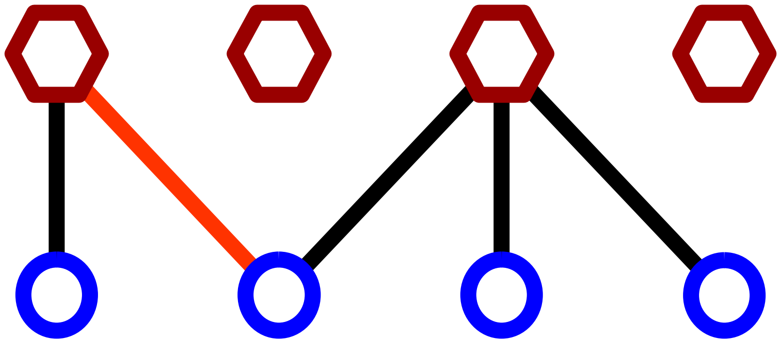
Workshop on
**Complex Networks and Social Dynamics**
Espoo 31st Aug.-2nd Sept. 2007

Tim Evans
Theoretical Physics

# Cultural Transmission and Network Rewiring

**T.S.Evans, A.D.K.Plato**

- *“Exact Solution for the Time Evolution of Network Rewiring Models”*
  Phys. Rev. E **75** (2007) 056101 [cond-mat/0612214]
- *“Network Rewiring Models”* (for ECCS07)
  arXiv:0707.3783



- A Simple Model
- Exact Solution
  - Equilibrium
  - Time Dependence
  - Phase Transition
- Generalisations
  - Network of Individuals
  - Voter Model
  - Minority Game
  - Different Update methods
  - Different Types of Individuals - `Two Tribes'
- Summary

# A Simple Model of Cultural Transmission

- Fixed population of ***E*** individuals
- Each person chooses one of ***N*** artifacts
  - Artifacts have no intrinsic benefit
  - e.g. pedigree dog, shoe style, name for baby
- At each time step, one random person updates their choice using one of two methods:-

  *(a)* COPYING someone else's choice

  *(b)* INNOVATING, picking an artifact at random
  it will be one no one else has chosen if *N* large

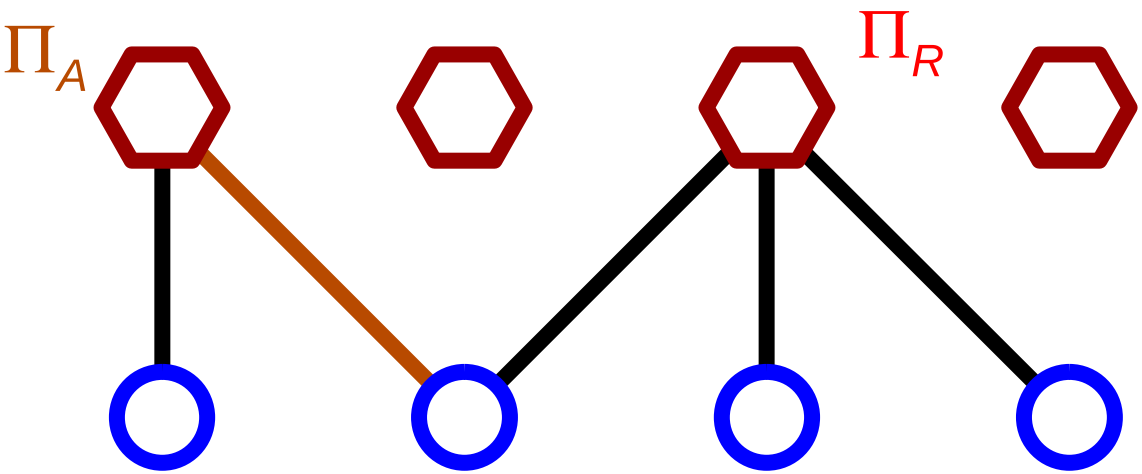
# The Model as Network Rewiring

- **Removal**: Choose an individual at random
  = choosing departure artifact with probability $\Pi_R = (k/E)$
  = preferential removal from artifacts
- **Attachment**: Choose an arrival artifact with probability

$$\Pi_A(k) = [(1-p_r)k + p_r\langle k\rangle]/E \qquad \langle k\rangle = (E/N)$$

copying probability — innovation probability

- **Rewire**: Only *after* these choices are made.




