W1

5.3 WICK'S THEOREM 1950

Key tool (but related to general properties for $\langle 0|T\phi\ldots\phi|0\rangle$ of gaussian noise)

→ GOAL: To turn $T(\hat{\phi}\ldots\hat{\phi})$ into product of c-numbers.

Notation

$$\phi_i \equiv \hat{\phi}_i(x_i)$$

(so some $\phi_i = \phi_j$ ↑

where $\hat{\phi}_i$ is any bosonic field (possibly some/all the same field)

$x_i \equiv x_i^\mu$ are distinct coordinates. May take some $x_i^\mu = x_j^\mu$ at end

Normal Ordering

- Split every field into two parts

$$\hat{\phi}_i = \hat{\phi}_i^{+} + \hat{\phi}_i^{-}$$

↑ plus sign NOT dagger † of h.c.

Arbitrary choice, so choose it to simplify calculation

- Expand into sum of products of $\phi_i^\pm$
- A NORMAL ORDERED expression, N, has, in each $\phi_i^\pm$ product, all $\hat{\phi}^-$ to the left of all $\hat{\phi}^+$ but order within ϕ^+ set unchanged, and " " ϕ^- " "

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$$\text{e.g. } N(\phi_1^- \phi_2^+ \phi_3^- \phi_4^+)$$

$$= \phi_1^- \phi_3^- \phi_2^+ \phi_4^+$$

Other cases.

different choice of ϕ^+, ϕ^- is better if

(i) Symmetry Breaking $\langle 0 | \hat{\phi} | 0 \rangle \neq 0$

(ii) Many Body Problems (see PS5)

- Thermal Field Theory

$$\sum_n \langle n | e^{-\beta E_n} \hat{a}^+ \hat{a} | n \rangle$$

$$= \frac{1}{e^{\beta \omega} - 1} \neq 0$$

(iii) Curved Space Time

W2

BEST CHOICE - To simplify calculations we need

Choose $\phi^\pm$ split s.t.

(i) $< N(\text{fields}) > = 0$

(ii) $[\phi_i^+, \phi_j^-] \propto \mathbb{1}$ a c-number

(iii) $[\phi_i^+, \phi_j^+] + [\phi_i^-, \phi_j^-] = 0$

Usual case

For vacuum expectation values $\langle \mathcal{O} \rangle = \langle 0 | \mathcal{O} | 0 \rangle$ choose

$$\phi^+(x) = \int \frac{d^3k}{\sqrt{2\omega}} e^{-ikx} a_k$$ ~ PURE ANNIHILATION op.

$$\phi^-(x) = \int \frac{d^3k}{\sqrt{2\omega}} e^{+ikx} a_k^+$$ ~ PURE CREATION op.

EFS This choice ensures (i), (ii), (iii) true

e.g. $\langle 0 | N(\text{fields}) | 0 \rangle = \sum \langle 0 | \phi^- \phi^- \phi^- \ldots \phi^+ \phi^+ \phi^+ | 0 \rangle$

$= 0$

Notation

$:(\text{fields}): = N(\text{fields})$ when $\phi_i^\pm$ split as above

Usual notn in books who only use split above

W 3.5

Two-Point Functions

Two arbitrary bosonic fields $\phi_1 \equiv \phi_1(x)$ $x_1 \equiv x$
$\phi_2 \equiv \phi_2(y)$ $x_2 \equiv y$

Define a CONTRACTION of two fields Δ s.t.

$$\overbrace{\phi_1(x)\,\phi_2(y)} = \Delta_{12}(x,y)$$
$$= T(\phi_1(x)\phi_2(y)) - N(\phi_1(x)\phi_2(y))$$

Shorthand

$$\overbrace{\phi_1\,\phi_2} = \Delta_{12} = T_{12} - N_{12}$$

To find Δ

$$N(\phi_1(x)\,\phi_2(y)) = \phi_1^+(x)\,\phi_2^+(y) + \phi_1^-(x)\,\phi_2^+(y) + \underbrace{\phi_2^-(y)\,\phi_1^+(x)}_{\text{Only term which switches order c.f. } \phi_1(x)\phi_2(y)} + \phi_1^-(x)\,\phi_2^-(y)$$

