||5.4|| FEYNMAN DIAGRAMS

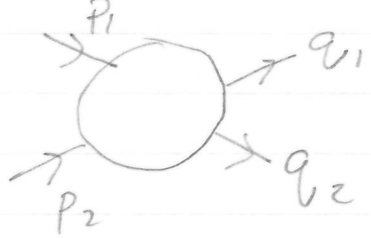
Tong §3.3.3 (3.25)

To illustrate

- We will use SYTh (Scalar Yukawa Theory)

$$H_{int} = -\int d^3x \, \mathcal{L}_{int}, \quad \mathcal{L}_{int} = -g\,\phi(x)\,\psi^\dagger(x)\psi(x)$$

- Consider $\psi\psi \to \psi\psi$ scattering [PS6, Q4]



$$\mathcal{M} = \langle f | S | i \rangle$$
$$\downarrow$$
$$G(y_1, y_2, z_1, z_2)$$

where, ...

Initial state: 2 ψ's momenta $\underline{p}_1$ & $\underline{p}_2$

Final state: 2 ψ's momenta $\underline{q}_1$ & $\underline{q}_2$

F2

See Handout $\mathcal{M} \to G$

$$\Rightarrow \mathcal{M} = \langle f | S | i \rangle$$

LEAVE OUT

$$= \prod_{i=1,2} \left(\int d^3 y \; e^{-ip_i y_i} \, 2\Omega_{p_i} \right) \quad \{ y_i^0 \to -\infty$$

$$\prod_{f=1,2} \left(\int d^3 z \; e^{+iq_f z_f} \, 2\Omega_{q_f} \right) \quad \{ z_f^0 \to +\infty$$

$$\times \underbrace{\langle 0 | T \psi(z_1) \psi(z_2) \, S \, \psi^+(y_1) \psi^+(y_2) | 0 \rangle}$$

We need the 4-point Green function G

where

$$G(y_1, y_2 \, z_1, z_2) = \langle 0 | T \psi(z_1) \psi(z_2) \psi^+(y_1) \psi^+(y_2) S | 0 \rangle$$

$$= \sum_n G_n (y_1, y_2 \; z_1 z_2)$$

Since $S = \exp\{ -i \int_{-\infty}^{+\infty} dt \, H_{int} \} = \exp\{ +i \int d^4 x \, \mathcal{L}_{int} \}$

we have

$$G_n = \langle 0 | T \psi(z_1) \psi(z_2) \psi^+(y_1) \psi^+(y_2)$$

$$\times \frac{(-ig)^n}{n!} \prod_{i=1}^{n} \left(\int d^4 x_i \; \psi^+(x_i) \psi(x_i) \phi(x_i) \right)$$

$$\times | 0 \rangle$$

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CUT - on previous page

(Example $\psi\psi \to \psi\psi$ in SYTh (PS6, Q4))

We need $G = \langle 0 | T\, \psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2) S | 0 \rangle$

$$= \sum_n G_n(y_1, y_2, z_1, z_2)$$

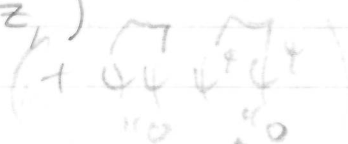
where $G_n = \langle 0 | T\, \psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2) \cdot \frac{(-ig)^n}{n!} \prod_{i=1}^{n} \left(\int d^4x_i\, \psi^\dagger(x_i)\psi(x_i)\phi(x_i) \right) | 0 \rangle$

(a) G_0 $O(g^0)$

EFS PS6, Q4ii — EFS find two contributions to G_0 corresponding to terms in $\mathcal{U}_0$ in

$$\Delta_\psi(y_1 - z_1)\,\Delta_\psi(y_2 - z_2) + \Delta_\psi(y_1 - z_2)\,\Delta_\psi(y_2 - z_1)$$

FREE FIELD PROP.