**N artifacts**

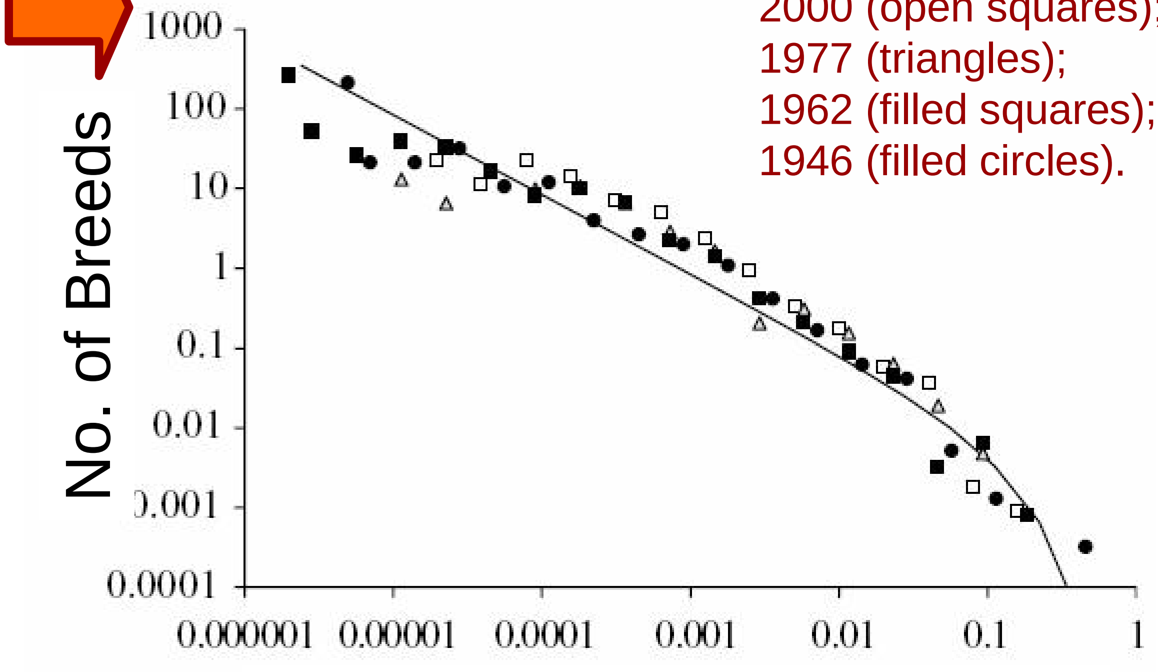
***n(k)*** artifact degree dist.

**E individuals**

# Evidence for this model

- Registrations of pedigree dogs
- Baby name registrations
- Changes in top 100 of popular music charts
- Applied to archaeological pot shards

[Herzog, **Bentley**, Hahn 2004]




2000 (open squares);
1977 (triangles);
1962 (filled squares);
1946 (filled circles).

Frequency of registrations of each breed of pedigree dog

See Neiman (1995); Bentley, Maschner (2000,2001); Bentley, Hahn, Shennan (2004); Bentley, Shennan (2003,2005); Hahn, Bentley (2003); Herzog, Bentley, Hahn (2004); Bentley, Lipo, Herzog, Hahn (2007).

# Relationship to Other Systems

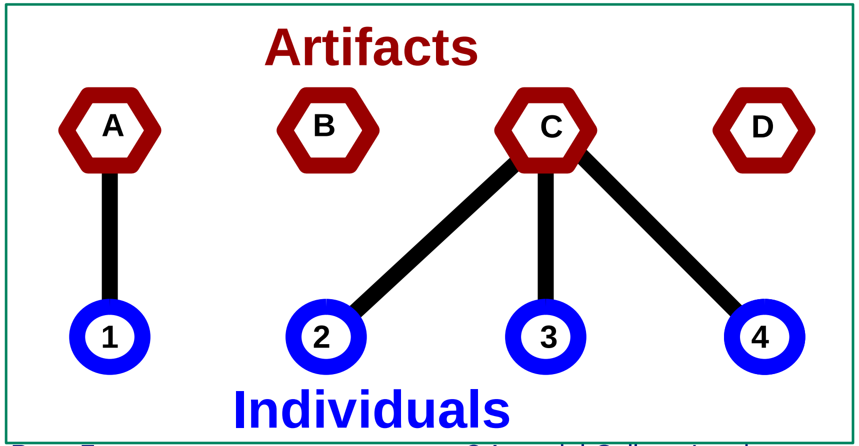
- Unipartite Graph Rewiring [Watts & Strogatz 1998]
- Gene Frequencies [Kimura & Crow, 1964]
  - Inheritence and Mutation
- Family Names [Zanette & Manrubia, 2001]
  - Inheritence and New Immigrants
- Language Extinction [Stauffer et al. 2006]
- Minority Game variant [Anghel et al, 2004]

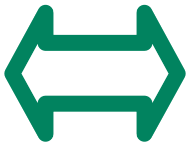
# Relationship to Statistical Physics Models

