Spectral Representation of Three Point Functions at Finite Temperature

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Abstract

We derive a series of relations between the six retarded and advanced three-point functions and the eight time-ordered three-point functions of real time finite-temperature field theory, valid for all types of field. We then show how to construct the spectral function for three-point functions at non-zero temperature by analogy with the well known spectral function for propagators.

In real time formulations of finite temperature field theory, the degrees of freedom are doubled leading to a two by two matrix structure [1]. This means that the spectral representation of the propagator is also a two by two matrix. It has relatively simple and well known matrix structure however, a consequence of the equilibrium nature of the problem [1]. Specifying this structure for the two-point function is equivalent to enforcing the KMS condition and vice versa. This form has been of the utmost use in studying real time finite temperature field theory e.g. in the cancellation of intermediate singularities [1].

In moving to higher order functions, it seems that one may be able to simplify the function if the matrix structure is again merely reflecting the thermal aspects of the problem. This would be of great interest for instance for the three-point functions of QCD at non-zero temperature [2]. In this note we will derive the structure of the three-point spectral function. We look first at the spectral representation of the two-point function and note how it is constructed in terms of the retarded functions. We then show how the three-point spectral function can be constructed in analogy with the propagator. In doing so we shall generalise to any order in any theory some results recently obtained by Kobes [3] for the simplest three-point 1PI diagram for a scalar field with a cubic self-interaction.

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We start from the generating functional in the path ordered approach to real time finite temperature field theory [1],

$$Z[j] = \langle\langle T_C exp\{\imath \int_C d\tau j^\dagger \phi + \phi^\dagger . j\}\rangle\rangle \tag{1}$$

where the fields $\phi(\tau)$,$\phi^\dagger(\tau)$ are Heisenberg fields with sources $j^\dagger(\tau)$, $j(\tau)$, analytically continued to take values in the complex time plane. It is convenient to run the curve $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4$ from $-\infty$ to $+\infty$ (C_1), $+\infty$ to $+\infty - \imath\bar{\alpha}\beta$ (C_2), $+\infty - \imath\bar{\alpha}\beta$ to $-\infty - \imath\bar{\alpha}\beta$ (C_3), and $-\infty - \imath\bar{\alpha}\beta$ to $+\infty - \imath\beta$ (C_4). In this note $\alpha = 1 - \bar{\alpha}$ is an arbitrary parameter reflecting some of the freedom of choice in the path C and physical results should not depend on it. The $\langle\langle ... \rangle\rangle$ indicate that the thermal expectation value is taken. Further,we will always suppress any spin or other indices on the fields as they do not effect our arguments which are based purely on thermal considerations. We will also write explicitly only the time and energy variables.

We then define

$$\phi^\mu(t) = \begin{cases} \phi(t), & t \in C_1 \quad \text{if } \mu = 1 \\ \phi(t - \imath\bar{\alpha}\beta), & t - \imath\bar{\alpha}\beta \in C_2 \quad \text{if } \mu = 2 \end{cases} \tag{2}$$

and likewise for $\phi^\dagger$, where $\mu = 1, 2$ is the thermal index. To account for the sign discussed in [4], we define

$$j^\dagger(t) = \begin{cases} j^\dagger(t), & t \in C_1 \quad \text{if } \mu = 1 \\ -j^\dagger(t - \imath\bar{\alpha}\beta), & t - \imath\bar{\alpha}\beta \in C_2 \quad \text{if } \mu = 2 \end{cases} \tag{3}$$

and likewise for $j(\tau)$. These definitions, after the usual path integral manipulations [1], lead to the same scalar propagator and Feynman rules as used in [3]. Thus the vertices, which have a single thermal label, merely take an additional minus sign when they have thermal label two, c.f. [4]. The Green functions are then

$$\Gamma_T^{\mu_1\mu_2\cdots\mu_N} := \left(\prod_{a=1}^{N} \frac{-\imath\partial}{\partial j_a^\dagger(t_a)^{\mu_a}} \right) Z\mid_{j=0} = \langle\langle T_C \phi_1^{\mu_1}(t_1)\phi_2^{\mu_2}(t_2)...\phi_N^{\mu_N}(t_N)\rangle\rangle. \tag{4}$$

When fermionic fields are present, the $\partial/\partial j$ are defined to act in an order so as to ensure the second definition is compatible. Here T_C indicates that the fields are path ordered with respect to their time argument, fields with arguments furthest along path C to the left.

