4.4 Propagators & 2-point functions

2pt ① START 2017

? More to KG sol^n in Classical Q.F.T.?

~~2. Greens Functions & Propagators~~

For linear p.d.e.s we can solve inhomogeneous pdes (cuf) (where $c\phi(x)$ not a solution if $\phi(x)$ is) using Green functions

Tong (2.100) p40

i.e. here

$$\left(\partial_{x_0}^2 - \nabla_x^2 + m^2\right) G(x,y) = -i\,\delta^4(x-y)$$

$\nabla_x^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ (not $\frac{\partial^2}{\partial y_i^2}$)

(convention could absorb into G)

Solution for ϕ at $(x_0, \underline{x})$ given kick at y^μ || External kick at $(y_0, \underline{y})$ driving system

This describes how disturbance from one kick (interaction) at y^μ PROPAGATES through system

i.e. free propagation

These will come from non-linear interactions

- Green functions G depend on B.C. (= Boundary Conditions)

e.g. · nothing before kick $\Rightarrow$ Retarded Propagator

· waves absorbed by kick $\Rightarrow$ Advanced Propagator

NOW ON PREVIOUS PAGE

Classically, we study propagation by solving the e.o.m, here the K-G equ^n.
As a PDE, we solve KG using
GREEN FUNCTIONS $G(x,y) = G(x-y)$ by symmetry
$G(x,y)$ is effect at x if hit system at y

$$\left(\partial_t^2 - \nabla^2 + m^2\right) G(x,y) = -i\delta^4(x-y)$$

Let $G(x-y) = \int d^4p\, G(p)\, e^{-ip(x-y)}$

$$\Rightarrow \circledast = \left(\partial_t^2 - \nabla^2 + m^2\right) G(x-y)$$

$$= \int d^4p\, \left(-p_0^2 + \underline{p}^2 + m^2\right) G(p)\, e^{-ip(x-y)}$$

so if $G(p) = \dfrac{i}{p^2 - m^2}$ we get

See δ fune handout

$$\circledast \overset{?}{=} \int d^4p\, e^{-ip(x-y)} \overset{?}{=} -i\delta^4(x-y) \text{ as required}$$

$$\text{so } G(x-y) \overset{?}{=} \int d^4p\, \frac{i}{p^2 - m^2}\, e^{-ip(x-y)}$$

Too QUICK! (a) PDE sol^n / Green functions depend on boundary conditions

(b) what do we do about poles at $p_0 = \pm\omega_p \Leftrightarrow p^2 = m^2$

$$G(x,y) = \int d^3\underline{p} \int_{-\infty}^{+\infty} dp_0\, \frac{1}{(p_0 - \omega_p)} \frac{1}{(p_0 + \omega_p)}\, i\, e^{-ip_0(x_0 - y_0)}\, e^{+i\underline{p}\cdot(\underline{x} - \underline{y})}$$

$$p^2 - m^2 = p_0^2 - \underline{p}^2 - m^2 = (p_0 - \omega_p)(p_0 + \omega_p)$$

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Answers Push poles off Re axis ⇔ Different B.C

Cauchy's Theorem

(For simple poles)

$$\oint dz\, f(z) = \sum_j 2\pi i\, R_j$$

where near $z = z_j$ $f(z)$ diverges as $f(z) = \dfrac{R_j}{z - z_j}$ (if simple poles)

(F. Prop. Handout, eqn 6)