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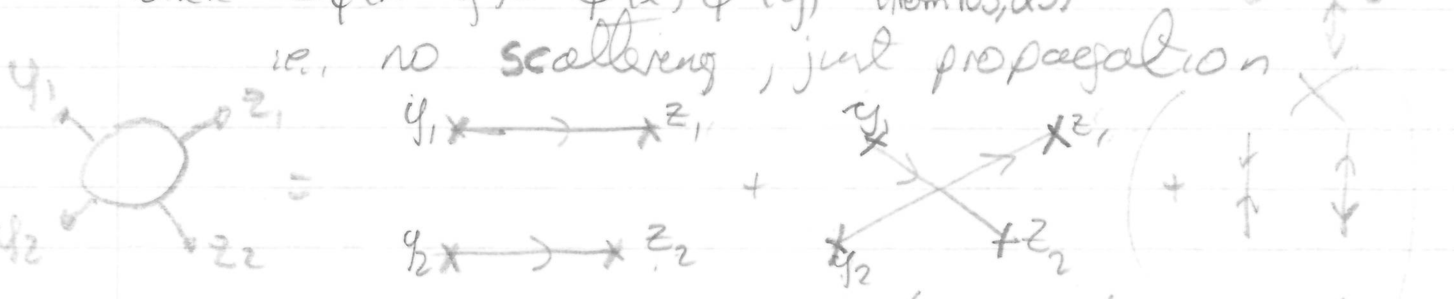
where $\Delta_\psi(x-y) = \psi(x)\psi^\dagger(y)$ (from PS5, Q3)

i.e. no scattering, just propagation



20L19 16/11/16 24/11/17 $\Rightarrow$ In momentum space find $\mathcal{M} \propto \delta^4(p_1 - q_1)\delta^4(p_2 - q_2) + \delta^4(p_1 - q_2)\delta^4(p_2 - q_1)$

PS6 Q4iii (b) G_1 $O(g^1)$

EFS All odd powers of g are zero, follows from odd number of ϕ's in expression + Wick's theorem $\Leftrightarrow$ $\phi_i\, \psi/\psi^\dagger = 0$

(c) G_2 $O(g^2)$

$$G_2(y_1, y_2, z_1, z_2) = \frac{(-ig)^2}{2!} \int d^4x_1 \int d^4x_2 \; \Gamma(x_1, x_2, y_1, y_2, z_1, z_2)$$

where

$$\Gamma = \langle 0 | T \, \psi(z_1)\psi(z_2) \, \psi^+(x_1)\psi(x_1)\phi(x_1) \, \psi^+(x_2)\psi(x_2)\phi(x_2) \, \psi^+(y_1)\psi^+(y_2) | 0 \rangle$$

10 field expectation value!

Choose standard split for $\langle 0 | \ldots | 0 \rangle$,
Apply Wick's theorem & use $\langle 0 | : \text{fields} : | 0 \rangle = 0$

$$\Rightarrow \Gamma = \langle 0 | T \psi(z_1)\psi(z_2)\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\psi^\dagger(y_1)\psi^\dagger(y_2) | 0 \rangle$$

$$\stackrel{(A)}{=} \psi(z_1)\psi(z_2)\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\psi^\dagger(y_1)\psi^\dagger(y_2)$$

+ Many other terms with 5 contractions [945 Terms in total]

$\Gamma \equiv \Gamma_A + (\text{rest})$

$$\Gamma_A = \Delta_\psi(z_1 - x_1)\,\Delta_\psi(z_2 - x_2)\,\Delta_\phi(x_1 - x_2)\,\Delta_\psi(x_1 - y_1)\,\Delta_\psi(x_2 - y_2)$$

<u>NOTES</u> (For our standard split)

(i) <u>Trivial Zero Terms</u>

From $[a, b^\dagger] = 0$, $[a, a] = 0$ etc find

EFS PS5, Q2 (iii)

a) $\psi\phi = 0$, $\psi^\dagger\phi = 0$ contractions of different fields zero

b) $\psi\psi = \psi^\dagger\psi^\dagger = 0$ – Complex field with itself are zero

$\Rightarrow$ c) <u>contractions non-zero only for field & (field)†</u> h.c.

$\psi(x)\psi^\dagger(y) = \Delta_\psi(x - y)$ same form as $\phi\phi$ except Ω_k, M not ω_k, m

$\phi(x)\phi(y) = \Delta_\phi(x - y)$

$\Rightarrow$ so many terms zero.

