# 4 QFT for Real Scalar Fields $\hat{\phi}$

We have seen the link between classical field theory $\phi$ and SHO through normal modes.
$\Rightarrow$ Focus on QHO quantised SHO

## 4.1 QHO = Quantum Harmonic Oscillator

Q.M. may be defined via
"ccr" = "$\underbrace{\text{canonical}}_{=\text{standard}}$ commutation relations"

for position & momentum operators $\hat{q}_i, \hat{p}_i$ for $i=1,\dots,N$ particles

$$[\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$$

($q_i = q_i$ in 1D case)

AND $[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$ one definition of $\hbar$

Classically
We work in normal modes, linear combinations of $q$ & $p$, repeat for $\hat{q}$ & $\hat{p}$

$$\left.\begin{aligned}\hat{Q}_i &= \sum_i U_{ij}\,\hat{q}_j \\ \hat{P}_i &= \sum_i U^*_{ij}\,\hat{p}_j\end{aligned}\right\}$$

N.B. $Q_i^\dagger = Q_i$, $P_i^\dagger = P_i$ $\Leftrightarrow U_{ij} = U^*_{ij}$

CW In PS1 Q3 in general

$$\Rightarrow [\hat{Q}_i, \hat{P}_j] = \sum_{k,l} U_{ik} U^*_{jl} \underbrace{[\hat{q}_k, \hat{p}_l]}$$

$$= \sum_{k,l} U_{ik} U^*_{jl}\, i\hbar\,\delta_{kl}$$

$$= i\hbar\,[\underline{U}\,\underline{U}^\dagger]_{ij}$$

Find PS1, Q3

$$\Rightarrow [\hat{Q}_i, \hat{P}_j] = i\hbar\,\delta_{ij} \quad \text{IFF } UU^\dagger = \mathbb{1}$$

since $\hat{q}, \hat{p}$ hermitian.

$\Rightarrow$ Unitary ($\Rightarrow$ linear) transformations preserve c.c.r.

- Unitary transformations are "canonical transformations" they preserve c.c.r

$\Rightarrow$ Can use $\{\hat{q}_i, \hat{p}_i\}$ OR $\{\hat{Q}_i, \hat{P}_i\}$ as both equally $\{$good$\}$ QM variables $\{$standard$\}$

- Best to use $\{\hat{Q}_i, \hat{P}_i\}$ for normal modes $\}$ e.g. 1D example showed this

- In QFT each normal mode $i$ corresponds to different $\underset{\sim}{k}$ value for a distinct degree of freedom for each channel for propagation.

Focus on one QHO mode with characteristic

$P = P^+$ assumed e.g. $P_k + P_{-k}$ in 1D case

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{Q}^2$$

NOTE NOT unique generalisation of $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$ due to OPERATOR ORDER

Best solved with dimensionless creation $\hat{a}^+$ & annihilation $\hat{a}$ operators

$$\hat{a} := \frac{1}{\sqrt{2}}\left(\frac{\hat{Q}}{\ell} + i\frac{\ell}{\hbar}\hat{P}\right),$$

$$\Rightarrow \hat{a}^+ := \frac{1}{\sqrt{2}}\left(\frac{\hat{Q}}{\ell} - i\frac{\ell}{\hbar}\hat{P}\right), \quad \left(\hat{a}^+ \neq \hat{a} \Rightarrow \underline{\text{not}} \text{ hermitian}\right)$$

where $\ell = \sqrt{\frac{\hbar}{m\omega}}$ is characteristic length (EFS check units)