Retarded Green Function G_R

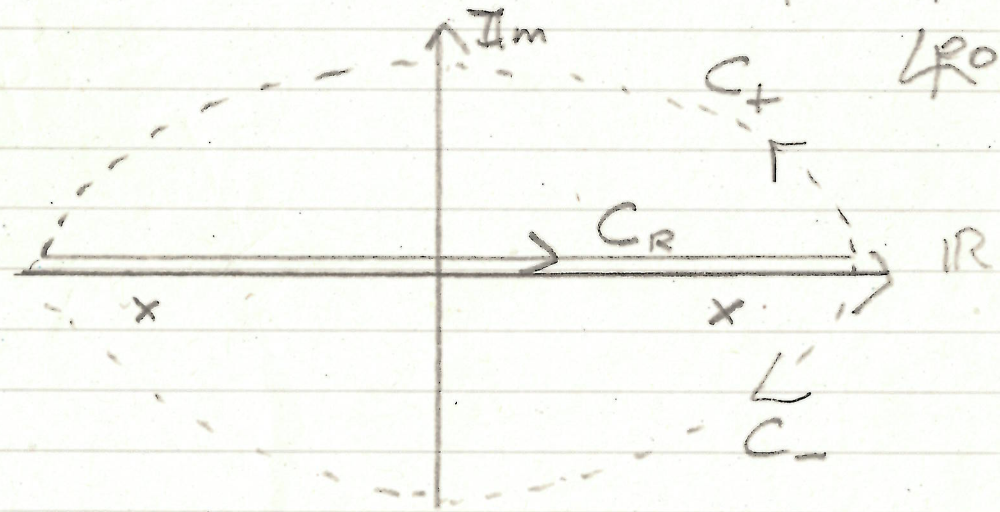
Boundary condition: $G_R(x,y) = 0$ if $x_0 < y_0$

Only reacts to delta function impulse at y^μ

Solution

$$G_R(x,y) = G_R(x-y) = \int d^4p \, \underbrace{\frac{i}{(p_0+i\varepsilon)^2 - \omega_p^2}}_{(p_0-\omega_p+i\varepsilon)(p_0+\omega_p+i\varepsilon)} e^{-ip(x-y)}$$

Note: • $0 < \varepsilon < 1$
• $\varepsilon \to 0^+$ at end.



eoL11 2/11/15 To do $\int dp_0$ we note

a) $(x_0-y_0) > 0$ $\quad \displaystyle\int_{C_-} dp_0 \, \underbrace{e^{-ip_0(x_0-y_0)}}_{e^{-i(-i\infty)(+ve)} \sim e^{-|\infty|}} = 0$

b) $(x_0-y_0) < 0$ $\quad \displaystyle\int_{C_+} dp_0 \, e^{-ip_0(x_0-y_0)} = 0$

$$\Rightarrow \int_{C_R} dp_0 \, \underset{\text{integrand}}{I} = \Theta(x_0-y_0) \int_{C_R+C_-} dp_0 \, I + \Theta(-x_0+y_0) \int_{C_R+C_+} dp_0 \, I$$

$$\int dp_0 \underset{\text{integrand}}{\mathcal{L}} = \underbrace{\theta(x_0-y_0)\int_{C_R+C_-} dp_0\,\mathcal{L}}_{\text{Two poles at}} + \theta(y_0-x_0)\underbrace{\int_{C_R+C_+} dp_0\,\mathcal{L}}_{\text{No poles inside here}}$$

$p_0 = \pm\omega_p$

Clockwise round pole

$$= \theta(x_0-y_0)\int d^3p \left[-1 \cdot \frac{1}{2\pi} \cdot 2\pi i \cdot \overbrace{\frac{i e^{+i\underline{p}(\underline{x}-\underline{y})}}{2\omega_p + i\varepsilon}}^{\text{Residue}} \right.$$

$$\left. -1 \cdot \frac{1}{2\pi} \cdot 2\pi i \cdot \frac{i e^{-i\underline{p}(\underline{x}-\underline{y})}}{-2\omega_p + i\varepsilon} \right]$$

change $\underline{p} \to -\underline{p}$

$$\Rightarrow G_R(x-y) = \theta(x_0-y_0)\int \tilde{d}^3 p \, e^{+i\underline{p}\cdot(\underline{x}-\underline{y})} \times$$

$$\left(\frac{1}{2\omega} e^{-i\omega(x_0-y_0)} - \frac{1}{2\omega_p} e^{+i\omega(x_0-y_0)} \right)$$