[*** Only contractions between a field & its h.c. are non-zero in standard cases.]

e.o.L20 30/11/15 (ii) Same contribution to G(t) from other terms in Γ

e.g. x_1 & x_2 terms from H_{int} are identical
so can swap roles

$$\Gamma_B = \psi(z_1)\psi(z_2)\psi^+(x_1)\psi(x_1)\phi(x_1)$$
$$\psi^+(x_2)\psi(x_2)\phi(x_2)\psi^+(y_1)\psi^+(y_2)$$

will give identical contribution to G as Γ_A
*** In fact permutation of internal vertex coord cancels $\frac{1}{n!}$ H_{int}^n

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Help!!! $9.7.5.3.1 = 945$ distinct pairs of 5 contractions here

~~ALL ONE DIAGRAM ONE NUMBER FACTOR~~

Combinatorics

CUT see later

(i) $\frac{1}{n!}$ from $S = T\sum\frac{(iHt)^n}{n!}$ is largely cancelled

by permutation of x_i in H_{int}
e.g. $x_1 \Leftrightarrow x_2$ in Γ_2 above

(ii) Each interaction term chosen with normalisation to match permutations of fields

e.g. $\frac{\lambda}{4!}\phi^4$ as $\underbrace{\phi(x)\phi(x)\phi(x)\phi(x)}$

4! ways of matching these contractions

$\frac{g}{1!}\psi^+\psi\phi$

↑ No identical fields

Coordinate Space Feynman Rules for GREEN FUNCTIONS $\langle 0 | T \text{fields} | 0 \rangle$

0/ Draw all topologically distinct diagrams using rules below.

Aim is one distinct algebraic contribution to G
= one distinct Feynman diagram.

1 Connect pairs of vertices with lines representing contractions Δ

(a) For REAL fields, no arrows on line

e.g. $\underset{x_i}{\bullet} - - - - - \underset{x_j}{\bullet} = \Delta_\phi(x_i - x_j) = \overbrace{\phi(x_i)\phi(x_j)}$

b) For COMPLEX fields = conserved charge add arrow to differentiate ψ & ψ^+

e.g. $\underset{x_i}{\overset{(\psi)}{\bullet}} \longleftarrow \underset{x_j}{\overset{(\psi^+)}{\bullet}} = \Delta_\psi(x_i - x_j) = \overbrace{\psi(x_i)\,\psi^+(x_j)}$

Arrow ensures $\bullet \leftarrow\!\rightarrow \bullet$ or $\bullet \rightarrow\!\leftarrow \bullet$ NOT DRAWN

$\overbrace{\psi\psi} = 0$ $\quad$ $\overbrace{\psi^+\psi^+} = 0$

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F7.2

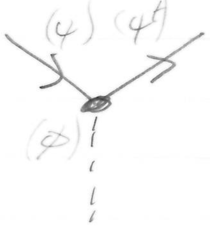
2 Each term in H_{int} represented by an "internal vertex" with

a) Unique coordinate x_i but NOT considered as a label when counting distinct diagrams

b) Integration $\int d^4x_i$

c) $(-i) * $ (coupling constant)

d) one leg for each field in term

e.g. SYTh $\phi\psi^\dagger\psi$ $\rightarrow$  $\equiv -ig\int d x_i$

3 Fields carrying arguments of Green function
e.g. used for initial/final states of U
represented by "External Vertex" with one leg and one coordinate label y_i^μ, z^μ
which is NOT integrated

$$\underset{y_i}{\times\!\!\longleftarrow} \equiv \psi(y_i)\ldots$$

$$y_i \times\!\!\longrightarrow \equiv \psi^+(y_i)\ldots$$

$$y_i \times\text{-----} \equiv \phi(y_i)\ldots$$

N.B. often no $\times\!\!-$ added, the end of any dangling leg = external vertex

(Coord. Space F. Rules continued)

4 Divide by the "Symmetry Factor S

remains $n!$ from exp expansion in here.

(PSp93) Symmetry Factor & Combinatorics

Many terms found expanding $\langle 0|T(\text{fields})|0\rangle$ using Wick give same expression but we can avoid over counting

