2.2 One-Dimensional Chain (1D Space + 1D time)

- Aim to show emergence of collective motion i.e. degrees of freedom not what you expect
- Continuum limit to find fields from particles
- Dispersion Relation for relativistic particles!

(No 5) Cl.6

Simple one-spatial dimension model

We use this to gain a some motivation for field

- N identical particles mass m
- Confined to lie in ring, $x_n \in \mathbb{R}^1$ coordinates $n = 0, 1, \ldots (N-1)$
- Ring Size $L = Na$
- Identical Pairwise interactions $U(x_n - x_m)$ $n \neq m$
- Identical External Periodic Potential
$$V(x) = V(x+a) \text{ where } a = \frac{L}{N}.$$

$\Rightarrow$ can deduce equilibrium positions are at
$$x_n = na$$
e.g. if U repulsive

$$\Rightarrow L = \sum_n \left\{ \frac{1}{2} m \dot{x}_n - \sum_{m<n} U(x_n - x_m) - \sum_n V(x_n) \right\}$$

Too difficult to solve, not quadratic in x_n
non-linear

PS2 EFS

$\Rightarrow$ small perturbation expansion to find (under reasonable assumptions)

$$L \simeq \sum_n \left\{ \frac{1}{2} m \dot{u}_n^2 - \frac{m\omega^2}{2} (u_{n+1} - u_n)^2 - \frac{m\Omega^2 u_n^2}{2} \right\}$$

where $x_n(t) = na + u_n(t)$ so $u_n(t)$ = perturbation of n^{th} mass

$$m\omega^2 = \left.\frac{\partial^2 U}{\partial x^2}\right|_{x=a}, \quad m\Omega^2 = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=0}$$

i.e. small perturbations look like masses on ring of perfect springs

e.o.L2, 13/10/14
L2 13/10/14

$$\Rightarrow L = \frac{1}{2} m \dot{\underset{\sim}{U}}^T \cdot \dot{\underset{\sim}{U}} - \frac{1}{2} \underset{\sim}{U}^T \cdot \underset{\approx}{M} \cdot \underset{\sim}{U}$$

where $\underset{\sim}{U} = \underset{\sim}{U}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix}$

$M = M^T$ & $M_{mn} \in \mathbb{R}$ Symmetric, Real

$\Rightarrow$ Real e/values

Generic Solution

$$\underline{U} = \sum_j y_j(t) \underset{\sim}{e}_j \quad \text{where } \underset{=}{M} \underset{\sim}{e}_j = \omega_j \underset{\sim}{e}_j$$

$$\Rightarrow L = m \sum_n \left\{ \frac{1}{2} (\dot{y}_j)^2 - \frac{1}{2} \omega_j^2 y_j^2 \right\}$$ SHO with freq λ_n

$$\Rightarrow y(t) = A_j \cos(\omega_j t) + B_j \sin(\omega_j t)$$

or

$$= \tilde{A}_j e^{i\omega_j t} + \tilde{B}_j e^{-i\omega_j t}$$

L2 7/10/16

* ALWAYS of this form for quadratic L

- THESE ARE THE NORMAL MODES of the system.
- No interaction between modes
- Each mode propagates independently carrying energy, momenta & other conserved quantities
  $\Rightarrow$ ONE MODE = ONE DEGREE of FREEDOM for system
- Normal modes represent COLLECTIVE MOTION, eigenvectors $\underset{\sim}{e}_j$ are linear combinations of many masses moving $\Rightarrow$ Microscopic particles $\neq$ Energy transmission

Practical Solution Here

See PS2

EXPLOIT SYMMETRY & guess solutions / normal modes are waves with wavelength

$$\frac{2\pi}{\lambda_j} = k_j = \frac{2\pi j}{L} \qquad j = 0, 1, \ldots, (N-1)$$

i.e. work with Fourier series = eigenmode expansion

Try $u_n(t) = \sum_j y_j(t)\, e^{i k_j n a}$ for mode j

$$\Rightarrow \left[\; = \sum_j y_j(t)\, e^{2\pi i n j a / L} \right]$$

Throwing away Im parts or use sin/cos

Leave out

$$\left[\Rightarrow L = \sum_j \left\{ \frac{1}{2} m \left(\dot{y}_j(t)\right)^2 - \underbrace{\left(\frac{m\omega^2}{2} 4\sin^2\left(\frac{2\pi j a}{L}\right) + \frac{1}{2} m\Omega^2 \right)}\, y_j^2 \right. \right]$$

find $$L = \sum_j \left(\frac{1}{2} m \dot{y}_j^2 - \frac{1}{2} m \omega_j^2 y_j^2(t) \right)$$

i.e. SHO with $y = Ae^{i\omega_j t}$ solution

$$\text{where} \quad \omega_j^2 = 4\omega^2 \sin^2\left(\frac{2\pi j a}{L}\right) + \Omega^2$$