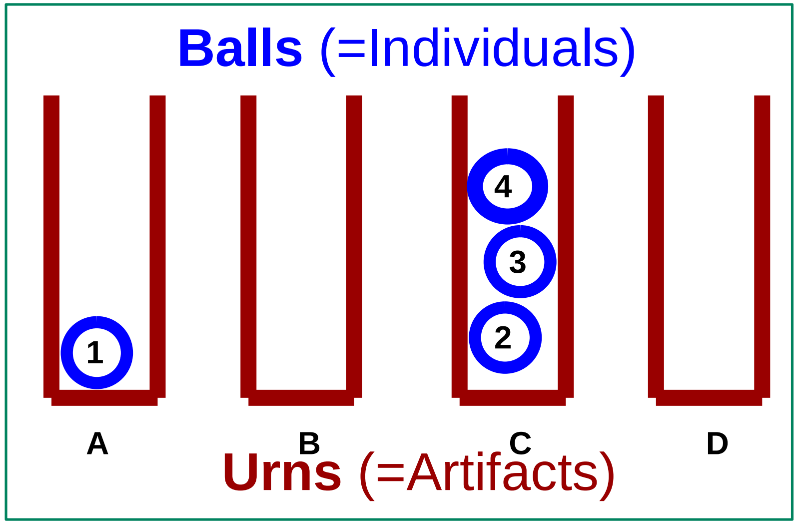
Some parameter values of other models are equivalent to our model:

- Urn Models [Bernoulli 1713, …, Ohkubo et al. 2005]
- Zero Range Processes (Misanthrope version) [review M.R.Evans & Hanney 2005; Pulkkinen & Merikoski 2005]
- Voter Models [Liggett 1999, …, Sood & Redner 2005]
- Backgammon/Balls-in-Boxes

applied to glasses [Ritort 1995], wealth distributions, simplicical gravity …





$$F_2(t) := \frac{1}{E(E+1)} \left. \frac{d^2 G(z,t)}{dz^2} \right|_{z=1}$$

- A Simple Model
- **Exact Solution**
  - Equilibrium
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$$G^{(m)}(z) = (1-z)^m F(a+m, b+m; c; z)$$

$$G(z,t) = \sum_{k=0}^{E} (z)^k \, n(k,t) = \sum_{m=0}^{E} c_m (\lambda_m)^t G^{(m)}(z)$$

# Mean Field Degree Distribution Master Equation

Mean field approximation very accurate for many models (low vertex correlations)

$$
\begin{aligned}
n(k,t+1) - n(k,t) = & + n(k+1,t)\Pi_R(k+1)[1-\Pi_A(k+1)] \\
& - n(k,t)\Pi_A(k)[1-\Pi_A(k)] \\
& - n(k,t)\Pi_R(k)[1-\Pi_R(k)] \\
& + n(k-1,t)\Pi_A(k-1)[1-\Pi_R(k-1)]
\end{aligned}
$$

**(1-Π) terms Invariably ignored**

**Number of edges attaching to a vertex of degree (k-1)**

**Probability of NOT reattaching to same vertex**

# Only Exactly Solvable Case

Attachment and removal probabilities, $\Pi_R$ and $\Pi_A$, must be time independent

$\Rightarrow$ normalisations can only use the constants of motion, $E$ and $\langle k \rangle$

(or $N=E/\langle k \rangle$)

– $\Pi_R(k) = (k / E)$ Choose random edge to be rewired

– $\Pi_A(k) = [(1-p_r)k + p_r \langle k \rangle ] / E$

Preferential Attachment
Copying
Inheritance

Random Attachment
Innovation
Mutation

# Exact Equilibrium Solution

$$n(k) = A \frac{\Gamma(k+\overline{K})}{\Gamma(k+1)} \frac{\Gamma(E-\overline{E}-\overline{K}-k)}{\Gamma(E+1-k)}$$

$$\overline{K} = \frac{p_r}{p_p}\langle k \rangle$$

$$\overline{E} = \frac{p_r}{p_p} E$$

*A* is ratio of four $\Gamma$ functions

- Simple ratios of $\Gamma$ functions
- Similar to those found for growing networks but second fraction is only found for network rewiring with correct master equation
- Only approximate solutions known previously

# Large Degree Equilibrium Behaviour – Large $p_r$ Case

For $p_r > p_* \sim 1/E$

(on average at least one edge attached to a randomly chosen artifact per generation)

$$\lim_{k \to \infty} \left[ n(k) \right] = k^{-\gamma} \exp(-\zeta k)$$

$$\gamma = 1 - \frac{p_r}{p_p} \langle k \rangle$$

**Power below one but in data *indistinguishable from one***

$$\zeta = -\ln(1 - p_r)$$

**Exponential Cutoff**

# Large Degree Equilibrium Behaviour – Small $p_r$ Case

For $p_r < p_* \sim 1/E$

(on average if all edges have been rewired once no edge is attached to a randomly chosen artifact per generation)