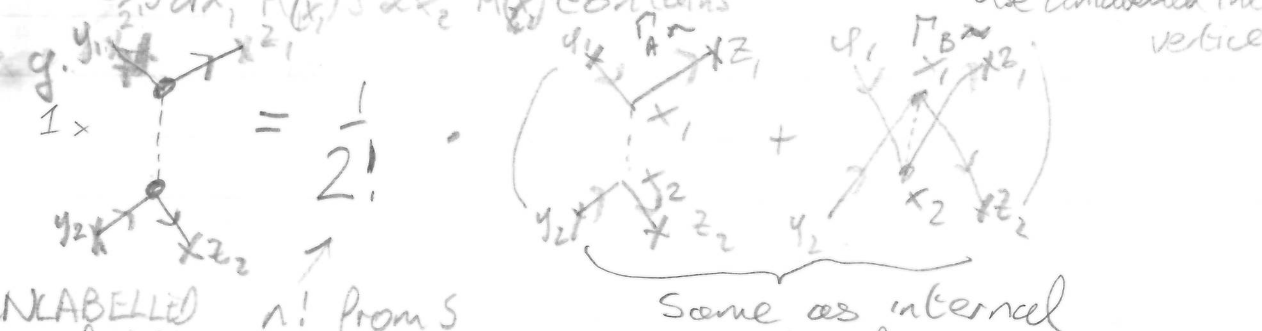
(i) $\frac{1}{n!}$ from $S = Te^{-iH_{int}t} = \sum_n \frac{(-iH(t))^n}{n!}$

This matches the $n!$ ways of assigning labels x_i to each vertex

$\Rightarrow$ DO not consider these permutations = $\frac{-i}{2!}\int dx_1 H(x_1) \int dx_2 H(x_2)$ contains

Use unlabelled int. vertices.

e.g.



UNLABELLED Internal Vertices

$n!$ from S

Same as internal coordinates are integrated over, dummy indices.

* DON'T LABEL or DON'T PERMUTE INTERNAL x_i LABELS

ONE DIAGRAM $\Leftrightarrow$ Two terms in Wick's theorem

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(ii) Permutations at each vertex

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Do not distinguish legs of same type at an internal vertex

PROVIDED you include this factor in definition of coupling constant

ie. $\mathcal{L}_{int} = \pm\, (cc)(\text{fields})\,/\,P$ $\quad P = \#$ Permutations fields which leave $\mathcal{L}_{int}$ unchanged.

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(ii) Permutations of Fields at Vertex

Example $\mathcal{L}_{int} = -\dfrac{\lambda}{4!}\phi^4$ $\qquad H_{int} = -\int d^3x\, \mathcal{L}_{int}$

$\Rightarrow$ $\overset{\Delta}{\bullet\!-\!\!-\!\bullet}$ $\quad y_x$ — $\quad$ $\times = -i\lambda\int d^4x$

One interaction factor produces terms like:

$$G \sim \cdots\left(-i\frac{\lambda}{4!}\int d^4x\ \overset{1}{\phi(x)}\,\overset{2}{\phi(x)}\,\overset{3}{\phi(x)}\,\overset{4}{\phi(x)}\right)\cdot$$

4 WAYS $\quad$ 3 WAYS $\quad$ 2 WAYS $\quad$ 1 WAYS

$$\cdots\phi(x_A)\cdots\phi(x_B)\cdots\phi(x_C)\cdots\phi(x_D)$$

(1A, or 1B, 1C, 1D) $\quad$ with $x_A \neq x_B \neq x_C \neq x_D$

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ALL give same factor of

$$= \frac{\cancel{4!}}{\cancel{4!}}\cdot -i\lambda\cdot\Delta(x-x_A)\,\Delta(x-x_B)\,\Delta(x-x_C)\,\Delta(x-x_D)$$

$\times \ldots$

ALL 4! terms in WICK give ONE contribution = ONE Feynman Diagram.

So DON'T DISTINGUISH IDENTICAL LEGS AT A VERTEX

$=$

NO NEED TO INCLUDE PERMUTATION CONSTANT IN F. RULES.

IF YOU INCLUDE SYMMETRY FACTOR $\frac{1}{P}$ IN COUPLING CONSTANT DEFN

So Here

$\times \equiv -i\lambda\int d^4x$ $\qquad$ NO 4!'s

Generally

$$\mathcal{L}_{int} = -\frac{(c.c)\ (\text{fields})}{(\#\ \text{invariant permutations of fields})} = \text{fields} \equiv -i\,(c.c.)\int d^4x_c$$

(c.c.) ONLY $\qquad$ NO other constants

e.g. $-\dfrac{ig}{4}(\psi^{+}\psi)(\psi^{+}\psi) = \times = -ig\int d^4x_c$

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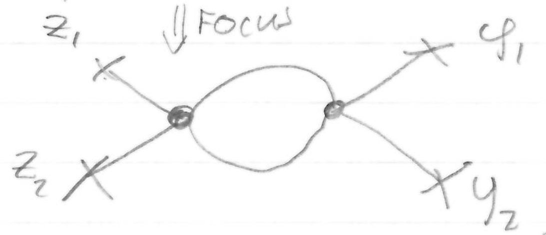
(iii) Symmetry Factor S Definition