<u>EFS</u> $G_R(x-y) = \theta(x_0-y_0)\left(D(x-y) - D(y-x)\right)$ ← change $\underline{p} \to -\underline{p}$

where $D(x-y) = \int \frac{\tilde{d}^3 p}{2\omega} e^{-ipx}$ $\quad p_0 = \omega > 0$

<u>NOTES</u>

1. Clearly solves K.G. eqn as $G(p) \sim \frac{1}{p^2-m^2}$.

2. B.C. are that $G_R(x-y) = 0$ if $x_0 < y_0$
   i.e. NO perturbations at x^μ before y^μ impulse
   ONLY " " " after " "
   $\Rightarrow$ retarded

3. Clearly Lorentz invariant from $\int d^4p$ form
   $\Rightarrow$ must be L.inv in $\int d^3p$ form but
   $\theta(x_0-y_0)$ is less obvious (EFS)

4. Clearly function of $(x-y)^\mu$ not just of x^μ & y^μ
   $\Rightarrow$ Translation invariance
   Anticipated as wrote $G(x-y)$

Causal Properties of G_R

For physical propagation, perturbation at y^μ can not produce any change at x^μ if
x & y space-like separated i.e. $(x-y)^2<0$

Consider $x_0=y_0$

$$\Rightarrow G_R(x-y) = \int \frac{d^3p}{2\omega} e^{+i\underline{p}\cdot(\underline{x}-\underline{y})} - \underbrace{\int \frac{d^3p}{2\omega} e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}}_{\text{Change } \underline{p}' = -\underline{p}}$$

$$= 0$$

- True for any $\underline{x}-\underline{y}$ when $x_0=y_0$

  $\Rightarrow$ True $\forall$ $(x-y)^2<0$ values

- L. Inv $\Rightarrow$ true for any $(x-y)^2<0$, whatever x_0, y_0 are

$\Rightarrow G_R$ is zero for unphysical separations

<u>N.B.</u> For $(x-y)^2>0$ find $G_R>0$ allowed

e.g. $\underline{x}=\underline{y} \Rightarrow G_R(x-y) \sim \int \frac{d^3p}{\omega}$ i.sin(ωt)

Green Functions G & QFT Two-Point Functions Δ, D

Classical Green function of K.G G
are linked to
Two-Point operators & expectation values of
free field QFT

WIGHTMAN FUNCTION $D(x,y)$

Define $D(x,y) = \langle 0 | \hat{\phi}(x)\, \hat{\phi}(y) | 0 \rangle$

$$\Rightarrow D(x,y) = \int \frac{d^3p}{\sqrt{2\omega_p}} \int \frac{d^3q}{\sqrt{2\omega_q}}\, e^{-ipx} e^{+iqy} \langle 0 | \hat{a}_p \hat{a}_q^+ | 0 \rangle$$

as $\hat{a}_p |0\rangle = \langle 0 | \hat{a}_q^+ = 0$

i.e. this encodes creation of quanta at y momentum q
removal " " " x " p

i.e. propagation of influence from y to x
→ hence term

Commute $\hat{a}\hat{a}^+$
$$\Rightarrow D(x,y) = \int \frac{d^3p}{2\omega}\, e^{-ip(x-y)}$$

This is not physical in itself as $D(x-y) \neq 0$
for $(x-y)^2 < 0$

Retarded Two-Point Function Δ_R

Since $G_R = \theta(x-y)\left(D(x-y) - D(y-x)\right)$

we have that

$$G_R(x-y) = \theta(x_0-y_0)\left(\langle 0|[\hat{\phi}(x), \hat{\phi}(y)]|0\rangle\right)$$

$$= \langle 0|\left(\theta(x_0-y_0)[\hat{\phi}(x), \hat{\phi}(y)]\right)|0\rangle$$

$G_R(x-y) = \Delta_R(x-y)$ Two-point vacuum expectation value

Retarded K.G. Green Function.

