Three Point Functions at Finite Temperature

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Abstract

We consider the calculations of three-point functions at finite temperature as they are usually performed in the literature. We show that as normally used, the Imaginary-Time and the Real Time finite temperature formalisms calculate expectation values of retarded and time ordered three point functions respectively. We also present a relation between these quantities that shows that they are not generally equal.

In the past in finite temperature field theory, attention has focused largely on two-point functions [1, 2]. Recently there has been increasing interest in three-point functions, especially for the calculation of the QCD coupling constant as it runs with temperature [3, 4, 5, 6].

Here we will look at only one issue, that of the apparent incompatibility between results for three-point functions obtained from the two different finite temperature formalisms, ITF (Imaginary-Time Formalism) [1, 2] or the newer RTF (Real Time Formalism) [2]. Typically in any one-loop three-point function calculation, one finds factors of n^3 in RTF calculations, where n is the number distribution, but only single n factors in the equivalent ITF calculation.

In the particular case of three-point functions in QCD, this results in very different high temperature behaviour. In RTF, one finds β^{-3} and β^{-2} behaviour in the static case (zero external energies) [3, 4], where β is the inverse temperature, while with ITF

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one finds only β^{-1} in the static case [5] and β^{-2} in the dynamic case (non-zero external energies) [5, 6]. Such differences have led to the suggestion that RTF and ITF may not be compatible after all [4]. However, with these QCD calculations there are many other issues to be resolved, such as gauge dependence and dependence on choice of momentum renormalisation point. We do not address these other issues here and we do not claim to have the complete solution in the case of the QCD running coupling constant. The results of this Letter, when applied to these QCD calculations, clarify only one problem, that of the possible incompatibility between ITF and RTF alluded to in [4].

The main results of this Letter apply to any theory. First of all we show that ITF, as *usually* used, calculates the retarded three-point function, whereas RTF gives the time-ordered three-point function. Secondly, we give an explicit relation, equation (20), between the retarded and time-ordered three-point functions. This relation shows retarded and time-ordered functions are not equal at non-zero temperature, contrary to the case at zero temperature [7] or even the case of the real parts of two point functions at non-zero temperature [1, 2]. The relation is the extension to three-point functions of the well known relation between the retarded and time-ordered finite temperature propagators [1, 2]. These results also clearly show that we do not expect calculations which use ITF and RTF in the standard way, e.g. the QCD calculations [3, 4, 5, 6], to give the same result, and that such calculations can differ by powers of β in the high temperature limit.

We first need to consider what expectation values are actually calculated in RTF and ITF. In RTF calculations, one looks at the diagram where all the external vertices are fixed to be type 1, and, by definition, this is the time ordered product [2]. To see what is calculated in ITF, we have to extend the work of Baym and Mermin [8] from two-point functions to three-point functions. We consider a three-point function of three (not necessarily equal) Heisenberg fields, $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, and we shall suppress any spin indices and any dependence on space or three-momentum variables as neither of these aspects affects our results.

We start by defining

$$\Gamma^T(t_1,t_2,t_3) = Tr\{e^{-\imath\tau H} T\ \phi_1(t_1)\phi_2(t_2)\phi_3(t_3)\}/\ Tr\{e^{-\imath\tau H}\}, \quad (1)$$

$$\Gamma_{abc}(t_1,t_2,t_3) = (-1)^p Tr\{e^{-\imath\tau H}\phi_a(t_a)\phi_b(t_b)\phi_c(t_c)\}/\ Tr\{e^{-\imath\tau H}\}, \quad (2)$$

where T stands for time ordering. In (2), p is the number of times one has to swap fermion fields in going from a 123 ordering to the abc ordering of Γ_{abc} of (2). Throughout this Letter we let (abc) be any permutation of (123), so that $t_a = t_2$, $n_b = n_3$ when we choose the (231) permutation for (abc). When $\tau = -i\beta$, we have the canonical ensemble which is assumed to be absolutely convergent. Thus Γ^T is analytic in the lower half τ plane. Using cyclicity of the trace, we see that $\Gamma_{abc}(t_1,t_2,t_3)$ converges only when $\{t_1,t_2,t_3\} \in A_{abc}$ where

$$A_{abc} := \{\{t_a,t_b,t_c\}\,|\,Im(t_c) \geq Im(t_b) \geq Im(t_a) \geq Im(t_c+\tau)\}. \quad (3)$$