From these definitions, it is straightforward derive the propagator for the a-th field

$$\begin{aligned} \Pi_a^{\mu\nu}(t_1 - t_2) &:= \langle\langle T_C \phi_a(t_1)^\mu \phi_a^\dagger(t_2)^\nu \rangle\rangle \\ &= \int dk\, e^{-\imath(t_1 - t_2)} \Pi_a^{\mu\nu}(k) \end{aligned} \tag{5}$$

by just using the KMS condition [1]. This is then found to be

$$\Pi_a(k) = U_a(k).\bar{\Pi}_a(k).\bar{U}_a^{-1}(k) \tag{6}$$

$$\bar{\Pi}_a(k) = \begin{pmatrix} \Pi_a^R(k) & 0 \\ 0 & -\Pi_a^A(k) \end{pmatrix} \tag{7}$$

where

$$U_a(k) := \begin{pmatrix} (1+\sigma_a n_a(k))^{1/2}/b_{Ra}(k)b_{La} & \sigma_a(n_a(k))^{1/2}b_{La}/b_{Ra}(k) \\ (n_a(k))^{1/2}b_{Ra}(k)/b_{La} & (1+\sigma_a n_a(k))^{1/2}.b_{Ra}(k)b_{La} \end{pmatrix} \tag{8}$$

$$\bar{U_a} = \tau_3.U_a.\tau_3 \tag{9}$$

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

$$\begin{aligned} n_a(k) &= (e^{\beta k} - \sigma_a)^{-1} && (11) \\ b^2_{Ra}(k) &= (e^{\beta k})^{\alpha - 1/2} && (12) \end{aligned}$$

We set $\sigma_a = +1(-1)$ if the a-th field is bosonic (fermionic). The form of the propagator does not depend on the real parameter b_L, though it is sometimes chosen to be $b_{La} = n_a^{\alpha-1/2}.b_{Ra}(k)$. Note that most equilibrium work is done with $\alpha = 1/2$ and with an equivalent form for the propagator that involves $n_a(k) = (exp\{\beta|k|\} - \sigma_a)^{-1}$ [1]. The Π^R, Π^A functions are the retarded and advanced propagators,

$$\Pi^R_a(k) = \int dt\, e^{\imath kt}\, \Pi^R_a(t) \tag{13}$$

and likewise for Π^A_a, with

$$\begin{aligned} \Pi^R_a(t_1 - t_2) &:= \theta(t_1 - t_2)[\Gamma_{12}(t_1 - t_2) - \Gamma_{21}(t_1 - t_2)], && (14) \\ \Pi^A_a(t_1 - t_2) &:= -\theta(t_2 - t_1)[\Gamma_{12}(t_1 - t_2) - \Gamma_{21}(t_1 - t_2)]. && (15) \end{aligned}$$

Here

$$\begin{aligned} \Gamma_{12}(t_1 - t_2) &= \langle\langle \phi_a(t_1)\phi^\dagger_a(t_2)\rangle\rangle, \\ \Gamma_{21}(t_1 - t_2) &= \pm\langle\langle \phi^\dagger_a(t_2)\phi_a(t_1)\rangle\rangle, && (16) \end{aligned}$$

and we take the $+$ $(-)$ sign if we have a bosonic (fermionic) propagator.

The key points to note are that there is a Bogoliubov transformation matrix for each leg of the function, and a core matrix which we denote with a bar ($\bar{\Pi}$ here). Two elements of this core matrix are zero corresponding to two algebraic identities. These identities are most naturally seen in the approach of Kobes and Semenoff [5, 3] and correspond to their 'largest time equation' and its complex conjugate, the 'smallest time equation'. The remaining entries of the core matrix are then each proportional to one of the retarded or advanced two point functions and represent the well known relation between retarded propagators and the elements of the real time matrix two-point function.

These are the elements that we shall try to use in generalising to the case of three-point functions. First we need to find all the relations between the eight functions of the real time calculations and the six retarded functions. The eight real time connected three-point vertex functions are

$$\Gamma^{\mu_1\mu_2\mu_3}_T(t_1, t_2, t_3) = \langle\langle T_C \phi^{\mu_1}_1(t_1)\phi^{\mu_2}_2(t_2)\phi^{\mu_3}_3(t_3)\rangle\rangle \tag{17}$$

where $\{\phi_a^\mu\}$ $a = 1, 2, 3$ are any three fields with spin and other indices suppressed except for the thermal index, μ, defined as in (2). Throughout this paper we let subscripts a, b, c be any permutation of $1, 2, 3$. These subscripts indicate with which field a quantity is associated with, e.g. $n_a = n_2$ and $t_b = t_3$ when we look at the $(abc) = (231)$ permutation. The three different expectation values of retarded three-point functions, R_a, and the three advanced functions, $\bar{R}_a$, are defined through the (anti-)commutators [6]

