through the factor of $n(t,p)$. Integrating over the species X momentum we obtain;

$$\frac{1}{a^3(t)}\frac{d\left(a^3 n(t)\right)}{dt} = -\int dp_X 4\pi p_X^2 C_{decay} \simeq -\frac{n(t)}{2M}\frac{\lambda^2}{4\pi} \tag{343}$$

using $\int dp_X 4\pi p_X^2 n(t,p) = n(t)$ and noting there will be no particle creation spontaneously from vacuum. Hence we simply obtain,

$$\frac{dN}{dt} = -\frac{\lambda^2}{8\pi M}N \tag{344}$$

Thermal end state:
However, in cosmology we are interested in the situation that the final state particles are in thermal equilibrium. Then the matrix element is not evaluated in vacuum, but in this thermal state. For a tree level process one finds,

$$\prod_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)}\,|M|^2_{thermal} \;=\; \prod_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)}\,|M|^2_{vacuum}\left(1 \pm F\right)_f \tag{345}$$

The factors F are the filling fraction of momentum states of the final state particles. In vacuum these are zero, but in a thermal product state we have the filling fraction for a species f which is a boson $(+)$ or fermion $(-)$ are;

$$\frac{1}{(2\pi)^3}\left(1 \pm F\right)_f = \frac{1}{(2\pi)^3}\left(1 \pm \frac{1}{e^{\frac{E-\mu}{kT}} \mp 1}\right) = \frac{1}{(2\pi)^3}\frac{1}{1 \mp e^{-\frac{E-\mu}{kT}}} \tag{346}$$

giving stimulated boson emission, and reduced fermion emission.

Returning to our previous example, this modifies the result. Now, using our same approximation $m \ll M \simeq E_X$

$$
\begin{aligned}
C_{decay} &= \frac{n(t,p)}{2E} \sum_{av\, p_X} \frac{\lambda^2}{(2\pi)^2} \int d\Omega_1 \left(\frac{|p_1|}{4E_2}\right) \left(\frac{1}{1-e^{-\frac{E_1}{kT}}}\right) \left(\frac{1}{1-e^{-\frac{E_2}{kT}}}\right) \\
&\simeq \frac{n(t,p)}{2M} \frac{\lambda^2}{4\pi} \left(\frac{1}{1-e^{-\frac{M}{2kT}}}\right)^2 \qquad (347)
\end{aligned}
$$

Now assuming our theory is T invariant (= CP invariant) then the backward process is simply given by detailed balance, so,

$$
\frac{dN}{dt} = \frac{\lambda^2}{8\pi M} \left(\frac{1}{1-e^{-\frac{M}{2kT}}}\right)^2 (N_{eq}(T) - N) \qquad (348)
$$

Typically if we are treating X non-relativistically we would be considering the situation $kT \ll M$, and then,

$$
\begin{aligned}
\frac{dN}{dt} &\simeq \frac{\lambda^2}{8\pi M} (N_{eq}(T) - N) \\
&= \Gamma (N_{eq}(T) - N) \qquad (349)
\end{aligned}
$$

exactly as we had discussed before. Hence in this simple case we have derived $\Gamma = \lambda^2/8\pi M$ and in fact is constant.

Let us briefly check our units. Γ is a rate and hence should have mass dimension $[\Gamma] = +1$. The Lagangian must contain the interation term $\lambda X Z_1 Z_2$ and should be mass dimension -4. Since X, Z_i are scalars with dimension 1, then $[\lambda] = 1$. Thus indeed we have $[\Gamma] = [\lambda^2] - [M] = +2-1 = +1$ as required.

Example 2: Now consider 2 - 2 scattering, where $X + Y \to W + Z$ and both Y, W and Z are thermally distributed. Now we have for the forward process,

$$
\begin{aligned}
C_{decay} &= \frac{n(t,p)}{2E} \int \frac{d^3p_Y}{(2E_Y)} n_Y \prod_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)} \sum_{av\, spins\, and\, p_X} |M|^2_{vac} (1 \pm F)_f (2\pi)^4 \delta^4(p_X - \sum_f p_f) \\
&= \frac{n(t,p)}{2E} \int \frac{d^3p_Y}{(2E_Y)} \frac{n_Y}{(2\pi)^2} \sum_{av\, spins\, and\, p_X} \int d\Omega_1 |M|^2_{vac} (1 \pm F)_1 (1 \pm F)_2 \left.\frac{|p_1|}{4E_2}\right|_{on-shell} \qquad (350)
\end{aligned}
$$

where n_Y would be a thermal distribution,

$$n_Y = \frac{1}{(2\pi)^3}\frac{1}{e^{\frac{E_Y-\mu_Y}{kT}} \mp 1} \tag{351}$$

Assume the process is dominated by a tree level scattering with coupling λ. Let us consider the situation where X has mass M_X, and Z has mass M_Z with $\Delta M = M_X - M_Z$. Consider $kT \ll M_X, M_Z$ and these to behave non-relativistically. Consider the other species, Y and W to be highly relativistic at the thermal scales of interest.