Using this we can analytically extend the definition of the time ordered product, Γ^T, to cover complex time arguments, and we define

$$\Gamma^T(t_1,t_2,t_3) = \Gamma_{abc}(t_1,t_2,t_3) \quad (4)$$

when

$$\{t_a, t_b, t_c\} \in A_{abc} \tag{5}$$

and

$$Re(t_a) \geq Re(t_b) \geq Re(t_c). \tag{6}$$

The cyclicity of the trace leads to the boundary condition

$$\Gamma_{cab}(t_1, t_2, t_3) = \sigma_c \Gamma_{abc}(t_1^{'}, t_2^{'}, t_3^{'}) \tag{7}$$

when $\{t_1, t_2, t_3\} \in A_{abc}$, and

$$\Gamma^T(t_1, t_2, t_3) = \Gamma^T(t_1^{'}, t_2^{'}, t_3^{'}) \tag{8}$$

when $\{t_1, t_2, t_3\} \in A_{abc}$ and $Re(t_a) \geq Re(t_b) \geq Re(t_c)$, where $t_a^{'} = t_a, t_b^{'} = t_b$, $t_c^{'} = t_c - \tau$ in (7) and (8). In this Letter, we shall set $\sigma_a = +1\,(-1)$ if the a-th field is bosonic (fermionic). Because the Γ_{abc} are bounded when $\{t_1, t_2, t_3\} \in A_{abc}$, its Fourier transform, γ_{abc}, exists so that

$$\Gamma_{abc}(t_1, t_2, t_3) = (2\pi)^{-3} \int dp_1 dp_2 dp_3 e^{-\imath(p_1 t_1 + p_2 t_2 + p_3 t_3)} \gamma_{abc}(p_1, p_2, p_3) \tag{9}$$

where $t_1, t_2, t_3 \in A_{abc}$. The boundary conditions on (9), when $\{t_1, t_2, t_3\} \in A_{abc}$, lead to

$$\gamma_{abc}(p_1, p_2, p_2) = \gamma_{cab}(p_1, p_2, p_3).f_c \tag{10}$$

where

$$f_a := \sigma_a e^{-\imath \tau p_a}. \tag{11}$$

Now we consider exactly what one calculates in ITF. To do this we consider the Fourier series of Γ^T for $t_i/\tau \in \Re e$, $-1 < t_i/\tau < 1$ and define $\gamma_{\nu_1,\nu_2,\nu_3}$ by

$$\Gamma^T(t_1, t_2, t_3) = \tau^{-3} \sum_{\nu_1,\nu_2,\nu_3} e^{-2\pi\imath(\nu_1 t_1 + \nu_2 t_2 + \nu_3 t_3)/\tau}.\gamma_{\nu_1,\nu_2,\nu_3} \tag{12}$$

where ν_a is an integer (half-integer) if the a-th field is bosonic (fermionic). Thus

$$\gamma_{\nu_1,\nu_2,\nu_3} = \delta_{\nu_1,\nu_2,\nu_3} \int_0^\tau dt_1 dt_2\, e^{2\pi\imath(\nu_1 t_1 + \nu_2 t_2)/\tau}.\Gamma^T(t_1, t_2, 0), \tag{13}$$

where we have used time translation invariance. From (13), after some manipulation, we find that

$$\gamma_{\nu_1,\nu_2,\nu_3} = \delta_{\nu_1,\nu_2,\nu_3} \Phi(z_1 = 2\pi\nu_1/\tau, z_2 = 2\pi\nu_2/\tau) \tag{14}$$

where

$$\begin{aligned}\Phi(z_1, z_2) = (2\pi)^{-2} \int & dp_1 dp_2 dp_3\, \delta(p_1 + p_2 + p_3) \\ \Big\{\gamma_{123}(p_1, p_2, p_3). & \left(\frac{\imath}{z_1 - p_1}.\frac{\imath}{z_1 - p_1 + z_2 - p_2} - f_1.\frac{\imath}{z_1 - p_1}.\frac{\imath}{z_2 - p_2} + \right. \\ & \left. f_1 f_2.\frac{\imath}{z_1 - p_1 + z_2 - p_2}.\frac{\imath}{z_2 - p_2}\right) + \\ \gamma_{321}(p_1, p_2, p_3). & \left(\frac{\imath}{z_1 - p_1}.\frac{\imath}{z_1 - p_1 + z_2 - p_2} - f_1^{-1}.\frac{\imath}{z_1 - p_1}.\frac{\imath}{z_2 - p_2} + \right. \\ & \left. f_1^{-1} f_2^{-1}.\frac{\imath}{z_1 - p_1 + z_2 - p_2}.\frac{\imath}{z_2 - p_2}\right)\Big\}.\end{aligned} \tag{15}$$

