5.4 FEYNMAN DIAGRAMS

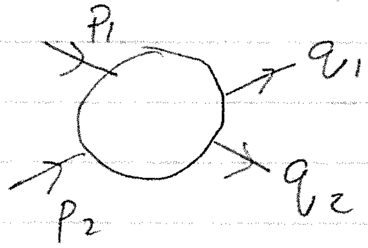
Tong §3.3.3 (3.25)

To illustrate

- We will use SYTh (Scalar Yukawa Theory)

$$H_{int} = -\int d^3x\, \mathcal{L}_{int}, \quad \mathcal{L}_{int} = -g\,\phi(x)\psi^\dagger(x)\psi(x)$$

- Consider $\psi\psi \to \psi\psi$ scattering



$$\mathcal{M} = \langle f \,|\, S \,|\, i \rangle$$

Initial state: 2 ψ's momenta $\underline{p}_1$ & $\underline{p}_2$

Final state: 2 ψ's momenta $\underline{q}_1$ & $\underline{q}_2$

$$\Rightarrow \mathcal{M} = \langle f | S | i \rangle$$

$$= \prod_{i=1,2} \left(\int d^3y \, e^{-ip_i y_i} \, 2\Omega_{p_i} \right) \quad \{ y_i^0 \to -\infty$$

$$\prod_{f=1,2} \left(\int d^3z \, e^{+iq_f z_f} \, 2\Omega_{q_f} \right) \quad \{ z_f^0 \to +\infty$$

$$\times \underbrace{\langle 0 | T \, \psi(z_1) \psi(z_2) \, S \, \psi^+(y_1) \psi^+(y_2) | 0 \rangle}$$

We need the 4-point Green function G

where

$$G(y_1 y_2 z_1 z_2) = \langle 0 | T \, \psi(z_1) \psi(z_2) \psi^+(y_1) \psi^+(y_2) S | 0 \rangle$$

$$= \sum_n G_n(y_1 y_2 z_1 z_2)$$

Since $S = \exp\{-i \int_{-\infty}^{+\infty} dt \, H_{int}\} = \exp\{+i \int d^4x \, \mathcal{L}_{int}\}$

we have

$$G_n = \langle 0 | T \, \psi(z_1) \psi(z_2) \psi^+(y_1) \psi^+(y_2)$$

$$\times \frac{(-ig)^n}{n!} \prod_{i=1}^{n} \left(\int d^4x_i \, \psi^+(x_i) \psi(x_i) \phi(x_i) \right)$$

$$\times | 0 \rangle$$

CUT - on previous page

(Example $\psi\psi \to \psi\psi$ in SYTh (PS6, Q4))

We need $G = \langle 0 | T\, \psi(z_1)\psi(z_2)\,\psi^\dagger(y_1)\psi^\dagger(y_2) S |0\rangle$

$$= \sum_n G_n(y_1, y_2, z_1, z_2)$$

where $G_n = \langle 0 | T\, \psi(z_1)\psi(z_2)\,\psi^\dagger(y_1)\psi^\dagger(y_2)\cdot \frac{(-ig)^n}{n!}\prod_{i=1}^{n}\left(\int d^4x_i\, \psi^\dagger(x_i)\psi(x_i)\phi(x_i)\right) |0\rangle$

(a) G_0 $O(g^0)$

EFS
PS6, Q4ii

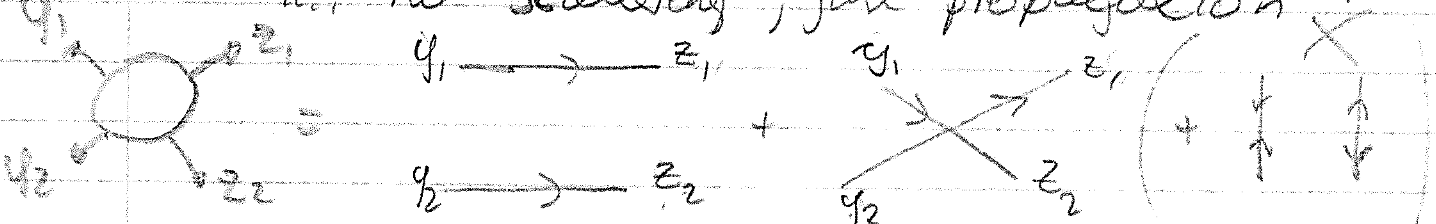
EFS find two contributions to G_0 corresponding to terms in $\mathcal{U}_0$ in

$$\Delta_\psi(y_1 - z_1)\,\Delta_\psi(y_2 - z_2) + \Delta_\psi(y_1 - z_2)\,\Delta_\psi(y_2 - z_1)$$

