

4.5 FREE COMPLEX SCALAR QFT $\hat{\Phi}$

Classically $\Phi(x) \in \mathbb{C}$ also obeys K-G but differs from $\phi(x) \in \mathbb{R}$ in that it has TWO (not one) degrees of freedom

i.e. $\Phi(x)$ represents two modes with spin 0, mass m

BUT As there have different Noether charges $\Rightarrow$ distinct particles we will identify them as
We can write these as two real fields

$$\Phi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i \phi_2(x)) \quad \phi_i(x) \in \mathbb{R}$$

Proceed as above so

$$\Rightarrow \hat{\Phi}_i(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} (\hat{a}_{ip} e^{-ipx} + \hat{a}_{ip}^\dagger e^{+ipx})$$

with $i=1,2$ but $\omega_p^2 = p^2 + m^2$ for BOM

Defⁿ of particle/mode

$\Rightarrow$ We have TWO types of 'particle' / mode

$\Leftrightarrow$ Independent, commuting sets of annihilation/creation operators

$$[\hat{a}_{ip}, \hat{a}_{jq}^\dagger] = \delta^3(p-q) \delta_{ij}$$

and

$$[\hat{a}_{ip}, \hat{a}_{jq}] = [\hat{a}_{ip}^\dagger, \hat{a}_{jq}^\dagger] = 0$$

DEFⁿ OF DISTINCT PARTICLES

However, $\Phi \in \mathbb{C}$ not ϕ_i are e/states of conserved $U(1)$ charge so better to work with $\hat{\Phi}$.

$$\Rightarrow \hat{\Phi}(x) = \frac{1}{\sqrt{2}} (\hat{\Phi}_1(x) + i \hat{\Phi}_2(x)) \quad \text{h.c.}$$

$\hat{\Phi}$ is NOT Hermitian

$$\hat{\Phi}^\dagger(x) = \frac{1}{\sqrt{2}} (\hat{\phi}_1(x) - i \hat{\phi}_2(x)) \neq \hat{\Phi}(x)$$

We find

$$\hat{\Phi}(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} \left(\frac{1}{\sqrt{2}} (\hat{a}_{1p} + i \hat{a}_2) e^{-ipx} + \frac{1}{\sqrt{2}} (\hat{a}_{1p}^\dagger + i \hat{a}_{2p}^\dagger) e^{+ipx} \right)$$

Suggests we consider

$$\hat{b}_p = \frac{1}{\sqrt{2}} (\hat{a}_{1p} + i \hat{a}_2) \Leftrightarrow (\hat{b}_p)^\dagger = \frac{1}{\sqrt{2}} (\hat{a}_{1p}^\dagger - i \hat{a}_2^\dagger)$$

$$\hat{c}_p = \frac{1}{\sqrt{2}} (\hat{a}_{1p} - i \hat{a}_2) \Leftrightarrow (\hat{c}_p)^\dagger = \frac{1}{\sqrt{2}} (\hat{a}_{1p}^\dagger + i \hat{a}_2^\dagger)$$

Tong
(2.69)
(t=0)

$$\Rightarrow \hat{\Phi}(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} \left(\hat{b}_p e^{-ipx} + \hat{c}_p^\dagger e^{+ipx} \right)$$

$\uparrow$
 NOT $\hat{b}_p^\dagger$

$\Rightarrow$ Our ^{complex} scalar field $\hat{\Phi}$ is made up of two ^{distinct} parts $\hat{b}$ - annihilates particle
 $\hat{c}^\dagger$ - creates DISTINCT anti-particle

As $(\hat{b})^\dagger \neq \hat{c}^\dagger$ we have DIFFERENT types of "particle" in ONE field

$\Rightarrow$ FIELD $\neq$ Particle (or anti-particle)
Particles. We can ONLY interpret $\hat{b}$ $\hat{c}$ as particle modes if they obey same QM relations.

EF5
PS4

Easy to check that they do

e.g. $[\hat{b}_p, \hat{b}_q^\dagger] = \delta^3(p-q) = [\hat{c}_p, \hat{c}_q^\dagger]$

All others zero

i.e. $[\hat{b}_p, \hat{c}_q^\dagger] = 0$

etc.

e.o.L12
1/11/16

EF5
PS4

Also $\hat{a}_{1p}^\dagger \hat{a}_{1p} + \hat{a}_{2p}^\dagger \hat{a}_{2p} = \hat{b}_{1p}^\dagger \hat{b}_{1p} + \hat{c}_p^\dagger \hat{c}_p$

$\Rightarrow \hat{H} = \int d^3p \omega_p (\hat{a}_{1p}^\dagger \hat{a}_{1p} + \hat{a}_{2p}^\dagger \hat{a}_{2p} + 1)$

$= \int d^3p \omega_p (\hat{b}_p^\dagger \hat{b}_p + \hat{c}_p^\dagger \hat{c}_p + 1)$

Some
dispersion
relation

#particles

#anti-particles

each carry same energy
also some 1/2

e.o.L13, 17/11/14

Conserved Charge - Why use Φ not ϕ_i ?