Consider the initial and final momenta; $p_X \simeq (M_X, 0, 0, 0)$, $p_Y \simeq (p, p, 0, 0)$, $p_Z \simeq (E_Z, p+q\cos\theta, q\sin\theta\cos\phi, q\sin\theta\sin\phi)$, $p_W \simeq (q, -q\cos\theta, -q\sin\theta\cos\phi, -q\sin\theta\sin\phi)$. Then 3-momentum is conserved. Since $kT \ll M_X$, then $p \sim kT \ll M_X$. Consider energy;

$$M_X + p = E_Z + q \simeq M_Z + q \quad \Longrightarrow \quad q \simeq \Delta M + p \tag{352}$$

Check;

$$E_Z \;=\; M_Z + \frac{(p+q\cos\theta)^2 + q^2\sin^2\theta}{2M_Z} + \ldots \tag{353}$$

But since $p^2 \ll M_X^2$, M_Z^2 and $q^2 = (\Delta M + p)^2 \ll M_Z^2$ also, then indeed $E_Z \simeq M_Z$. The kinematics is thus much simplified so there is no angular dependence.

$$\begin{aligned}
C_{decay} &\simeq \frac{n_X}{2E_X}\int\frac{d^3p_Y}{2E_Y}\frac{n_Y}{(2\pi)^2}\int d\Omega_W\lambda^2\left(\frac{1}{1-e^{-\frac{E_W}{kT}}}\right)\left(\frac{1}{1-e^{-\frac{E_Z}{kT}}}\right)\frac{|p_W|}{4E_Z}\Bigg|_{on-shell} \\
&\simeq \frac{n_X}{2M_X}\int_0^\infty\frac{4\pi p^2 dp}{2p}\frac{1}{(2\pi)^5}\left(\frac{1}{e^{\frac{p}{kT}}-1}\right)4\pi\lambda^2\left(\frac{1}{1-e^{-\frac{\Delta M+p}{kT}}}\right)\frac{|\Delta M+p|}{4M_Z} \\
&\simeq \frac{n_X}{M_X M_Z}\frac{\lambda^2}{4(2\pi)^3}\int_0^\infty dp\frac{p\,(\Delta M+p)}{\left(e^{\frac{p}{kT}}-1\right)\left(1-e^{-\frac{\Delta M+p}{kT}}\right)}
\end{aligned} \tag{354}$$

In fact we have to be careful with the lower limit if $\Delta M < 0$. Then $\int_{-\Delta M}^{\infty}$ are the limits due to the on-shell constraint. Let $x = p/kT$, and $y = \Delta M/kT$, then,

$$\begin{aligned}
C_{decay} &\simeq \frac{n_X}{M_X M_Z}\frac{\lambda^2}{4(2\pi)^3}(kT)^3\int_0^\infty dx\frac{x\,(y+x)}{(e^x-1)\,(1-e^{-y-x})} \\
&\simeq \frac{n_X}{M_X M_Z}\frac{\lambda^2}{4(2\pi)^3}(kT)^3 f(\frac{\Delta M}{kT})
\end{aligned} \tag{355}$$

where one can evaluate $f(y)$. For example, if $|\Delta M| \ll kT$ then $f(y) \simeq f(0) = \pi^2/3$, and $\Delta M \ll M_X, M_Z$ so,

$$C_{decay} \sim \frac{n_X \lambda^2}{M_X^2} (kT)^3 \qquad |\Delta M| \ll kT \tag{356}$$

On the other hand, for $kT \ll |\Delta M|$ then $f(y) \simeq \pi^2 y/6$, so,

$$C_{decay} \sim \frac{n_X \lambda^2 \, \Delta M}{M_X M_Z} (kT)^2 \qquad kT \ll |\Delta M| \tag{357}$$

In addition to this process we must also include the 1-body decay process, $X \to Y + W + Z$. Suppose that $\Delta M < 0$. In this case the 1-body decay cannot occur. Suppose also that $|\Delta M| \ll kT$. Then, integrating over $d^3 p_X$, would lead to a Boltzmann equation for the real space density of X, n,

$$\begin{aligned} \frac{1}{a^3} \frac{d\,(a^3 n)}{dt} &= \frac{\lambda^2}{96\pi} \frac{(kT)^3}{M_X^2} \left(n_{eq}(T) - n(t)\right) \\ &= \Gamma \left(n_{eq}(T) - n(t)\right) \end{aligned} \tag{358}$$

Again we check our units. Γ has mass dimension $[\Gamma] = +1$. The Lagangian must contain the interaction term $\lambda XYZW$ and should be mass dimension -4. Since the particles are all scalars with dimension 1, then $[\lambda] = 0$. Temperature has mass dimension $[kT] = +1$. Thus indeed we have $[\Gamma] = [\lambda^2] + [(kT)^3] - [M^2] = 0 + 3 - 2 = +1$ as required.