$\left(+ \ \psi\psi \ \ \psi^\dagger\psi^\dagger \ \to 0 \ \ \to 0\right)$

where $\Delta_\psi(x-y) = \psi(x)\psi^\dagger(y)$

i.e. no scattering, just propagation



eoL19
26/11/16 $\Rightarrow$ In momentum space find $\mathcal{M} \propto \delta^4(p_1-q_1)\,\delta^4(p_2-q_2) + \delta^4(p_1-q_2)\,\delta^4(p_2-q_1)$

PS6
Q4(iii)

(b) G_1 $O(g^1)$

EFS All odd powers of g are zero, follows from odd number of ϕ's in expression + Wick's theorem $\Leftrightarrow$ $\phi_i\ \psi/\psi^\dagger = 0$

(c) $\underset{\sim}{G_2}$ $\underset{\sim}{O(g^2)}$

$$G_2(y_1, y_2, z_1, z_2) = \frac{(-ig)^2}{2!} \int d^4x_1 \int d^4x_2 \; \Gamma(x_1, x_2, y_1, y_2, z_1, z_2)$$

where

$$\Gamma = \langle 0 | T \, \psi(z_1) \psi(z_2) \, \psi^+(x_1) \psi(x_1) \, \phi(x_1) \, \psi^+(x_2) \psi(x_2) \, \phi(x_2) \, \psi^+(y_1) \psi^+(y_2) | 0 \rangle$$

$\uparrow$

10 field expectation value!

Apply Wick's theorem & use $\langle 0 | :\text{fields}: | 0 \rangle = 0$

$$\Rightarrow \Gamma = \langle 0 | T \psi(z_1) \psi(z_2) \psi^+(x_1) \psi(x_1) \phi(x_1) \psi^+(x_2) \psi(x_2) \phi(x_2) \psi^+(y_1) \psi^+(y_2) | 0 \rangle$$

$$\overset{(A)}{=} \psi(z_1) \psi(z_2) \psi^+(x_1) \psi(x_1) \phi(x_1) \psi^+(x_2) \psi(x_2) \phi(x_2) \psi^+(y_1) \psi^+(y_2)$$

+ Many other terms with 5 contractions [945 Terms in total]

$\Gamma \equiv \Gamma_A + (\text{rest})$

$$\Gamma_A = \Delta_\psi(z_1 - x_1) \Delta_\psi(z_2 - x_2) \Delta_\phi(x_1 - x_2) \Delta_\psi(x_1 - y_1) \Delta_\psi(x_2 - y_2)$$

NOTES

(i) Trivial Zero Terms

From $[a, b^+] = 0$, $[a, a] = 0$ etc find

EFS PS5

a) $\psi \phi = 0$ contractions of different
$\psi^+ \phi = 0$ fields zero

b) $\psi \psi = \psi^+ \psi^+ = 0$ – Complex field with itself are zero

$\Rightarrow$ c) contractions non-zero only for field & (field)$^+$ h.c.

$$\psi(x) \psi^+(y) = \Delta_\psi(x-y)$$ same form as $\phi\phi$ except Ω_k, M not ω_k, m

$$\phi(x) \phi(y) = \Delta_\phi(x-y)$$

$\Rightarrow$ so many terms zero.

[** Only contractions between a field & its h.c. are non-zero in standard cases.]

e.o.L20 30/11/15 (ii) Same contribution to G(...) from other terms in Γ

e.g. x_1 & x_2 from H_{int} are identical so can swap roles.

$$\Gamma_B = \psi(z_1)\,\psi(z_2)\,\psi^+(x_1)\,\psi(x_1)\,\phi(x_1)\,\psi^+(x_2)\,\psi(x_2)\,\phi(x_2)\,\psi^+(y_1)\,\psi^+(y_2)$$

will give identical contribution to G as Γ_A

*** In fact permutation of internal vertex coord cancels $\frac{1}{n!}$ H_{int}^n

Help!!! $9.7.5.3.1 = 945$ distinct pairs of 5 contractions here

CUT

~~All One Diagram: One Number Factor~~

Combinatorics

CUT see later

(i) $\frac{1}{n!}$ from $S = T e^{\Sigma} \frac{(iHt)^n}{n!}$ is largely cancelled

by permutation of x_i in H_{int}
e.g. $x_1 \Leftrightarrow x_2$ in Ω_2 above

(ii) Each interaction term chosen with normalisation to match permutations of fields

e.g. $\frac{\lambda}{4!}\phi^4$ as $\underbrace{\phi(x)\,\phi(x)\,\phi(x)\,\phi(x)}$

4! ways of matching these contractions

$\frac{g}{1!}\psi^+\psi\phi$

↑ No identical fields.