$\Rightarrow$ c.c.r become $[\hat{a}, \hat{a}^+] = 1$

AND $[\hat{a}, \hat{a}] = [\hat{a}^+, \hat{a}^+] = 0$

EFS PS2 Q4

$$\Rightarrow \hat{q} = \frac{l}{\sqrt{2}}(\hat{a} + \hat{a}^+) \quad , \quad \hat{p} = \frac{i\hbar}{\sqrt{2}l}(\hat{a}^+ - \hat{a})$$

$$\Rightarrow \hat{H} = \frac{1}{2m}\frac{\hbar^2}{2l^2}(\hat{a} - \hat{a}^+)(\hat{a}^+ - \hat{a}) + \frac{1}{2}m\omega^2\frac{l}{2}(\hat{a}^+ + \hat{a})(\hat{a} + \hat{a}^+)$$

$$\Rightarrow \hat{H} = \frac{1}{2}\hbar\omega(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+) = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

Value depends on operator order in original $\hat{H}$ } Zero Point Energy

If have many normal modes $\hat{Q}_k, \hat{P}_k \Leftrightarrow \hat{a}_k, \hat{a}_k^+$

ONE OSCILLATOR PER NORMAL MODE

$$\Rightarrow \hat{H} = \sum_k \hbar\omega_k\left(\hat{a}_k^+\hat{a}_k + \frac{1}{2}\right)$$

as each mode independent = commutes with others

k = mode label, usually the wavenumber will do rather than an integer

<u>SOLUTION</u> for Energy Eigenvalues

PS1

Energy eigenstates are $|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle \quad (n = 0, 1, 2, \ldots)$

They contain $n$ "quanta" $\quad \hat{a}^+\hat{a}|n\rangle = n|n\rangle$

Energy e/value is $E_n = \frac{1}{2} + n\hbar\omega$

# 4.2 QFT for free scalar fields

Tong §2.1 p21
PS §2.3 p19

QM $[\hat{q}_i(t), \hat{p}_j(t)] = i\hbar\delta_{ij}$ , $[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

Deviation of particle $i$ in geographical space → Conjugate Momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$

QFT ⇓

Deviation of field $i$ in field space → Conjugate Momentum $\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$

AXIOM $[\hat{\phi}_i(\underline{x}, t), \hat{\Pi}_j(\underline{x}', t)] = i\hbar\delta_{ij}\,\delta^3(\underset{\sim}{x} - \underset{\sim}{x}')$

AND $[\hat{\phi}_i(x, t), \hat{\phi}_j(x', t)] = [\hat{\Pi}_i(x, t), \hat{\Pi}_j(x', t)] = 0$

Notes

① These are EQUAL TIME COMMUTATION RELATIONS = ETCR

They DEFINE QFT, axiom

② QM $\hat{q}_i(t)$ ⟶ QFT $\hat{\phi}_i(\underline{x}, t)$

Geographical Space quantised

Geographical space as coordinate

QM = 1+0 Dimensional Space Time QFT ⟺ 3+1 Dimensional Space-Time QFT

③ 1D Ring $q_n(t) = \left(\frac{a}{\sqrt{\mu}}\right) \phi(x = na, t)$

Finite # N Masses Q/SHO QM $\xrightarrow{N \to \infty}$ $(N = \infty)$ Infinite # Oscillators QFT

## Dynamics

We want $k^2 = m^2$ so classical sol$^n$ of KG.

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2} m \phi^2 + \frac{1}{2}(\underset{\sim}{\nabla}\phi)^2 \quad \leftarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m\phi$$

$$\phi(\underset{\sim}{x}, t) = \int \frac{d^3k}{2\omega_k}\left(e^{-ikx} A_k + e^{+ikx} B_k\right)$$

$B_k = A_k^*$ as $\phi \in \mathbb{R}e$

$$kx = \omega_k t - \underset{\sim}{k} \cdot \underset{\sim}{x}$$

$$\Big\Downarrow \text{Quantise}$$

<u>Axiom</u>

$$\hat{\phi}(\underset{\sim}{x}, t) = \int \frac{d^3k}{2\omega_k}\left(e^{-ikx} \hat{A}_k + e^{-ikx} \hat{A}_k^{+}\right)$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}}\left(e^{-ikx} \hat{a}_k + e^{+ikx} \hat{a}_k^{+}\right)$$

where $\hat{A}_k = \sqrt{2\omega_k}\,\hat{a}_k$ ; <u>EFS</u> $\hat{\phi} = \hat{\phi}^{+}$ hermitian
c.f. $\phi = \phi^*$

$$\Rightarrow \hat{\Pi}(x,t) = \partial_t \hat{\phi} = \int d^3k \, -i\sqrt{\frac{\omega_k}{2}}\left(e^{-ikx}\hat{a}_k + e^{+ikx}\hat{a}_k^{+}\right)$$