W.4

Two-Point Functions Two fields $\phi_1 \equiv \phi_1(x)$
$\phi_2 \equiv \phi_2(y)$

Define CONTRACTION of two fields Δ

$$\Delta \text{ s.t. } T\phi_1(x)\phi_2(y) = N\phi_1(x)\phi_2(y) + \overset{\text{Def}^n \text{ of } \Delta}{\Delta_{12}(x,y)}$$

(Also denoted $\Delta(x,y) = \phi(x)\phi(y)$)

$$\Rightarrow N\phi_1(x)\phi_2(y) = \overbrace{\phi_1^+(x)\phi_2^+(y)}^{\text{Order fixed}} + \overbrace{\phi_1^-(x)\phi_2^+(y)}$$

$$+ \underbrace{\phi_2^-(y)\phi_1^+(x)} + \underbrace{\phi_1^-(x)\phi_2^-(y)}_{\text{Order fixed}}$$

Only term different from $\phi(x)\phi(y)$

$\underline{x_0 > y_0}$

$$T\phi_1(x)\phi_2(y) = \phi_1(x)\phi_2(y)$$

$$\Rightarrow \Delta_{12}(x,y) = \phi_1^+(x)\phi_2^-(y) - \phi_2^-(y)\phi_1^+(x)$$

$$= [\phi_1^+(x), \phi_2^-(y)]$$

$\underline{x_0 < y_0}$

$$T\phi_1(x)\phi_2(y) = \phi_2(y)\phi_1(x)$$

Exploit $[\phi^+, \phi^+] = [\phi^-, \phi^-] = 0$ (see PS5 for general version)

$$\Rightarrow \Delta_{12}(x,y) = [\phi_2^+(y), \phi_1^-(x)]$$

$$\Rightarrow \Delta_{12}(x,y) = \theta(x_0 - y_0)[\phi_1^+(x), \phi_2^-(y)]$$

$$+ \theta(y_0 - x_0)[\phi_2^+(y), \phi_1^-(x)]$$

Now if $[\phi^+, \phi^-] \propto \hat{\mathbb{1}}$ a c-number

e.g. when $\phi^+ \sim \int \hat{a}$ $\phi^- \sim \int \hat{a}^\dagger$

$\Rightarrow$ CONTRACTION is a c-number
Invariably choose this to be so

If it is a c-number AND if $\langle N(\text{fields})\rangle = 0$ then

$$\langle T\phi_1(x)\,\phi_2(y)\rangle = \overset{=0}{\langle N\phi_1(x)\,\phi_2(y)\rangle} + \Delta_{12}(x,y)$$

NOT^N

If $\phi_1 = \phi_2 = \phi$

$$\Rightarrow \Delta_F(x-y) = \Delta(x-y) = \overbracket{\phi(x)\,\phi(y)}$$

NOTATION

FEYNMAN PROPAGATOR = CONTRACTION
in this case

e.o. L17
18/11/15
17/11/16

General Fields

Same result for best choice of split

$$\overbracket{\phi_i(x)\,\phi_j(y)} = \Delta_{F\,ij}(x-y)\,\hat{\mathbb{1}} = \Delta_{ij}\,\hat{\mathbb{1}} = \underbrace{T\hat{\phi}_i\hat{\phi}_j}_{\text{MANY contractions zero if fields unrelated}} - N\hat{\phi}_i\hat{\phi}_j$$

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See PS5

WICK'S THEOREM

$$T\,\phi_1\phi_2\dots\phi_n = \sum_{\substack{\text{ALL POSSIBLE}\\ \text{CONTRACTIONS}\\ 0 \text{ to } \frac{n}{2}}} N(\phi_1 \dots \overline{\phi_i \dots \phi_j} \dots \phi_n)$$

etc

Def^n A CONTRACTION of a pair of fields means replacing those operators by a factor of Δ and is denoted as $\overline{\phi_1 \dots \phi_2}$

i.e. $$\hat{A}\,\overbrace{\hat{\phi}_1(x_1)\,\hat{B}\,\hat{\phi}_2}(x_2)\,\hat{C}$$