Coordinate Space Feynman Rules for GREEN FUNCTIONS

$\langle 0 | T(\text{fields}) | 0 \rangle$

- Draw all topologically distinct diagrams using rules below.

  Aim is one distinct algebraic contribution to G = one distinct Feynman diagram.

Handout   All poss. diags | Diagram = ...?   * F. rules for G.F. in space   F7.2

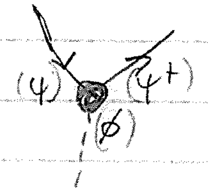
P.S p94   Coordinate Space Feynman Rules for $G = \langle 0 | T \text{ fields} | 0 \rangle$

KEY AIM: ONE DIAGRAM CAPTURES ALL CONTRIBUTIONS OF SAME VALUE

1 Each term in H_{int} factor represented by a vertex with

a) Unique coordinate x_i
b) Integration $\int d^4x_i$ ← provided symm accounted for! see below,
c) $(-i) * (\text{coupling constant})$ e.g. $-ig$ in SYTh $S = e^{-i\int dt H_{int}}$
d) legs (= end of edge) for each field in term

e.g. SYTh = Scalar Yukawa Theory, $H_{int} = g\int d^3x\, \psi^\dagger\psi\phi$



$-ig \int d^4 x_i$

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2 Connect pairs of vertices with lines representing Δ_F contractions

e.g. $\underset{x_i}{\overset{(\phi)}{\bullet}} - - - \underset{x_j}{\overset{(\phi)}{\bullet}} = \Delta_F(x_i - x_j) = \overbrace{\phi(x_i)\phi(x_j)} = \langle 0 | T\, \phi(x_i)\phi(x_j) | 0 \rangle$

a) If charged field, $\psi \neq \psi^+$, add arrow pointing FROM ψ^+ TO ψ - Needed to avoid $\boxed{\psi\psi \text{ or } \psi^+\psi^+}$

$\underset{(\psi)}{\overset{x_i}{\bullet}} \longleftarrow \underset{(\psi^+)}{\overset{x_j}{\bullet}} = \Delta_F(x_i - x_j) = \overbrace{\psi(x_i)\psi^+(x_j)}$

$= \langle 0 | T\, \psi(x_i)\psi^+(x_j) | 0 \rangle$

$= \int d^4p\, e^{-ip(x-y)} \dfrac{i}{(p^2 - m^2 + i\epsilon)}$

3 For initial/final states have external vertex with one leg & one coordinate y_i / z_i but no integration

$\overset{\psi^+}{\times\!\!\rightarrow}$

$\underset{y_i}{\times\!\leftarrow}$ $\overbrace{\psi(y_i) \cdots}$

$\underset{z_i}{\times - - -}$ $\overbrace{\phi(z_i) \cdots}$

Note often no "$\times$" added, just line ending not at a vertex

[There is a $\int d^4y\, e^{-ipy}\, 2\omega$ factor when converting to $M(\{E_A\})$ with p^μ INTO diagram]

(Coord. Space F. Rules continued)

4 Divide by the "Symmetry Factor S

— $n!$ from exp expansion in here.

(PSp93) <u>Symmetry Factor & Combinatorics</u>

Many terms found expanding $\langle 0 | T(\text{fields}) | 0 \rangle$ using Wick give same expression but we can avoid over counting

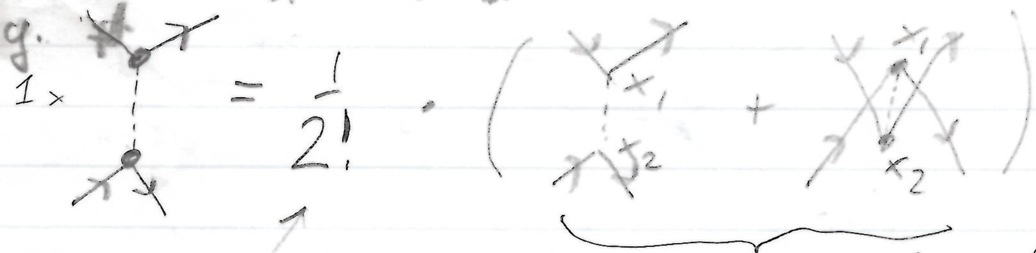
(i) $\frac{1}{n!}$ from $S = Te^{-iH_{int}t} = \sum \frac{(-iHt)^n}{n!}$

This cancels with $n!$ ways of assigning labels x_i to each vertex

$\Rightarrow$ DO not consider these permutations

$\sim \frac{1}{2!} \int dx_1 H(x_1) \int dx_2 H(x_2)$ contains

e.g.