$$\begin{aligned} R_a(t_1, t_2, t_3) &= R(\phi_a(t_a) \mid \phi_b(t_b)\phi_c(t_c)) \\ &= \theta(t_a - t_b)\theta(t_b - t_c)[\Gamma_{abc} - \Gamma_{bac} - \Gamma_{cab} + \Gamma_{cba}] + \\ &\quad \theta(t_a - t_c)\theta(t_c - t_b)[\Gamma_{acb} - \Gamma_{cab} - \Gamma_{bac} + \Gamma_{bca}], \\ \bar{R}_a(t_1, t_2, t_3) &= R(\phi_b(t_b)\phi_c(t_c) \mid \phi_a(t_a)) \\ &= \theta(t_c - t_b)\theta(t_b - t_a)[\Gamma_{abc} - \Gamma_{bac} - \Gamma_{cab} + \Gamma_{cba}] + \\ &\quad \theta(t_b - t_c)\theta(t_c - t_a)[\Gamma_{acb} - \Gamma_{cab} - \Gamma_{bac} + \Gamma_{bca}], \end{aligned} \tag{18}$$

where

$$\begin{aligned} \Gamma_{abc} &= \Gamma_{abc}(t_1, t_2, t_3) \\ &= (-1)^p \langle\langle \phi_a(t_a)\phi_b(t_b)\phi_c(t_c) \rangle\rangle \end{aligned} \tag{19}$$

In (19), p is the number of times one has to swap fermion fields in going from a 123 ordering to the abc ordering of Γ_{abc} of (19). Note that it was shown in [7] that the six R_a,$\bar{R}_a$ are indeed the results obtained from the *usual* Imaginary-Time formalism calculation of the three point functions where the the simplest analytic continuation of the external energies is used.

In (17), Γ_T^{111} is the time ordered expectation value of the fields at real times that was calculated in [7] in terms of the retarded and advanced functions (18). The other elements of $\Gamma_T^{\mu_1\mu_2\mu_3}$ can be calculated in the same manner, that is we use the definitions (17) and (18), the boundary condition that equilibrium fields must satisfy, the KMS condition [1], and work in Fourier space. We define the Fourier transforms of our three-point functions through

$$g(k_1, k_2, k_3) = \int dt_1 dt_2 dt_3 \, e^{\imath(k_1 t_1 + k_2 t_2 + k_3 t_3)} \, g(t_1, t_2, t_3). \tag{20}$$

The KMS condition in our notation is

$$\Gamma_{abc}(k_1, k_2, k_3) = \Gamma_{cab}(k_1, k_2, k_3).f_c, \tag{21}$$

where

$$f_a = \sigma_a e^{-\beta k_a}. \tag{22}$$

Comparing (17) with (18) in Fourier space and using (21), we find after some algebra that

$$\Gamma_T^{111}(k_1, k_2, k_3) = \sum_{\substack{a=1 \\ a \neq b,c; b<c}}^{3} \sigma_a n_b n_c (R_a + \bar{R}_a f_a)$$

$$
\begin{aligned}
\Gamma_T^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)|_{\substack{\mu_a=2\\ \mu_{b,c}=1}} &= (\sigma_a f_a)^{\bar{\alpha}}\Big\{R_a n_b n_c \sigma_a + R_b n_a n_c \sigma_b f_a^{-1} \\
&\quad + R_c n_a n_b \sigma_c f_a^{-1} + \bar{R}_a n_b n_c \sigma_a \\
&\quad + \bar{R}_b n_a n_c \sigma_b f_b + \bar{R}_c n_a n_b \sigma_c f_c\Big\} \\
\Gamma_T^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)|_{\substack{\mu_a=1\\ \mu_{b,c}=2}} &= (\sigma_a f_a)^{-\bar{\alpha}} f_a \left\{R_a n_b n_c \sigma_a + R_b n_a n_c \sigma_b f_b\right. \\
&\quad + R_c n_a n_b \sigma_c f_c + \bar{R}_a n_b n_c \sigma_a \\
&\quad \left. + \bar{R}_b n_a n_c \sigma_b f_a^{-1} + \bar{R}_c n_a n_b \sigma_c f_a^{-1}\right\} \\
\Gamma_T^{222}(k_1,k_2,k_3) &= \sum_{\substack{a=1\\ a\neq b,c; b<c}}^{3} \sigma_a n_b n_c (R_a f_a + \bar{R}_a)
\end{aligned} \tag{23}
$$

where $n_a = n_a(k_a)$ of (11) and here we always choose $b < c$ and $a \neq b, c$. The R_a and $\bar{R}_a$ are the six different expectation values of retarded and advanced three-point functions (18) in Fourier space,

$$
R_a = R_a(k_1,k_2,k_3), \tag{24}
$$

and likewise for $\bar{R}_a$.

