2.3 Symmetry and Classical Field Theory

- The ACTION $S = \int dt\, L = \int d^d x\, \mathcal{L}$ describes the whole of classical dynamics
- Any solution to classical e.o.m. is a turning point in S e.g. local maximum
- Want S to be invariant under physical symmetry.
  $\Leftrightarrow$ symmetries map sol^n. to e.o.m to other related (same energy) sol^n.

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Classical Field Theory

SYMMETRY (cont)

Symmetry only mixes objects (field, particles, ...) which lie in same irreducible representation of the group.
(and vice-versa irrep ⇒ group of related objects)

Lorentz/Poincaré group labels irreps by

a) MASS, m^2, 0 or $\mathbb{R}^+$
b) SPIN, s, 0, $\frac{1}{2}$, 1, ...

e.g. Two degrees of freedom of photon (polarisation) linked in one "particle" because space-time symmetry mixes them ($m=0$, $S=1$)
Three ° freedom of Z^0 (Long, + 2x Transverse)
(since $m>0$, $s=1$)

Relativistic Space-Time Symmetry

– Poincaré Group

(i) Translations $x^\mu \to x'^\mu = x^\mu + a^\mu$

↑ constant

(ii) Rotations/Boosts

$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

No detailed study here, only results quoted

Ex. <u>Relativistic Scalar Field Theory</u> $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ Lorentz subgroup

$S = \int dt\, L = \int d^4x\, \mathcal{L}$

<u>EFS</u> • $\int d^4x = \int d^4x'$ invariant (EFS)

$\Rightarrow$ Work with Lagrangian density

• <u>K.E.</u> $\dot{\phi}^2 = (\partial_t \phi)^2$ NOT invariant

But we have $\left(\frac{\partial \phi}{\partial t}\right)^2 - c^2\left(\frac{\partial \phi}{\partial x}\right)^2$ in 1D example

c=1? <u>EFS</u> $\Rightarrow (\partial_\mu \phi)(\partial^\mu \phi) = \left(\frac{\partial \phi}{\partial t}\right)^2 - \overset{1}{\cancel{c^2}}\left(\frac{\partial \phi}{\partial x}\right)^2 - \dots$ (with c=1 here)

(Active)

This is invariant <u>IFF</u> $\phi(x) \xrightarrow{\Lambda} \phi'(\cancel{x'}) = \phi(\underbrace{\Lambda x}_{x'})$

i.e. coordinates change but function is UNCHANGED/INVARIANT

This is CALLED a (LORENTZ) SCALAR FIELD

c.f. $A^\mu(x) \to A'^\mu(y) = \Lambda^\mu{}_\nu A(y)$

Equivalent to having ZERO SPIN

Tong (1.27) p12 • <u>Potential Terms</u> (= No derivatives)

$\int d^4x\, \phi^2(x) \xrightarrow{\text{Transfmn}} \int d^4x' (\phi'(x'))^2$ (ϕ unchanged)

$= \int d^4x' (\phi(x'))^2$

$= \int d^4x (\phi(x))^2$ change label

$\Rightarrow$ Action invariant

$\Rightarrow$ our 1D toy model has given us a generic form of a relativistic scalar (=L.inv) (spin 0) field theory

Action $S = \int d^4x\, \mathcal{L}$, Lagrangian Density $\mathcal{L} = \frac{1}{2}\{(\partial_\mu \phi)(\partial^\mu \phi) - m^2\phi^2\}$

$\frac{1}{2}$: convention

m^2: assert: MASS^2 of particle ($c = \hbar = 1$ now)

if $\phi(x) \in \mathbb{R}$ = ONE degree of freedom

e.o L4, 17/10/14

(See P.S. ch2.2 p16-17)

For later use, we find the Hamiltonian density $\mathcal{H}$

Momentum Density $\Pi(x) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$, $L = \int d^3x\, \mathcal{L}$

$$\Rightarrow \mathcal{H} = \Pi\dot{\phi} - \mathcal{L} \quad \text{where } H = \int d^3x\, \mathcal{H} = \text{Hamiltonian}$$

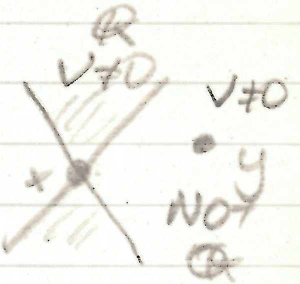
$$= \Pi^2 - \frac{1}{2}\{\underbrace{\dot{\phi}^2}_{``\Pi^2"} - (\nabla\phi)^2 - m^2\phi^2\}$$

(ON PS2)

$$\Rightarrow \mathcal{H} = \underbrace{\left\{\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + m^2\phi^2\right\}}_{\mathcal{H}}$$

c.f. $\frac{1}{2}\frac{p^2}{2m} + \frac{1}{2}\omega^2 x^2$

Other Limitations on Field Theories



1. Locality

Symmetry often allows terms such as
$\int d^4x \int d^4x' (\phi(x))^2 \, V(x-x') \, (\phi(x'))^2$

However this is action at a distance for general V
- NOT OK for fundamental physics (restrict V)
- OK for low energy / velocity approximation e.g. condensed matter physics.

2. Renormalisation

We can only do calculations with polynomials of fields (usually) up to limited power
- $\underset{\text{MAX for 3+1}}{\phi^2, \phi^4}$, $\underset{\text{NOT ok in 3+1, MAX in 2+1}}{\phi^6}$ terms may be OK depending on space-time dimension
- $\sin(\phi^2)$ NOT helpful

eol.4 16/10/15

3. Hermitian

To ensure real probabilities need real action & $\mathcal{L}=\mathcal{L}^*$

Here simple as $\phi(x) \in \mathbb{R}$, $x^\mu \in \mathbb{R}$ $\Rightarrow \mathcal{L} \in \mathbb{R}$

eol.4 11/10/16

Complex action $\Rightarrow$ instabilities, non. equilibrium

eo.L4 10/10/17