UNLABELLED $\quad$ $n!$ from S $\quad$ Same as internal coordinates are integrated over, dummy indices.

* DON'T LABEL or DON'T PERMUTE x_i LABELS

ONE DIAGRAM $\Leftrightarrow$ Two terms in Wick's theorem

C.O. L20 4/12/15 (ii) Permutations at each vertex

Do not distinguish legs of same type at a vertex

IF you Include this factor in definition of coupling constant

Example? $\lambda\phi^4$ $\quad$ ie. $\mathcal{L}_{int} = \frac{1}{P}(cc)(\text{fields})$ $\quad$ $P = \#$ Permutations fields which leave $\mathcal{L}_{int}$ unchanged.

(ii) Permutations of Fields at Vertex (cont)

Example $\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$ $H_{int} = -\int d^3x\, \mathcal{L}_{int}$

One interaction factor produces terms like:

$$G \sim \cdots \left(-i \frac{\lambda}{4!} \int d^4x \; \overset{1}{\phi(x)} \overset{2}{\phi(x)} \overset{3}{\phi(x)} \overset{4}{\phi(x)} \right).$$

4 WAYS   3 WAYS   2 WAYS   1 WAYS

$$\cdots \phi(x_A) \cdots \phi(x_B) \cdots \phi(x_C) \cdots \phi(x_D)$$

or (1A, 1B, 1C, 1D)

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ALL give same factor of

$$= \frac{4!}{4!} \cdot -i\lambda \cdot \Delta(x-x_A)\Delta(x-x_B)\Delta(x-x_C)\Delta(x-x_D)$$

ALL 4! terms in WICK give ONE contribution = ONE Feynman Diagram.

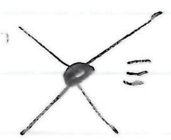
So DONT DISTINGUISH IDENTICAL LEGS AT A VERTEX

=

NO NEED TO INCLUDE PERMUTATION CONSTANT IN F. RULES.

IF YOU INCLUDE SYMMETRY FACTOR $\frac{1}{P}$ IN COUPLING CONSTANT DEFN

Here

 $\equiv -i\frac{\lambda}{1}\int dx$ NO 4!'s

Generally

$$\mathcal{L}_{int} = -\frac{(c.c)\,(\text{fields})}{(\#\text{ permutations of fields})}$$ =

 fields $\equiv -i(c.c.)\int d^4x_i$

(c.c.) ONLY

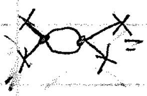
NO other constants

e.g. $-\frac{ig}{4}(\psi^{+}\psi)^2 \equiv$  $= -ig\int d^4x_i$

(iii) Symmetry Factor S Definition [$\frac{1}{S}$ in F. Rules!]

Def^n S = Number of permutations of internal lines which give same diagram

e.g. suppose we consider
$$x_C = x_D = \tilde{x}$$
in above example?


$\Rightarrow$ Must be from another internal vertex

$$\underbrace{\tfrac{1}{2!}\phi(z_1)\phi(z_2)}_{(i)\ 8}\ \tfrac{1}{4!}\phi(x)\phi(x)\phi(x)\phi(x)\ \tfrac{1}{4!}\phi(\tilde{x})\phi(\tilde{x})\phi(\tilde{x})\phi(\tilde{x})\ \phi(y_1)\phi(y_2)$$

A B ①②③④ $\tilde{1}\ \tilde{2}\ \tilde{3}\ \tilde{4}$ / C D

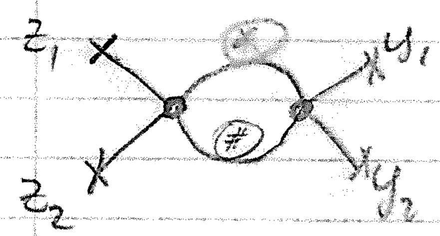
Contraction labels above the fields: (i) 8, (ii) 4, (v) 2, (vi) 1, (iv) 3, (iii) 4

