$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$

Equation of Motion: The Klein-Gordon Equation

[Tong §1.1.1 p8-9] *Also* see my online Classical Field *Notes

$$-\frac{\partial\mathcal{L}}{\partial\phi} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right) = 0 \qquad \text{c.f.} \quad \frac{d}{dt}\frac{\partial L}{\partial\dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

gives the Euler-Lagrange equations for a real relativistic scalar field as

$\partial_t \Rightarrow \partial_\mu$

so treating ϕ & $(\partial_\mu\phi)$ as independent (ϕ & $\dot{\phi}$ is original)

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \Rightarrow$

Here $\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$, $\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = (\partial^\mu\phi)$

$$\Rightarrow \text{e.o.m. are} \quad \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

K-G eqn = The KLEIN-GORDON Equation
- linear PDE
- describes non-interacting relativistic scalar (spin 0) particles of mass m

To see this look at
Normal modes are plane waves of form

$$e^{ikx} \equiv e^{ik_\mu x^\mu} = e^{ik_0 t - \underline{k}\cdot\underline{x}}$$

where $k_0^2 = \underline{k}^2 + m^2 = \omega^2$

e^{-ipx} sign e.g. TONG (p.95) p39

i.e. if $\phi(k) = \int d^4x \, \phi(x) e^{+ikx}$

$\phi(x) = \int d\!\!{}^-{}^4k \, \phi(k) e^{-ikx}$ where $d\!\!{}^-{}^n k = \frac{d^n k}{(2\pi)^n}$

$\Rightarrow$ find $(k^2 - m^2)\phi(k) = 0$ where $k^2 = k_\mu k^\mu$

$\Rightarrow k^2 = k^\mu k_\mu = m^2$ is soln. & $k_\mu = (k_0, \underline{k})$

$\Rightarrow$ Each $\phi(k)$ unmixed with others & has dispersion reln of relativistic particle of mass m.

<u>Key property of KG Solutions</u> is $k^2 = m^2$

Note that both $k_0 = +\omega_k = \left|\sqrt{\underline{k}^2 + m^2}\right|$

AND $k_0 = -\omega_k = -\left|\sqrt{\underline{k}^2 + m^2}\right|$

$$\left(\text{I treat } \omega_k = \left|\sqrt{\underline{k}^2+m^2}\right| \geq 0 \right)$$

are allowed.

The $k_0 < 0$ are just waves with opposite time evolution to $k_0 > 0$ solutions
(There is a $t \to -t$ symmetry)

Incoming vs outgoing waves

- Not a problem but k_0 can NOT be the energy of a physical particle!
- $|k_0| = \omega_k = \left|\sqrt{\underline{k}^2+m^2}\right| \geq 0$ IS
- We will link $k_0 < 0$ solutions to the ANTI-PARTICLE counterparts of matching $k_0 > 0$ PARTICLE solution

	Particle		Anti-particle
$e^{ik_0 t} \to$	$e^{+i\omega t}$	vs	$e^{-i\omega t}$
	$k_0 > 0$		$k_0 < 0$
Energy	$\omega = \left|\sqrt{\underline{k}^2+m^2}\right|$		same

$\uparrow$ same mass $m > 0$

General Solutions

$$\phi(k) = \frac{1}{2\omega_k}\,\tilde{\delta}(k_0-\omega_k)\,A_k + \frac{1}{2\omega_k}\,\tilde{\delta}(k_0+\omega)\,B_k$$

- $\tilde{\delta}(k) = 2\pi\,\delta(k)$ with $B_k = A_k^*$ if $\phi \in \mathbb{R}$
- Normalisation 2π's & ω_k's chosen for convention later

$$\Rightarrow \phi(x) = \underbrace{\int \frac{\tilde{d}^3k}{2\omega_k}}\left(A_k e^{-i\omega t + i\underline{k}\underline{x}} + B_k e^{+i\omega t + ik x}\right)$$

where $\int \tilde{d}^n k = \int \frac{dk}{(2\pi)^n}$, $A_k^* = B_k$ if $\phi \in \mathbb{R}$

$$\int \frac{\tilde{d}^3k}{2\omega_k} = \int \tilde{d}^4k\,\tilde{\delta}(k_0^2-\omega_k^2) = \underbrace{\int \tilde{d}^4k\,\tilde{\delta}(k^2-m^2)}_{\text{clearly Lorentz invariant}}$$

e.o L5, 20/10/14

Green's Functions & Propagators

See later, Handout δ functions

Q? More Green Functions here?

2.4 Complex Scalar Fields

Consider two real scalar fields of same mass ($N=2$), $O(2)$ symmetry ie. rotations in 2D

$$S = \frac{1}{2}\int d^4x \sum_{i=1}^{2}\left\{(\partial_\mu \phi_i)(\partial^\mu \phi_i) - m^2 \phi_i \phi_i\right\}$$

Symmetry is a rotation in 2-Dimensional internal (not space-time) field space

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}.$$

(reflection also possible) * So $\phi_1 \phi_2$ MIXED by symmetry not e/state

eoL5 20/10/15

14/10/16 However we have seen how we can also represent rotations in 2D as complex numbers

$$x+iy = re^{i\theta} \to re^{i\theta'} = x'+iy'$$

$\Delta\theta = (\theta' - \theta)$

$$\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}\cos(\Delta\theta) & \sin(\Delta\theta)\\ -\sin(\Delta\theta) & \cos(\Delta\theta)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$\Rightarrow$ Try $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

↑ convention, normalisation

EFS

Find

$$S = \int d^4x \left((\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 |\Phi|^2\right)$$

↑ NO° $\frac{1}{2}$ in standard normalisation, $S = S^*$

Equations of motion

Treat Φ & Φ^*, $\partial_\mu \Phi$ & $\partial_\mu \Phi^*$ as independent

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} = 0 \quad \& \quad \underbrace{\frac{\partial \mathcal{L}}{\partial \Phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^*} = 0}_{\mathcal{L} = \mathcal{L}^* \Rightarrow \text{just complex conjugate}}$$

Find (PS2)

$$\partial^\mu \partial_\mu \phi(x) + m^2 \Phi(x) = 0 \quad \overset{F.T.}{\Rightarrow} \quad (-k^2 + m)\Phi(k) = 0$$

Check can get by combination of e.o.m. for ϕ_1 & ϕ_2

General solution

$$\Phi(x) = \int \frac{d^3k}{2\omega_k} \left(e^{-ikx} A_k + e^{+ikx} B_k^* \right)$$

$\Uparrow$

$B_k \neq A_k$ as $\Phi \in \mathbb{C}$