N.B.

$$\theta(x_0-y_0)[\hat{\phi}(x), \hat{\phi}(y)] = \Delta_R(x-y)\,\mathbb{1}$$

unit operator

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Tong §2.7.1 — Feynman Propagator Δ_F

The most important for QFT, only see this later

Tong (2.93)-(2.95)

$$\Delta_F(x-y) := \langle 0 | T\hat{\phi}(x)\hat{\phi}(y) | 0 \rangle$$

where
$$T(\hat{A}(t)\hat{B}(t')) = \begin{cases} \hat{A}\hat{B} & \text{if } t > t' \\ \hat{B}\hat{A} & \text{if } t < t' \end{cases}$$

BIGGEST time SMALLEST

T = Time Ordering, puts operators in order of their time, largest (smallest) times on left (right)

$$\overset{T}{\Rightarrow} \Delta_F(x-y) = \theta(x_0-y_0)\, D(x-y) \text{ ①} + \theta(y_0-x_0)\, D(y-x) \text{ ②}$$

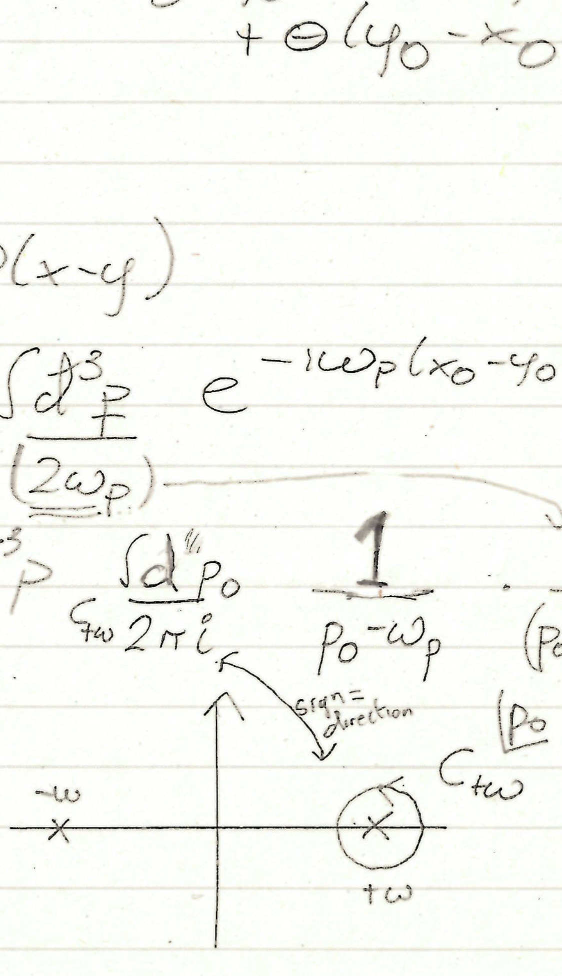
1st Term

$$① = \theta(x_0-y_0)\, D(x-y)$$
$$= \theta(x_0-y_0) \int \frac{d^3 \underline{p}}{(2\omega_p)}\, e^{-i\omega_p(x_0-y_0) + i\underline{p}\cdot(\underline{x}-\underline{y})}$$

↓↓ i.

$$= \theta(x_0-y_0) \int d^3 p \int_{C_{+\omega}} \frac{dp_0}{2\pi i} \frac{1}{p_0-\omega_p} \cdot \frac{1}{(p_0+\omega_p)} e^{-ip_0(x_0-y_0)} \times e^{+i\underline{p}\cdot(\underline{x}-\underline{y})}$$