<u>Axiom</u> & $\hat{H} = \frac{1}{2}\hat{\Pi}^2 + \frac{1}{2} m \hat{\phi}^2 + \frac{1}{2}(\underset{\sim}{\nabla}\phi)^2$

Annihilation & Creation Ops.

EFS PS4 ETCR $\Rightarrow$ $[\hat{a}_k, \hat{a}^+_p] = \delta^3(\underline{k}-\underline{p})$

$$[\hat{a}_k, \hat{a}_p] = [\hat{a}^+_k, \hat{a}^+_p] = 0$$

and $\hat{H} = \int d^3p \, \omega_p \left(\hat{a}^+_p \hat{a}_p + \frac{1}{2}\right)$

i.e. • our free quantum field is a sum of an infinite number of QHO, one for each momentum. cf. 1D Ring $\hat{q}_n(t) \sim \phi(x=na, t)$ $N \to \infty$

• Each degree of freedom at momentum $\underline{p}$ $\Leftrightarrow$ One QHO

N.B. without interactions number of particles/quanta in each mode is conserved $[\hat{H}, \hat{a}^+_k \hat{a}_k] = 0$

earlier?

N.B. 1D ring had $\hat{q}_n(t) = \left(\frac{a}{\sqrt{\mu}}\right) \phi(x=na, t)$

deviation in field space

deviation in geographical space

N masses $\to \infty$ for continuum

N oscillators

(4.2 cont)

## Note 5 State Space

TONG p30 (§2.4)

Full space of states is spanned by all possible states formed by acting creation operators on vacuum

$$|0\rangle \quad \text{"Empty" vacuum}$$
$$\hat{a}^\dagger_p|0\rangle = |\underline{p}\rangle \quad \text{1-particle state}$$
$$\hat{a}^\dagger_p\hat{a}^\dagger_q|0\rangle = |\underline{q},\underline{p}\rangle \quad \text{2-particle state}$$
$$\vdots$$
$$\hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\cdots\hat{a}^\dagger_{p_n}|0\rangle = |\underline{p_1}\,\underline{p_2}\cdots\underline{p_n}\rangle \quad \text{n-particle state}$$

- In practice we always work with LHS i.e. $\left\{\begin{matrix}\text{creation}\\ \text{annihilation}\end{matrix}\right\}$ operators acting on $\left\{\begin{matrix}|0\rangle\\ \langle 0|\end{matrix}\right\}$ i.e. $\langle 0|\ldots|0\rangle$ = VACUUM EXPECTATION VALUES

THEN Commute annihilation (creation) operators to right (left) to use $\hat{a}_p|0\rangle = 0$ ($\langle 0|\hat{a}^\dagger_p = 0$).

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- Normalisation

$$\langle 0|0\rangle = 1$$

$$\Rightarrow \langle \underline{p}|\underline{q}\rangle = \langle 0|\underbrace{\hat{a}_p\hat{a}^\dagger_q}_{\text{commutation Rel}^n}|0\rangle$$
$$= \langle 0|\left(\delta^3(\underline{p}-\underline{q}) + \hat{a}^\dagger_q\hat{a}_p\right)|0\rangle$$
$$= \delta^3(\underline{p}-\underline{q})\langle 0|0\rangle$$
$$= \delta^3(\underline{p}-\underline{q})$$

4.3

Tong 2.6 p35

## 4.3 TIME & QFT

So far all work has been at fixed times.

e.g. $\hat{a}^\dagger_p |0\rangle = |\underline{p}\rangle$ has no time

ETCR

Three ways used in QFT to introduce time, first two are:-

Ⓐ Schrödinger Picture = S. Pic.