S = # Permutations of internal lines which give same diagram

e.g. Consider $\frac{\lambda}{4!}\phi^4$



SWAP 2 INTERNAL LINES $\Leftrightarrow S=2$

Comes from term in Wick expansion of form

$$\underbrace{\phi(z_1)}_{(i)\ 8}\ \phi(z_2) \cdot \frac{1}{2!} \cdot \frac{1}{4!}\ \underbrace{\phi(x)}_{(ii)\ 3}\phi(x)\phi(x)\underbrace{\phi(x) \cdot \frac{1}{4!}\phi(\tilde{x})\phi(\tilde{x})}_{(v)\ 2}\phi(\tilde{x})\underbrace{\phi(\tilde{x})\cdot\phi(y_1)}_{(iii)\ 4}\underbrace{\phi(y_2)}_{(iv)\ 3}$$

$\uparrow$ $\uparrow$

$z_1 = x_A,\ z_2 = x_B$

But $\tilde{x} = x_C = x_D$ in previous example

(i) 8 WAYS to connect $\phi(z_1)$ to internal vertex field e.g. use x here as example

(ii) 3 WAYS to connect $\phi(z_2)$ to internal vertex field of SAME index as z_1 connected to e.g. $\phi(x)$ here

(iii) 4 ways to connect $\phi(y_1)$ to field of OTHER internal coordinate i.e. here $\phi(\tilde{x})$

(iv) 3 ways to connect $\phi(y_2)$ to another field of same coordinate as in (iii), i.e. here $\phi(\tilde{x})$

(v) Only 2 distinct ways to connect EACH OF last two $\phi(x)$ to one of the 2 remaining $\phi(\tilde{x})$

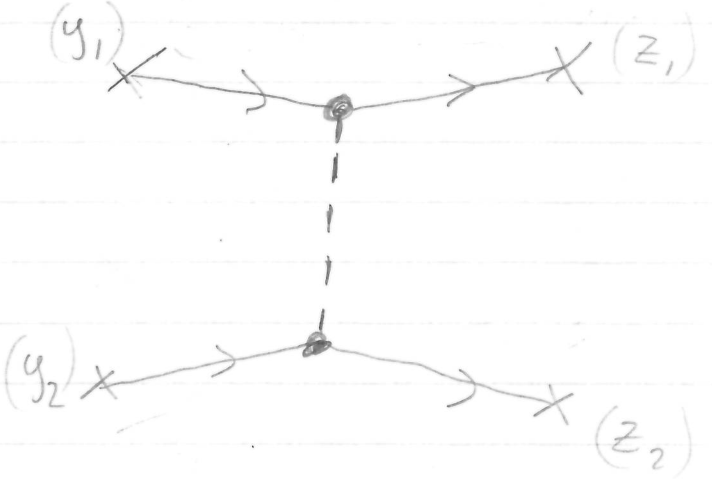
$$\Rightarrow \frac{1}{2!}\frac{1}{4!}\frac{1}{4!}\ 8\times3\times4\times3\times2 = \frac{1}{2} = \frac{1}{S}$$

Theorem

Each distinct contribution to M/G corresponds to a topologically distinct Feynman diagram

e.g. $\left(\frac{\Pi_A + \Pi_B}{2!}\right) =$

$\psi\psi \to \psi\psi$



$(S=1)$

$(i) \longrightarrow (f)$

(See PS5 for more examples PS6), $\exists$ More G_2 diagrams See PS6, Q4

$$= (-ig)^2 \int d^4x_1\, d^4x_2$$

$$* \; \Delta_\psi(x_1-y_1)\, \Delta_\psi(z_1-x_1)\, \Delta_\phi(x_1-x_2)\, \Delta_\psi(x_2-y_2)\, \Delta_\psi(z_2-x_2)$$

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Do this for next year

RECIPE for one diagram

1. Write down external legs for initial/final = "external" state fields
2. Write down n vertices if working at order n.
   Do this in all possible ways if more than one type of vertex
3. Connect up legs of external lines/vertices in all possible distinct ways.