# INTERACTING QFT $\phi^4$

Consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$$

$\Rightarrow$ Classical e.o.m $\quad \partial^\mu \partial_\mu \phi + m^2 \phi + \underbrace{\frac{\lambda}{3!}\phi^3}_{\text{Non-linear}} = 0$

$\Rightarrow$ solutions of free equation different $p^\mu$ are MIXED

See this in QFT as $\hat{\phi}^4$ term contains (using free fields)

$$\hat{\phi}^4 \sim \underbrace{\hat{a}^+_{p_1} \hat{a}^+_{p_2} \hat{a}^+_{p_3} a_{p_4}}_{\#\text{s of } \phi \text{ particles \& their momenta changing}} \delta(p_1 + p_2 + p_3 - p_4) \quad \text{etc}$$

|| These NON-LINEARITIES capture INTERACTIONS between particles. ||

5.1

Tong §3.1 Interaction Picture

We can solve free QFT EXACTLY so makes sense to build interacting theory around this.

Suppose we split (in ANY picture)

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}$$

(DROP HATS from now on)

| Full | Solvable | Difficult |
|---|---|---|
| | Free | Interactions/Non Linearities |
| | Quadratic | Cubic or higher |
| | $O(\hat{\phi}^n)$ $n \leq 2$ | $\hat{\phi}^n$ $n \geq 3$ |

In the INTERACTION PICTURE (Subscript I)
- operators carry just $H_{0S}$ time dependence

Tong (3.11) p51 $\Rightarrow$ $$O_I(t) := e^{iH_{0S}t}\, O_S\, e^{-iH_{0S}t} \quad (H_0 = H_{0,S})$$

cf $$O_H(t) = e^{iHt}\, O_S\, e^{-iHt} \quad \hat{H} = \hat{H}_S = \hat{H}_H$$

- states carry rest of time dependence

$$|\psi, t\rangle_I = e^{+iH_{0S}t}\, |\psi, t\rangle_S$$

as then

Expectation values unchanged

$$ {}_S\langle\psi, t|\, O_S\, |\psi, t\rangle_S = {}_S\langle\psi, t|\left(e^{-iH_{0S}t}\, O_I(t)\, e^{+iH_{0S}t}\right)|\psi, t\rangle_S$$

$$= {}_I\langle\psi, t|\, O_I(t)\, |\psi, t\rangle_I$$

Note convention here I, H, S Pic identical at $t=0$

## Important Cases

(i) $H_0$

$$H_{0I}(t) = e^{iH_{0S}t} H_{0S} e^{-iH_{0S}t} = H_{0S}$$

$H_{0S}$ is time independent

$* \Rightarrow$ BUT $H_{int\,I}$, $H_I$ are NOT as $[H_S, H_{0S}]$, $[H_{intS}, H_{0S}] \neq 0$

(ii) $\phi_I(t)$

$$\phi_I(t) = e^{iH_{0S}t} \phi_S e^{-iH_{0S}t}$$

$$= \left(\phi_H(t)\right)\Big|_{H_{int}=0}$$

i.e. $\phi_I(t)$ is exactly as we had for free field case for our quadratic $H_0$

Usual $H_0 \Rightarrow \quad \phi_I(t) = \int \frac{d^3k}{\sqrt{2\omega_k}} \left(a_k e^{-ikx} + a_k^+ e^{+ikx}\right)$

$k_0 = +\omega_k \geq 0$

(iii) Products

Let $\mathcal{O}_{12S} = \mathcal{O}_{1S}\mathcal{O}_{2S}$

$$\Rightarrow \mathcal{O}_{12I}(t) = e^{iH_{0S}t} \mathcal{O}_{1S} \underset{e^{-iH_{0S}t} \cdot e^{+iH_{0S}t}}{\wedge} \mathcal{O}_{2S} e^{-iH_{0S}t}$$

$$= \mathcal{O}_{1I}(t) \cdot \mathcal{O}_{2I}(t)$$

(iv) $H_{int}$ Suppose $H_{intS} = \int d^3x \left(\lambda/4!\right) \phi_S^4$

$$\Rightarrow H_{int,I} = \int d^3x \left(\frac{\lambda}{4!}\right) \cdot \phi_I(t)$$

(iv) $H_{int}$

e.g. Suppose $H_{int,S} = \int d^3x \left(\frac{\lambda}{4!}\right) \phi_S^{\,4}$

$$\Rightarrow \quad H_{int,I}(t) = \int d^3x \left(\frac{\lambda}{4!}\right) \left(\phi_I(t)\right)^4$$

(a) $H_{int,I}$ SAME FORM as $H_{int}$ but $\phi \to \phi_I$

(b) $H_{int,I}$ TIME DEPENDENT

$\phi_I(t)$ has free field form

(c) $[H_{int,I}(t), H_{int,I}(t')] \neq 0$ in general

afterall $[\phi(t), \phi(t')] \neq 0$ usually for free fields

## Typical Calculation

We want to know $\mathcal{U} = \langle f, t_f | i, t_f \rangle$
the amplitude of the overlap of the initial state evolved in time to $t_f$
to compare with possible final state measured at time $t_f$.