An important comment is that the form of this expressions is quite constrained on dimensional grounds. Since for a tree level process $C_{decay} \propto \lambda^2$, and in this case also $C_{decay} \propto n_Y \sim (kT)^3$, then at low temperatures, but not so low that ΔM is important, then the only possibility for the most relevant mass scale is M_X so,

$$\Gamma \sim \frac{\lambda^2 (kT)^3}{M_X^2} \tag{359}$$

Suppose instead that the process was occurring at very high temperatures so $M_X \ll kT$, and all the species were essentially relativistic. In this case then the only relevant mass scale is given by kT. We could precisely work out the expression, but it must go as,

$$\Gamma \sim \lambda^2 (kT) \tag{360}$$

simply on dimensional grounds.

An important example of such a process is the 2-2 scattering $n + e \to p + \nu$ mediated by the weak interaction. Note now the particles are fermions, not bosons, and this is a more complicated matrix element. For $kT \ll M_W$ then this vertex is described by the weak Fermi interaction vertex, coupling $G_W \sim 10^{-5}/(GeV^2)$. Suppose we consider temperatures $M_{n,p} \ll kT \ll M_W$, then it is clear that the decay rate of neutrons,

$$\Gamma \sim G_W^2 (kT)^p \tag{361}$$

and since $[G_W] = -2$ and $[\Gamma] = +1$ then,

$$\Gamma \sim G_W^2 (kT)^5 \tag{362}$$

Such a process will be important in nucleosynthesis later.

Example 3: Again consider 2 - 2 scattering but now of a complex scalar particle X and antiparticle $\bar{X}$, mass M, into 2 scalar particles mass m. Assume particle number conservation is preserved by the interactions and so $n(t,p) = n_X(t,p) = n_{\bar{X}}(t,p)$. Assume a tree level vertex with coupling λ. Then,

$$C_{decay} = \frac{n(t,p)}{2E} \int \frac{d^3p'}{2E_{p'}} \frac{n(t,p')}{(2\pi)^2} \sum_{av\, spins\, and\, p_X} \int d\Omega_1 \left|M\right|^2_{vac} (1+F)_1 (1+F)_2 \left.\frac{|p_1|}{4E_2}\right|_{on-shell} \tag{363}$$

so that integrating over p,

$$\frac{1}{a^3(t)} \frac{d\left(a^3 n(t)\right)}{dt} = \text{creation} - \int \frac{d^3p}{2E} n(t,p) \int \frac{d^3p'}{2E_{p'}} n(t,p') \sum_{av} \ldots \tag{364}$$

Again suppose our X particles are highly non-relativistic so that $p \ll E \sim M$. Also again suppose that $M \gg m$ so the products are highly relativistic.

Then, $p_1^i \simeq -p_2^i$ and so $E_1 \simeq E_2 \simeq M$ by momentum and energy conservation. The filling factors,

$$\begin{aligned} (1+F)_1 (1+F)_2 &= \left(\frac{1}{1-e^{-\frac{E_1}{kT}}} \right) \left(\frac{1}{1-e^{-\frac{E_2}{kT}}} \right) \\ &\simeq \left(\frac{1}{1-e^{-\frac{M}{kT}}} \right)^2 \sim 1 \end{aligned} \tag{365}$$

again assuming that $kT \ll M$. Then we find,

$$\begin{aligned} \frac{1}{a^3(t)} \frac{d\left(a^3 n(t)\right)}{dt} &\simeq \text{creation} - \frac{n^2(t)}{4M^2} \frac{1}{(2\pi)^2} 4\pi\lambda^2 \frac{M}{4M} \\ &= \frac{\lambda^2}{16\pi M^2} \left(n_{eq}^2(T) - n^2(t) \right) \\ &= \langle \sigma v \rangle \left(n_{eq}^2(T) - n(t)^2 \right) \end{aligned} \tag{366}$$

In this example we have derived that,

$$\langle \sigma v \rangle = \frac{\lambda^2}{16\pi M^2} \tag{367}$$

and is therefore constant in time.

Let us again check our units. σv is a rate per volume, and hence should have mass dimension $[\sigma v] = -2$. The Lagangian must contain the interation term $\lambda X \bar{X} Z_1 Z_2$ and since $X, \bar{X}, Z_i$ are scalars with dimension 1, then $[\lambda] = 0$ ie. is dimensionless. This gives, $[\sigma v] = [\lambda^2] - [M^2] = 0 - 2 = -2$ as required.

Suppose instead we were interested in a high temperature limit, where all particles were relativistic, so $M_X \ll kT$. In this case, simply on dimensional grounds we would expect,

$$\langle \sigma v \rangle = \frac{\lambda^2}{(kT)^2} \tag{368}$$