- Work with operators with no explicit time
- States carry all time

e.g. $\Rightarrow$ $\hat{\phi}_S(\underline{x}) := \hat{\phi}(\underline{x}, t=0)$ Standard Arbitrary choice of reference time ($t=0$ arbitrary choice)

S. Pic. No explicit t dependence NOTE

$$= \int \frac{d^3k}{\sqrt{2\omega_k}} \left( \hat{a}_k e^{+i\underline{k}\cdot\underline{x}} + \hat{a}^\dagger_k e^{-i\underline{k}\cdot\underline{x}} \right)$$

(For free field only)

and for any state $\psi$

$$i\frac{d|\psi, t\rangle_S}{dt} = \hat{H}_S |\psi, t\rangle_S$$

Schrödinger Equation another fundamental axiom.

(same $\forall$ states)

This is valid but unhelpful approach in QFT especially with relativity as Sch. Equ$^n$ gives time a special role.

## (B) Heisenberg Picture = H, Pic

PS(2.43) Tong(2.77)

- Let operators carry all time
- States have no explicit time dependence

Time Evolution Axiom = S.Eqn

$$\Rightarrow \quad \hat{O}_H(\underline{x},t) = e^{i\hat{H}_S t}\, \underbrace{\hat{O}_H(\underline{x},t=0)}_{\hat{O}_S(\underline{x})}\, e^{-i\hat{H}_S t}$$

(Same for ALL operators)

* N.B. $e^{\hat{A}} = \sum_n \frac{\hat{A}^n}{n!}$

and $|\psi\rangle_H = |\psi, t=0\rangle_S$ NOTE NO TIME

NOTE (i) $H_H = H_S \Rightarrow$ drop subscript for H

(ii) that time evolution is equivalent to

Tong(2.78) PS(2.44)

$$\frac{d\hat{O}_H}{dt} = i[\hat{H}, \hat{O}_H]$$

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(iii) Matching S.Pic & H.Pic at t=0 is the standard choice

### Consistency

Can work in either picture, completely equivalent

e.g. to find $\langle O\rangle$ at time t in state $\psi$ we need

$$\left(\langle\psi|\hat{O}|\psi\rangle\right)_{\text{time } t} = {}_S\langle\psi(t,\underline{x})|\,O_S(\underline{x})\,|\psi(t,\underline{x})\rangle$$

$$= {}_S\langle\psi(0,x)|e^{+i\hat{H}t}\cdot O_S(\underline{x})\cdot e^{-i\hat{H}t}|\psi(0,\underline{x})\rangle_S$$

$$= {}_H\langle\psi(x)|\,O_H(\underline{x},t)\,|\psi(x)\rangle_H$$

## Free Field Operator $\hat{\phi}(\underline{x}, t)$

TONG 2.4.1 p29; PS (2.46)

We write $\hat{\phi}(\underset{\sim}{x}, t) = \int \frac{d^3k}{2\omega_k} \left( \hat{a}_k e^{-ikx} + \hat{a}_k^+ e^{+ikx} \right)$

$$kx = \omega t - \underset{\sim}{k} \cdot \underset{\sim}{x}$$

For free field this is completely consistent with

$$\phi_H(\underline{x}, t) = \phi(\underline{x}, t)$$

$$\phi_S(\underline{x}) = \phi(\underline{x}, t=0)$$

$$H = \sum \hbar\omega_k \left( \hat{a}_k^+ \hat{a}_k + \frac{1}{2} \right)$$

Proof EFS/PS4 Q2 $e^{i\hat{H}t} \hat{a}_k^+ e^{-i\hat{H}t} = e^{+i\omega_k t} \hat{a}_k^+$

$\Downarrow$ h.c.?

$$e^{i\hat{H}} \hat{a}_k e^{-iHt} = e^{-i\omega_k t} \hat{a}_k$$