e.g. $i$ = protons at LHC
$f$ = leptons measured as output

Our pet theory predicts $|\mathcal{U}|^2$ as probability for this process

$$\Rightarrow \mathcal{U} = {}_S\langle f, t_f | e^{i\hat{H}_S(t_f - t_i)} | i, t_i \rangle_S = \langle f, t_f | i, t_f \rangle$$

as S. Equ$^n$ $\quad i\frac{d}{dt}|\psi, t\rangle_S = \hat{H}_S |\psi, t\rangle_S$

but picture irrelevant so

$$\mathcal{U} = {}_I\langle f, t_f | i, t_f \rangle_I$$

$$= \underbrace{{}_I\langle f, t_f|}_{\substack{\text{Measured} \\ \text{output}}} \underbrace{\mathcal{U}(t_f, t_i)}_{\underline{\text{Find}}} \underbrace{|i, t_i\rangle_I}_{\text{Input state}}$$

where $|\psi, t\rangle_I = \mathcal{U}(t, t_0) |\psi, t_0\rangle_I$

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Properties of $\mathcal{U}$ $\quad |\psi, t\rangle_I = \mathcal{U}(t, t_0)|\psi, t_0\rangle_I$

(i) $\mathcal{U}(t, t) = \mathbb{1}$

(ii) $\mathcal{U}(t_2, t_1)\,\mathcal{U}(t_1, t_0) = \mathcal{U}(t_2, t_0)$

(iii) Unitary

Normalisation constant

$\Rightarrow {}_I\langle \chi, t | \psi, t\rangle_I = {}_I\langle \chi, t_0 | (\mathcal{U}(t, t_0))^\dagger \,\mathcal{U}(t, t_0) |\psi, t_0\rangle_I$

$= \langle \chi, t_0 | \psi, t_0\rangle$ needed

$\Rightarrow [\mathcal{U}(t, t_0)]^\dagger\, \mathcal{U}(t, t_0) = \mathbb{1}$ UNITARY

(iv) $\Rightarrow \mathcal{U}(t, t_0)^\dagger = \mathcal{U}(t_0, t)$ from (i)

N.B. (iii) ${}_I\langle \chi, t | \psi, t\rangle_I = {}_I\langle \chi, t_0 | (\mathcal{U}(t, t_0))^\dagger\, \mathcal{U}(t, t_0)|\psi, t_0\rangle_I$

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Time Evolution Equations

To find $U(t_1, t_2)$ need time evolution of $|\psi, t\rangle_I = e^{iH_{0S}t} |\psi, t\rangle_S$

S. Eqn $\quad i\frac{d}{dt} |\psi, t\rangle_S = H_S |\psi, t_S\rangle$

LHS $\quad i\frac{d}{dt}\left(e^{-iH_{0S}t} |\psi, t\rangle_I\right) = H_{0S} e^{-iH_{0S}t} |\psi, t\rangle_I$

$$+ e^{-iH_{0S}t}\, i\frac{d}{dt} |\psi, t\rangle_I$$

RHS $\quad H_S |\psi, t\rangle_S = H_S e^{-iH_{0S}t} |\psi, t\rangle_I$

$$\Rightarrow \quad i\frac{d}{dt} |\psi, t\rangle_I = \underbrace{e^{iH_{0S}t} (H_S - H_{0S}) e^{-iH_{0S}t}} |\psi, t\rangle_I$$

$$i\frac{d}{dt} |\psi, t\rangle_I = H_{int, I}(t) |\psi, t\rangle_I$$

$$\Rightarrow \quad i\frac{d}{dt} U(t, t_0) = H_{int, I}(t)\, U(t, t_0)$$

So I.pic state evolution controlled by $\hat{H}_{int\, I}$ made from I pic fields $\hat{\phi}_I$ of known (free $H_0$) time evolution

## Time Evolution Sol$^n$

$$i\frac{d}{dt}|\psi,t\rangle_I = \hat{H}_{int,I}(t)\,|\psi,t\rangle_I$$

Looks like "$|\psi,t_2\rangle = e^{-i\int_{t_1}^{t_2} dt\, H_{int,I}(t)}\,|\psi,t_1\rangle_I$"

WRONG!

$\hat{H}_{int,I}$ is an operator $\Rightarrow$ operator ordering issues

Correct answer

Let $\varepsilon \to 0^+$ $\quad (1 \gg \varepsilon > 0)$

$$\Rightarrow \frac{1}{\varepsilon}\left(i|\psi,t+\varepsilon\rangle_I - i|\psi,t\rangle_I\right) \simeq \hat{H}_{int,I}(t)|\psi,t\rangle_I$$

$$\Rightarrow |\psi,t+\varepsilon\rangle_I \simeq \left(1 - i\varepsilon \hat{H}_{int,I}(t)\right)|\psi,t\rangle_I$$

$$\simeq \exp\{-i\varepsilon \hat{H}_{int,I}(t)\}\,|\psi,t\rangle_I$$

$$\Rightarrow |\psi,t+2\varepsilon\rangle_I \simeq e^{-i\varepsilon H_{int,I}(t+\varepsilon)}\,e^{-i\varepsilon H_{int}(t)}\,|\psi,t\rangle_I$$

$$\Rightarrow |\psi,t+N\varepsilon\rangle_I \simeq e^{-i\varepsilon H_{int,I}(t+(N-1)\varepsilon)}\cdot e^{-i\varepsilon H_{int,I}(t+(N-2)\varepsilon)}$$

$$\dots e^{-i\varepsilon H_{int,I}(t+\varepsilon)}\,e^{-i\varepsilon H_{int,I}(t)}$$

$$\cdot\,|\psi,t\rangle_I$$

$$\Rightarrow |\psi,t+N\varepsilon\rangle_I = T\exp\left\{-i\varepsilon\sum_{n=0}^{N-1} H_{int,I}(t+n\varepsilon)\right\}|\psi,t\rangle_I$$

$$= T\exp\left\{-i\int dt\, H_{int,I}(t)\right\}|\psi,t\rangle_I$$

Why no Campbell-Baker-Hausdorf terms

since $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$ True

AND $= T e^{A+B}$ True

if $A, B$ have times

EFS Prove (See PS5)