Full solution for one mode

$$u_n(t) = A\, e^{\pm i\omega(k)t - i k_j x} \qquad x = na$$

with Dispersion Relation $\quad \omega^2(k) = 4\omega^2 \sin^2(ka) + \Omega^2$

$$\simeq (4\omega^2 a^2) k^2 + \Omega^2$$

Cheng p4
Stone p7

Continuum Limit – Emergence of Fields

Suppose: spacing between original masses, $a \to 0$
($N \to \infty$ if L finite)

Let $u_n(t) = a\eta(\underset{\sim}{x} = na, t)$ (dimensionless)

Position of n^{th} mass

$$\underline{P.E} = \sum_n \left\{ \frac{1}{2} m\omega^2 \left(u_{n+1}(t) - u_n(t) \right)^2 + \frac{m\Omega^2 u_n^2}{2} \right\}$$

$$= a\sum_n \left\{ \frac{1}{2} \cdot \frac{m\omega^2 a^4}{a} \left(\frac{\eta((n+1)a, t) - \eta(na, t)}{a} \right)^2 + \frac{m\Omega^2}{2a} a^2 \eta^2 \right\}$$

if $\omega_n = \omega \; \forall n$

As $a \to 0$ P.E $\to \int dx \left\{ \frac{1}{2} \cdot K \left(\frac{\partial \eta}{\partial x}(x,t) \right)^2 + \frac{1}{2} \Lambda \eta^2 \right\}$

$(K = m\omega^2 a^3)$ $\quad (\Lambda = m\Omega^2 a)$

$$\underline{K.E} = \sum_n \frac{m}{2} \dot{u}_n^2$$

$$= a\sum_n \frac{1}{2}(ma)\left(\frac{\dot{u}_n}{a} \right)^2$$

$a \to 0$ K.E. $\to \int dx \cdot \frac{1}{2} \mu \cdot \left(\dot{\eta}(x = na, t) \right)^2$

($\mu = ma$ – mass parameter – $\frac{\mu c}{\hbar}$ is dimless)

Since $p_n = m\dot{u}_n = \mu\dot{\eta}(x = na, t)$

Define $\Pi(\underline{x}, t) = \mu\dot{\eta}$ $\quad$ Not Needed

$$\Rightarrow \Pi(x = na, t) = a\, p_n(t)$$

Put this together to find $(x=na)$

$$\lim_{a\to 0} L = \int dx \left\{ \frac{1}{2\mu} \left(\dot{\eta}(x,t)\right)^2 - \frac{1}{2}K\left(\frac{\partial}{\partial x}\eta(x,t)\right)^2 - \frac{1}{2}\Lambda\left(\eta(x,t)\right)^2 \right\}$$

If we absorb μ into definition of field

$$\phi(x,t) := \sqrt{\mu}\, \eta(x,t)$$

Dimension full $[M^{\frac{1}{2}}L^{\frac{1}{2}}]$

we find $L = \int dx\, \mathcal{L}$ with Lagrangian <u>density</u>

Stone p8 1.53 $$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}c^2\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\Omega^2\phi^2$$

with $c^2 = \dfrac{K}{\mu} = \omega^2 a^2$ = (speed)^2 of waves

$$\Omega^2 = \frac{\Lambda}{\mu} = \frac{M^2c^4}{\hbar^2} \quad \propto (\text{mass})^2 \text{ term}$$

<u>N.B.</u> don't assume a etc finite. They may need to scale with a.
Only physically measureable quantities must be finite!

(Was #5) Cl.10½

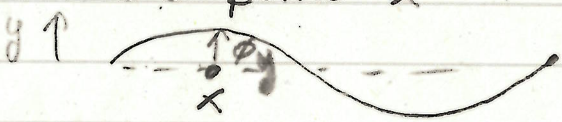
Interpretation

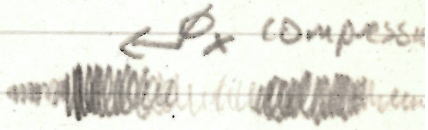
$u_n(t)$ = deviation in 1-Dimensional space of n^{th} particle at $x = na$

$u_n(t) = x_n(t) - na$

$\phi(\underbrace{x=na, t}_{\text{space-time } 1+1\text{dim}})$ = deviation in one direction 1-dim. string (or membrane $d=2$, etc) at point x

deviation





Note:- could have ϕ_y & ϕ_z not just compression waves ϕ_x
ϕ need not represent deviations in SAME direction as its argument
SO

$\phi_i(\underbrace{\underline{x}, t}_{\text{space-time location}})$ (Field representing particles)

deviation in i-th direction some "field" space which need not be observable!

* * * BIG ASSUMPTION LEAP OF FAITH. * * *

e.g. "Higgs" (in fact needs 4 components, only one directly observed)
"Cooper Pair"
"PION" - needs 3 components
π^+, π^0, π^-
3 degrees of freedom.
"4-vector potential" - A^0, A^1, A^2, A^3
for E/M fields
(only 2 physical)

oL3 14/10/14
co L3 14/10/15