Now there are many other definitions of Φ that satisfy (14). For instance, we can insert $\sigma_1 e^{\imath z_1 \tau}$ and $\sigma_2 e^{\imath z_2 \tau}$ anywhere in the expression (15) for Φ and it will still satisfy (14). However, (15) is the unique analytic continuation that is both analytic off the $z_1, z_2 \in \Re e$ plane and that tends to zero as either z_1 or z_2 tend to infinity in any direction in the complex plane. The proof that it is unique is the same as in [8], because we can consider Φ as a function of just one complex variable at a time, holding the others constant. The question is now what is (15) near the $z_1, z_2 \in \Re e$ plane. The simplest approach, and the one almost always used in the literature, is to set $z_1 = k_1 + \imath\epsilon_1$, $z_2 = k_2 + \imath\epsilon_2$, though there are many other possibilities for the analytic continuation of external energies in ITF. Then there are just six different ways of approaching this plane. The six different choices for the ϵ's and the results are shown in in table 1. Essentially, we get all the different

Result	R_1	$\bar{R}_1$	R_2	$\bar{R}_2$	R_3	$\bar{R}_3$
ϵ_1/ϵ	+2	-2	-1	+1	-1	+1
ϵ_2/ϵ	-1	+1	+2	-2	-1	+1

Table 1: The results obtained for different choices for ϵ_1, ϵ_2 in$\Phi(k_1 + \imath\epsilon_1, k_2 + \imath\epsilon_2)$

types of three point retarded function, which we define through the (anti-)commutators [7]

$$\begin{aligned}
R_a &= R(\phi_a(t_a) \mid \phi_b(t_b)\phi_c(t_c)) \\
&= \theta(t_a - t_b)\theta(t_b - t_c)[\Gamma_{abc} - \Gamma_{bac} - \Gamma_{cab} + \Gamma_{cba}] + \\
&\quad \theta(t_a - t_c)\theta(t_c - t_b)[\Gamma_{acb} - \Gamma_{cab} - \Gamma_{bac} + \Gamma_{bca}], \qquad (16) \\
\bar{R}_a &= R(\phi_b(t_b)\phi_c(t_c) \mid \phi_a(t_a)) \\
&= \theta(t_c - t_b)\theta(t_b - t_a)[\Gamma_{abc} - \Gamma_{bac} - \Gamma_{cab} + \Gamma_{cba}] + \\
&\quad \theta(t_b - t_c)\theta(t_c - t_a)[\Gamma_{acb} - \Gamma_{cab} - \Gamma_{bac} + \Gamma_{bca}], \qquad (17)
\end{aligned}$$

where $\Gamma_{abc} = \Gamma_{abc}(t_1, t_2, t_3)$ etc. This is what is usually calculated in ITF, i.e. the simplest analytic continuation to real energies is used.

The retarded function, which is the usual ITF result, should be compared with the expectation value of the time ordered product, which is the RTF result. The Fourier transform of the latter is

$$\begin{aligned}
\Gamma^T(k_1, k_2) = (2\pi)^{-2} \int dp_1 dp_2 dp_3 \delta(p_1 + p_2 + p_3) \\
\Big\{ \gamma_{123}(p_1, p_2, p_3) \quad & \left(\frac{\imath}{k_1 - p_1 + \imath\epsilon} . \frac{\imath}{k_1 - p_1 + k_2 - p_2 + \imath\epsilon} - \right. \\
& f_1 . \frac{\imath}{k_1 - p_1 - \imath\epsilon} . \frac{\imath}{k_2 - p_2 + \imath\epsilon} + \\
& \left. f_1 f_2 . \frac{\imath}{k_1 - p_1 + k_2 - p_2 - \imath\epsilon} . \frac{\imath}{k_2 - p_2 - \imath\epsilon} \right) + \\
\gamma_{321}(p_1, p_2, p_3) \quad & \left(\frac{\imath}{k_1 - p_1 - \imath\epsilon} . \frac{\imath}{k_1 - p_1 + k_2 - p_2 - \imath\epsilon} - \right. \\
& f_1^{-1} . \frac{\imath}{k_1 - p_1 + \imath\epsilon} . \frac{\imath}{k_2 - p_2 - \imath\epsilon} +
\end{aligned}$$

$$f_1^{-1} f_2^{-1} . \frac{\imath}{k_1 - p_1 + k_2 - p_2 + \imath\epsilon} . \frac{\imath}{k_2 - p_2 + \imath\epsilon} \Big) \Big\} . \qquad (18)$$

where ϵ is small, real and positive. We now make use of (10) and

$$\frac{\imath}{x \pm \imath\epsilon} = \frac{\imath PP}{x} \pm \pi\delta(x), \qquad (19)$$

where PP indicates the principal part, and then find after some manipulation that

$$\Gamma^T(k_1, k_2) = [(1 - F_1^{-1})(1 - F_2^{-1})(1 - F_3^{-1})]^{-1} \sum_{a=1}^{3} (1 - F_a^{-1}).R_a(k_1, k_2) + \\ [(1 - F_1)(1 - F_2)(1 - F_3)]^{-1} \sum_{a=1}^{3} (1 - F_a).\bar{R}_a(k_1, k_2) \qquad (20)$$

where $F_a := \sigma_a e^{-\imath\tau k_a}$ and $k_3 := -k_1 - k_2$.