$$= \hat{A}\hat{B}\hat{C}\,\underbrace{\Delta_{12}(x_1,x_2)}_{}$$

$$= T(\phi_1(x_1)\,\phi_2(x_2))$$

$$- N(\phi_1(x_1)\,\phi_2(x_2))$$

So Wick's Theorem is:-

$$T\,\phi_1\phi_2\cdots\phi_m = \sum_{\substack{\text{ALL}\\\text{possible}\\\text{contractions}}} N\ \phi_1\phi_2\cdots\phi_n$$

$$= N(\phi_1\phi_2\cdots\phi_n)$$ 0 contractio

$$+$$

$$+\Big\{N(\overbrace{\phi_1\phi_2}\phi_3\cdots\phi_n)$$ 1 contracti

$$+$$

$$N(\overbrace{\phi_1\phi_2\phi_3}\phi_4\cdots\phi_n)$$

Note always coef. of 1!

$$+$$

$\vdots$ $\binom{n}{2}$ terms in total $\Big\}$

$$+$$

$$\Big\{N(\overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}\,\phi_5\cdots\phi_n)$$ 2 contractio

$$+$$

$\vdots$ $\frac{1}{2}\frac{n!}{2!2!(n-4)!}$ terms $\Big\}$

$$+$$

$\vdots$

$$\Big\{\overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}\cdots\overbrace{\phi_{n-1}\phi_n}$$ $\left(\frac{n}{2}\right)$ contractios

$$+$$

$\vdots$ Total $[(n-1)(n-3)\ldots 1]$ terms $\Big\}$

$\parallel$

$n!!$

DONE Leave out

Note

$$N(\phi_1\cdots\overbrace{\phi_i\cdots\phi_j}\cdots\phi_n)$$

$$=\overbrace{\phi_i\phi_j}\,N(\phi_1\cdots\cancel{\phi_i}\cdots\cancel{\phi_j}\cdots\phi_n)$$

$\uparrow$

as c-number so commutes

eo L18
1/12/14

Why use Wick's Theorem?

If $\langle N(\phi_1 \dots \phi_n) \rangle = 0$ ∀ fields, ∀n

$$\Rightarrow \mathcal{M} \sim \sum_{\{\phi_i\}} \langle T \phi_1 \dots \phi_n \rangle$$

c. numbers

$$\sim \sum_{\{\phi_i\}} \Bigg(\sum_{\substack{\frac{n}{2} \\ \text{contractions}}} \big(\overbrace{\phi\phi} \dots \overbrace{\phi\phi} \big)$$

$$+ \underbrace{N(\text{at least 2 fields})}_{= 0} \sum_{\substack{< \frac{n}{2} \\ \text{contractions}}} \overbrace{\phi\phi}\overbrace{\phi\phi}\dots$$

$$\sim \sum_{\{\phi_i\}} \sum_{\substack{\frac{n}{2} \\ \text{contractions}}} \underbrace{\Delta . \Delta \dots \Delta}_{\text{Pure Numbers.}}$$

↑ expand S matrix

↑ Wick

Proof of WICK'S THEOREM

Assume (i) $[\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0$ for simplicity

(ii) $[\phi_i^+, \phi_j^-] \propto \mathbb{1}$ c-number necessary

(iii) $\phi_i = \phi_i(x_i)$ ALL times unequal

① T ordering is symmetric

$$T(\phi_1 \phi_2 \dots \phi_n) = T(\phi_{a_1} \phi_{a_2} \dots \phi_{a_n})$$

where $\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$ is permutation

Why? T ordering fixes same order in both cases.

② N ordering is symmetric

$$N(\phi_1 \phi_2 \dots \phi_n) = N(\phi_{a_1} \phi_{a_2} \dots \phi_{a_n})$$

Why? (a) N fixes order between ϕ^+ & ϕ^- but

(b) N leaves order within ϕ^+ fixed

But ϕ^+ commute within themselves, (ditto ϕ^-)

$$\underset{\curvearrowright}{\phi_1^+ \phi_2^- \phi_3^+} = \phi_2^- \underset{\curvearrowright}{\phi_1^+ \phi_3^+}$$