Classically $J^\mu = i\Phi(\partial^\mu \Phi^*) - i\Phi^* \partial^\mu \Phi = (c, \mathbf{j})$

Consider $\hat{Q} = \int d^3x \hat{j}^0$

$= \int d^3x (i\hat{\Phi}^\dagger \partial_t \hat{\Phi} - i\partial_t \hat{\Phi}^\dagger \hat{\Phi}) = \hat{Q}_A + \hat{Q}_B$

Leave
for
PS4,
Q1.

$\Rightarrow \hat{Q}_A = \int d^3x \int \frac{d^3p d^3q}{\sqrt{4\omega_p \omega_q}} \{ (\hat{b}_p^\dagger e^{+ipx} + \hat{c}_p e^{-ipx}) (\hat{b}_p e^{-ipx} e^{-i\omega_p t} + \hat{c}_q^\dagger e^{+iqx} e^{+i\omega_q t}) \}$

$\hat{Q}_A = \frac{\int d^3p \omega_p}{2\omega_p} (\hat{b}_p^\dagger \hat{b}_p - \hat{c}_p \hat{c}_p^\dagger - \hat{b}_p^\dagger \hat{c}_p^\dagger + \hat{c}_p \hat{b}_p)$

$\hat{Q}_B = \hat{Q}_A^\dagger = \frac{\int d^3p \omega_p}{2} (\hat{b}_p^\dagger \hat{b}_p - \hat{c}_p \hat{c}_p^\dagger - \hat{c}_p^\dagger \hat{b}_p + \hat{b}_p^\dagger \hat{c}_p)$

cancel.

$\hat{\Phi}$.4

$$\Rightarrow \hat{Q} = \int d^3p \left(\underset{\uparrow}{\hat{b}_p^\dagger} \hat{b}_p - \hat{c}_p^\dagger \underset{\uparrow}{c_p} \right) + \text{constant}^\infty$$

particles created by $\hat{b}^\dagger$

anti-particles - quanta created by $\hat{c}^\dagger$

EFS

Suggests interpretation as expected, $\hat{\Phi}$ removes +1 charge
* EFS find as $a_i, a_i^\dagger$ - find $a, a^\dagger$ etc

Final check - is $\langle \hat{Q} \rangle$ time independent for any state?

In topic $\langle \text{any } \hat{Q} | \hat{Q} | \text{any other} \rangle = \langle \hat{Q} \rangle$ & for infinitesimal

From earlier discussion we saw $\frac{\partial \hat{O}_H}{\partial t} = i [\hat{H}, \hat{O}_H]$

For Free Fields

$$[\hat{H}, \hat{Q}] = \int d^3p \int d^3q \omega_p \left[(\hat{b}^\dagger \hat{b} + \hat{c}^\dagger \hat{c}), (\hat{b}^\dagger \hat{b} - \hat{c}^\dagger \hat{c}) \right]$$

$$(i) [\hat{b}^\dagger \hat{b}, \hat{b}^\dagger \hat{b}] = 0 \text{ as } [\hat{A}, \hat{A}] = 0$$

$$(ii) [\hat{b}^\dagger \hat{b}, \hat{c}^\dagger \hat{c}] = 0 \text{ as } \hat{b} \text{ \& \; } \hat{c} \text{ commute}$$

$$\Rightarrow -i \frac{d}{dt} \hat{Q}(t) = 0 \Rightarrow \frac{d}{dt} \langle \hat{Q} \rangle = 0$$

$\Rightarrow \langle \hat{Q} \rangle$ constant in any state.

(# particles - # anti-particles) conserved and this is linked via Noether's theorem to the $U(1)$ phase symmetry

For interactions must check this...

Again Field $\hat{\Phi}$ does not correspond to a particle
However field $\hat{\Phi}$ IS an e/STATE of $\hat{H}, \hat{H}, \hat{Q}$

EFS $[\hat{Q}, \hat{\Phi}] = q \hat{\Phi}$ or see $\hat{\Phi}$ either removes +1 particle or adds -1 anti-p