Now we can try to write the three-point function as

$$
\Gamma_T^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = (\prod_{a=1}^{3} U_a(k_a)^{\mu_a\nu_a})\bar{\Gamma}^{\nu_1\nu_2\nu_3}(k_1,k_2,k_3) \tag{25}
$$

in analogy with (6). It is then straightforward to show that

$$
\begin{aligned}
\bar{\Gamma}^{111}(k_1,k_2,k_3) &= 0, \\
\bar{\Gamma}^{\nu_1\nu_2\nu_3}(k_1,k_2,k_3)|_{\substack{\nu_a=2\\ \nu_{b,c}=1}} &= \bar{R}_a \frac{n_b^{1/2} n_c^{1/2}}{(1+\sigma_a n_a)^{1/2}} \sigma_a . \frac{b_{Lb} b_{Lc}}{b_{La}} \\
\bar{\Gamma}^{\nu_1\nu_2\nu_3}(k_1,k_2,k_3)|_{\substack{\nu_a=1\\ \nu_{b,c}=2}} &= R_a \frac{n_b^{1/2} n_c^{1/2}}{(1+\sigma_a n_a)^{1/2}} . \frac{b_{La}}{b_{Lb} b_{Lc}} \\
\bar{\Gamma}^{222}(k_1,k_2,k_3) &= 0.
\end{aligned} \tag{26}
$$

Thus the three-point spectral function can be written in an way analogous to the two-point spectral function. There is an appropriate Bogoliubov transformation for each leg and a core function $\bar{\Gamma}$. The core matrix has two zero entries, one for each of the two algebraic identities noted by Kobes and Semenoff [5, 3], their largest and smallest time equations. Each of the remaining entries is proportional to one of the retarded or advanced functions and corresponds to expressing each retarded and advanced function in terms of the various functions of the real time formalism, $\Gamma_T^{\mu_1\mu_2\mu_3}$ of (17). It is then simple to express each retarded function in terms of the real-time functions by pre-multiplying (25) by inverse Bogoliubov matrices and using (26).

The generalisation of the results of Kobes [3] can easily be extracted from (25). First we must go from the connected functions considered here to the 1PI case of Kobes. We

merely remove the full propagators from each leg using the form (6) for the propagators and we find

$$\Gamma_{1PI}^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) \;=\; (\prod_{a=1}^{3}\bar{U}_a(k_a)^{\mu_a\nu_a})\bar{\Gamma}_{1PI}^{\nu_1\nu_2\nu_3} \tag{27}$$

where

$$\bar{\Gamma}_{1PI}^{\nu_1\nu_2\nu_3} \;=\; \left(\prod_{a=1}^{3}\bar{\Pi}_a^{-1}(k_a)^{\nu_a\nu_a}\right)\bar{\Gamma}^{\nu_1\nu_2\nu_3}. \tag{28}$$

One identifies $\Gamma_{1PI}^{112}-\Gamma_{1PI}^{221}$ ($\Gamma_{1PI}^{112}+\Gamma_{1PI}^{221}$) etc. with the real (imaginary) parts of Kobes' functions.

The results given in (25) and (27) are very general as they apply to three-point functions of any theory, whatever its interactions (provided there is an even number of fermions). This is because it relies on the usual equilibrium boundary condition, the KMS condition (21), that all fields must satisfy in equilibrium. Further, (25) and (27) will also be satisfied by any field or any approximation to a field that respects the KMS condition. Thus the equations in (25) apply both to full connected three-point functions and to any one diagram in the Feynman expansion of full connected three-point functions. Likewise (27) applies to both full 1PI functions and to any individual 1PI diagram.

One use of these relations lies in the fact that, as was shown in [7], the *usual* analytic continuation of external energies in ITF (Imaginary-Time Formalism) [1, 8] means one obtains the retarded functions in ITF, while it is simplest to extract the time-ordered function from RTF (Real Time Formalisms) [1]. Thus (25) and (27) relate all the results of the *usual* RTF calculations to the *usual* ITF three-point results, and vice versa. Note that this means that ITF and RTF as *usually* used give different results but that this is because they are being used to calculate different types of function and it is these functions that are unequal not that the formalisms are giving different answers for the same functions [7]. Since it is possible to use ITF and RTF in ways other than are standard in the literature in order to extract other functions, we must always note how each formalism is being used. Overall then ITF and RTF differ only in the relative ease with which each formalism can calculate a given quantity. As discussed in [7], the choice between ITF and RTF is merely one of computational convenience, not fundamental difference in the physics. The real question is not whether ITF or RTF is 'correct' but whether we need retarded or time-ordered functions, and this will be decided by the physical context of any given problem.

Beyond this use in comparing the results obtained most easily from ITF and RTF calculations, knowing the form of the three-point spectral function may be as useful as that of the two-point function in studying the structure of the real-time theory to arbitrarily high orders, c.f. the use of the two-point spectral function in checking that the theory is free of certain singularities [1].

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