1 then 3

①+② ⇒ Δ_{12} symmetric too. ⇒ WICK's theorem is symmetric

③ HENCE

W.L.O.G. choose $t_1 > t_2 > \dots > t_n$

$$T \phi_1 \phi_2 \dots \phi_n = \phi_1 \phi_2 \dots \phi_n$$

④ Assume Wick is true for (n-1) fields or less

$$T\phi_1\phi_2\ldots\phi_n = \phi_1 \, T\phi_2\phi_3\ldots\phi_n$$

$$= \phi_1 \sum_{\substack{\text{ALL} \\ \text{CONTRACTIONS} \\ \text{OF } 2..n \text{ fields}}} \underbrace{N(\overline{\phi_1 \ldots\ldots \phi_n})}_{(\Delta\ldots\Delta)N(\overset{\text{rest of}}{\text{fields}})}$$

For each term $\phi_1 (\Delta\Delta\ldots\Delta) N(\overset{\text{Rest of}}{\text{fields}})$ we need to add term with no contractions & all terms with one ϕ_1 contraction

$$\phi_1\Delta\ldots\Delta N(\text{r.o.f.}) = \Delta\Delta\ldots\Delta \, N(\phi_1 . (\text{r.o.f}))$$ Ⓐ

$$+\Delta\Delta\ldots\Delta \sum_{(1,j)} N(\overline{\phi_1 \quad (\text{r.o.f.})})$$ Ⓑ

So consider any one of these $\phi_1 N(\text{r.o.f})$ terms.

$$\phi_1 N(\phi_2\ldots\phi_m) = (\phi_1^+ + \phi_1^-) N(\phi_2\ldots\phi_m)$$

$$= \phi_1^+ N(\phi_2\ldots\phi_m)$$

$$+ N(\underset{\uparrow}{\phi_1^-}\phi_2\ldots\phi_m)$$

Since ϕ_1^- will always be 1st under N operation

$\Rightarrow$ This term is part of A

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We still want $N(\phi_1^+\phi_2\ldots\phi_m)$ for A and all Δ_{1j} contractions for B.

$$\phi_1^+ N(\phi_2\ldots\phi_m) = \overset{Ⓒ}{N(\phi_2\ldots\phi_m)\phi_1^+}$$

$$+ \overset{Ⓓ}{[\phi_1^+, N(\phi_2\ldots\phi_m)]}$$

Now

Ⓒ $= N(\phi_2\phi_3\ldots\phi_m)\phi_1^+ = N(\phi_2\ldots\phi_m\,\phi_1^+)$

N leaves ϕ_1^+ on far right

$= N(\phi_1^+\phi_2\ldots\phi)$ = other half of term Ⓐ

since N will push ϕ_1^+ to front of ϕ_j^+'s but order within ϕ_j^+ irrelevant as they all commute, so can push it to back = far left.

Other Ⓑ terms must come from Ⓓ above.

$$[A, BC] = [A,B]C + B[A,C]$$

$$\Rightarrow [A, B_1B_2\ldots B_m] = \sum_j B_1B_2\ldots B_{j-1}[A,B_j]B_{j+1}\ldots B_m$$

Apply to any term in $N\phi_2\ldots\phi_m \sim \sum \phi^-\ldots\phi^-\phi^+\ldots\phi^+$

to see we get a sum over

$$Ⓓ = \sum_j N(\phi_2\ldots\underbrace{[\phi_1^+,\phi_j]}\ldots\phi_m)$$

$$= [\phi_1^+,\phi_j^-] \quad \text{as } [\phi^+,\phi^+]=0$$

$$= \Delta_{1j} \quad \text{as } t_1 > t_j$$

So we have

$$\phi_1 N \phi_2 \dots \phi_m = N \phi_1 \phi_2 \dots \phi_m + \sum_{\substack{(1,j)\\ \text{contractions}}} N(\overbrace{\phi_1 \phi_2 \dots \phi_j} \dots \phi_m)$$

Do this for each N term in $T\phi_2 \dots \phi_n$ which have $(n-1-2m)$ fields under N and m contractions.

Each normal ordered term gets

Ⓐ ϕ_1 absorbed to front of normal ordering

& Ⓑ ϕ_1 contracted to one field in normal ordering.

This gives us WICK for n fields if true for $(n-1)$ fields.

True for $n=2$ - definition of Δ_{ij}

$\Rightarrow$ True $\forall n$

QED.