[$3C = \Delta(x_3 - x_C)$]

Normally we have 4 distinct terms to cancel $\frac{1}{4!}$:

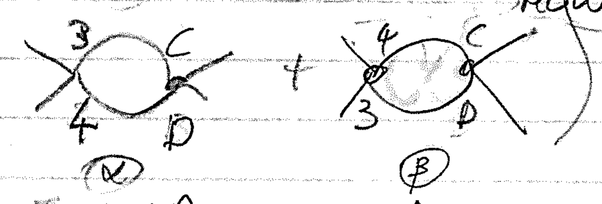
$3C, 4D, \tilde{1}\tilde{A}, \tilde{2}\tilde{B}$
$3C, 4D, \tilde{2}\tilde{A}, \tilde{1}\tilde{B}$
$4C, 3D, \tilde{1}\tilde{A}, \tilde{2}\tilde{B}$
$4C, 3D, \tilde{2}\tilde{A}, \tilde{1}\tilde{B}$

Here only two distinct terms in Wick
$3\tilde{1}, 4\tilde{2}$
$3\tilde{2}, 4\tilde{1}$
$\Rightarrow \frac{1}{2}$ left over



One Diagram

$\equiv$

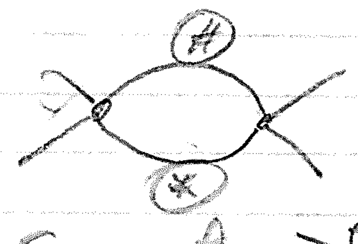


SHOW WICK (i) … (vi)

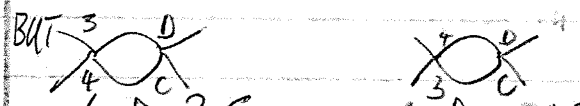
3C.4D Term $\rightarrow$ IN WICK EXPANSION $\leftarrow$ 3D.4C Term

BUT Switch 3 & 4

↓ $* \Leftrightarrow \#$



Symmetry $\Rightarrow S = 2$



4D 3C = β 4D 3C = α SAME AGAIN

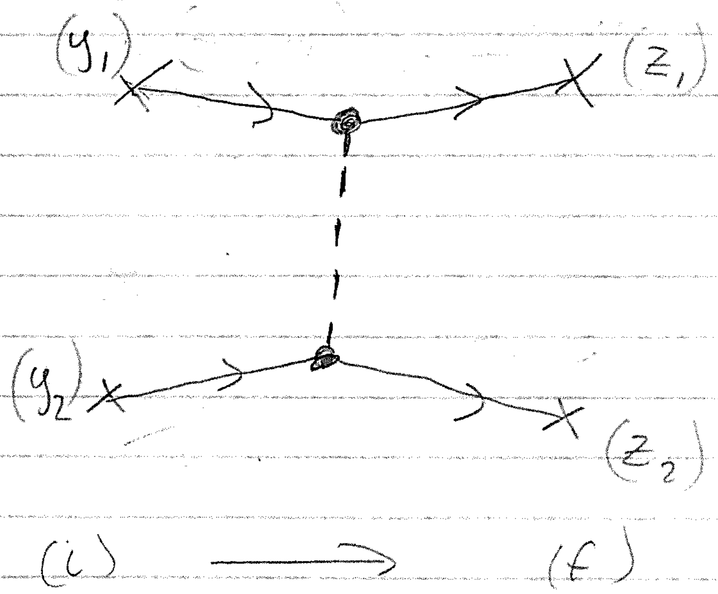
NOT a new term in WICK EXPANSION
$\Rightarrow S = 2$

Theorem

Each distinct contribution to M/G corresponds to a topologically distinct Feynman diagram

eg. $\left(\frac{\Pi_A + \Pi_B}{2!}\right) =$

$\psi\psi \to \psi\psi$



$(g=1)$

(See PS5 for more examples PS6), $\exists$ More O_2 diagrams See PS6, Q4

$$= (-ig)^2 \int d^4x_1\, d^4x_2$$

$$* \; \Delta_\psi(x_1-y_1)\,\Delta_\psi(z_1-x_1)\,\Delta_\phi(x_1-x_2)\,\Delta_\psi(x_2-y_2)\,\Delta_\psi(z_2-x_2)$$

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Do this for next year

RECIPE

1. Write down external legs for initial/final state fields
2. Write down n vertices if working at order n.
   Do this in all possible ways if more than one type of vertex
3. Connect up legs of external lines/vertices in all possible distinct ways.