Equation (20) will hold for any combination of three fields, irrespective of spin, as it relies on the boundary condition (10) that is true for all fields. It will also be satisfied by any field or any approximation to a field that respects the boundary condition (10). Thus equation (20) applies both to full connected three-point functions and to any one diagram in the Feynman expansion of full connected three-point functions.

We have thus shown two key results. First is the extension from two- to three-point functions of the result of Baym and Mermin [8], that is that in ITF after the physical analytic continuation in the external energies variables, one finds that we are calculating function Φ of (14). The simplest analytic continuation of this function, that is almost always used in the ITF literature, is seen to give the retarded function as a final result, as table 1 shows. The second result is equation (20), the generalistion to three-point functions of the well known relation between retarded and time-ordered propagators [1, 2].

One immediate observation can now be made, if we also remember that in the usual RTF calculations, where the (111) component is taken, we are calculating the time-ordered function, Γ^T. Our results show that *the usual RTF and ITF three-point calculations are not simply equal but are related by equation (20).*

In the case of the QCD calculations our results mean that there will be some big differences between the RTF calculations [3, 4] and the ITF calculations [5, 6]. This is because they all use the finite temperature formalisms in the standard way and are thus calculating time-ordered and retarded functions which are unequal as equation (20) shows. However, it is not possible to make a detailed quantitative comparison between the published results because they have not all been done at the same external momentum. In any case, there are several other unresolved issues that may cause problems with these QCD calculations, so we have yet to fully understand the case of the QCD running coupling constant at finite temperature.

We emphasise that there are ways of using ITF and RTF other than the usual way that is considered here and that is also used in the literature. For example, one has many other analytic continuations in ITF other than the simple one considered here and which is used in the literature. Thus, we must always note how a formalism is being used so that it is clear what sort of function is being calculated. This is important because, as our results for three-point functions have shown, the *usual* use of ITF and RTF leads to different types of function being calculated and further that different types of function are not in general equal at non-zero temperature.

However, ITF and RTF as a whole must give the same results for the same quantity as their path integral derivations show they are identical in content [2]. They differ only in the relative ease with which each formalism can calculate a given quantity. As we have shown above for the three-point functions, it is easiest to extract the retarded functions from ITF but the time-ordered ones from RTF. Thus the choice between ITF and RTF is merely one of computational convenience, not of any fundamental difference in the physics. The real question is not whether ITF or RTF is 'correct' but whether we need retarded or time-ordered functions in a particular problem, and this will be decided by the physical context.

Finally, we note that we can separate the complex equation (20) into two parts by using (19), one part containing only $(PP)^2$ and δ^2 terms while the other contains the $\imath PP.\delta$ terms. Unlike the case of the two-point functions however, these do not necessarily correspond to the real and imaginary parts of equation (20) because the equation also contains γ functions which are not necessarily real for three-point functions. We can not therefore make general statements about the real and imaginary parts of (20). This is to contrasted with the situation for propagators [1, 2]. There one can show, by using Hermitean properties and energy conservation, that $\gamma^*_{12}(k_1, k_2) = \gamma_{12}(-k_2, -k_1) = \gamma_{12}(k_1, k_2)$ (in an obvious generalisation of the notation to two-point functions with $\phi_1 = \phi_2^\dagger$) so $\gamma_{12} \in \Re e$. Thus, all the factors of $\imath$ in the case of the two-point retarded and time ordered functions come from the split given in (19). For the three point functions, even with $\phi_1 = \phi_3^\dagger$ and $\phi_2 = \phi_2^\dagger$, we can only show that $\gamma^*_{123}(k_1, k_2, k_3) = \gamma_{123}(-k_3, -k_2, -k_1) = \gamma_{123}(k_1 + k_2, -k_2, -k_1)$, so that γ_{123} etc. are not necessarily real and we can not do the simple split of (20) into real and imaginary parts that is possible for two-point functions.

Since this work was completed, we have become aware of the complementary work of Kobes [9] in which he derives an expression for the usual ITF calculation in terms of the various real-time functions for the case of a one-loop diagram in a scalar field theory with cubic self-interaction.

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