sign = direction



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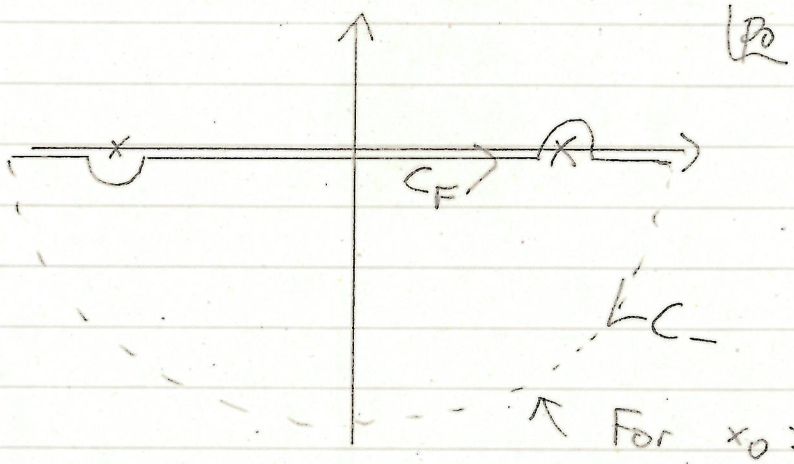
Reverse and
Distort contour to



Only pole at $p_0 = +\omega_p$ inside

For $x_0 > y_0$, take $K_- \to \infty$

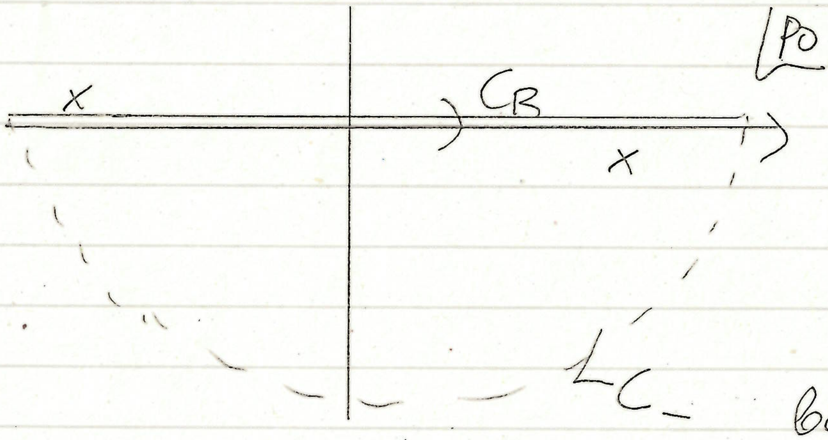
$$|e^{-ip_0(x_0-y_0)}| \to 0$$

$$\Rightarrow \int_{C_-} dp_0 (\sim) = 0$$

so can drop this.

* $\frac{1}{i} = -i$ matches contour reverse sign.

$$\Rightarrow ① = \theta(x_0-y_0) \int d^3p \int_{C_F} dp_0 \frac{i}{(p_0-\omega)} \frac{1}{(p_0+\omega)} e^{-ip(x-y)}$$



OR
push poles above/below by $i\varepsilon$ but leave $\int dp_0$ along real axis

$$① = \theta(x_0-y_0) \int d^3p \int_{-\infty}^{+\infty} dp_0 \frac{i}{(p_0-\omega-i\varepsilon)(p_0+\omega+i\varepsilon)} e^{-ip(x-y)}$$

(Take $\varepsilon \to 0^+$ at end of calc
$0 < \varepsilon \ll 1$)

2pt (12)

$$\Rightarrow (1) = \theta(x_0 - y_0) \int d^4p \, \frac{i}{\underbrace{p_0^2 - (\omega_p - i\varepsilon)^2}_{p^2 - \omega^2 + 2i\omega\varepsilon}} e^{-ip(x-y)}$$

Integrate for real p_0

$2i\omega\varepsilon$: +ve so $\equiv$ to $i\varepsilon$

EFS Likewise 2nd term comes when we complete to upper half ⊙ valid when $x_0 < y_0$

$$\Rightarrow \Delta_F(x-y) = \int d^4p \, \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$$

EFS $\left(\partial_t^2 - \nabla^2 + m^2\right) \Delta_F(x-y) = -i\,\delta(x-y)$

with B.C. $e^{-i\omega t}$ only as $t \to +\infty$
$e^{+i\omega t}$ " " $t \to -\infty$

EFS

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