Degree distribution rises near $k=E$

⇒In extreme case $p_r=0$ all the edges are attached to ONE artifact

- a CONDENSATION or FIXATION

Blows up

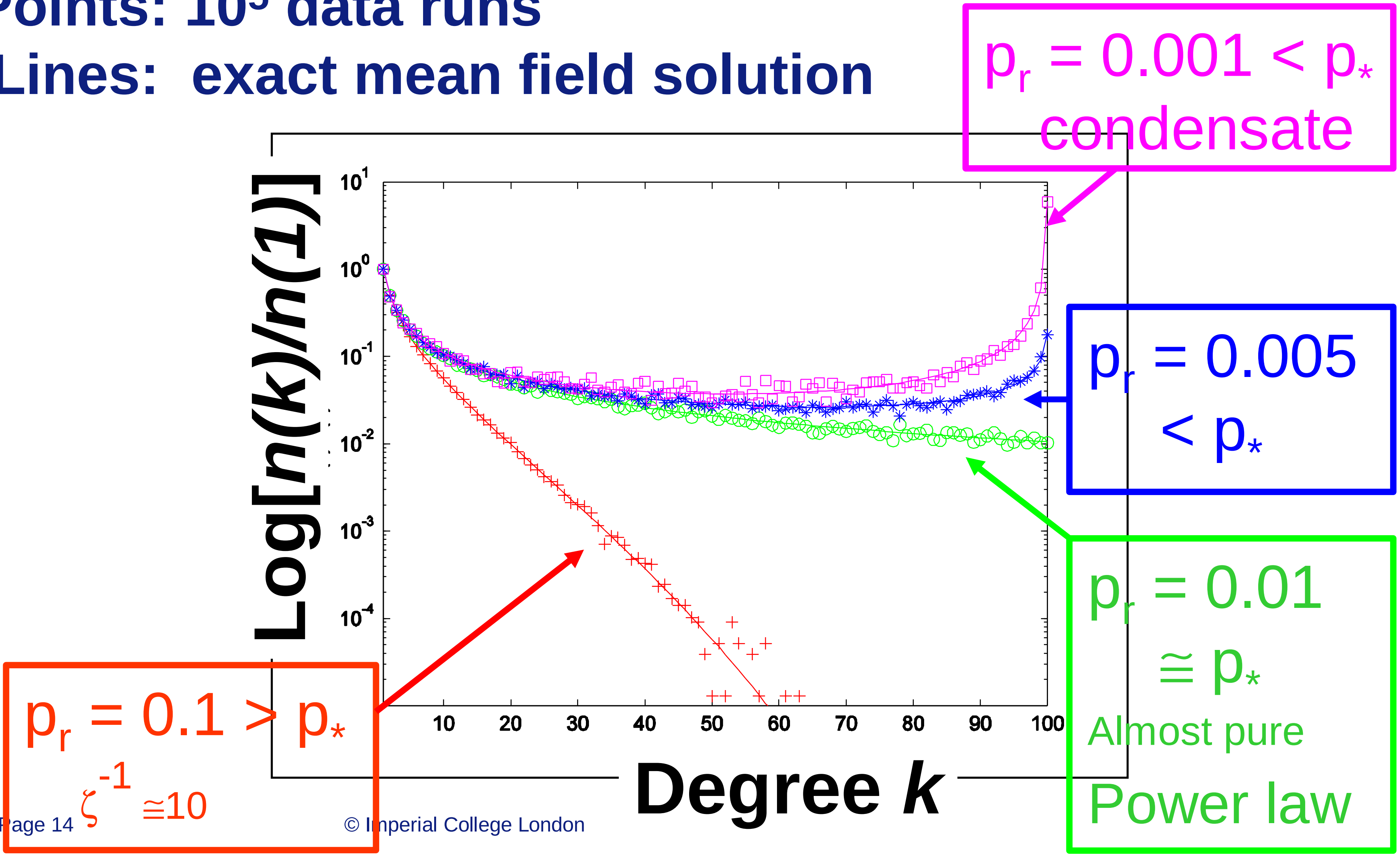
$$n(k) = A\left(\frac{\Gamma(k+\overline{K})}{\Gamma(k+1)}\right)\left[\frac{\Gamma(E-\overline{E}-\overline{K}-k)}{\Gamma(E+1-k)}\right]$$

# Equilibrium Behaviour Results

**N=E=100**

**Points: $10^5$ data runs**

**Lines: exact mean field solution**



## Exact Solution

$$\lambda_m = 1 - m(m-1)\frac{p_p}{E^2} - m\frac{p_r}{E}$$

($m$=0,1,2,..$E$)

- Use generating function.
  - It splits into $(E+1)$ eigenfunctions, given by Hypergeometric functions
  - Simple eigenvalues
- $\langle k^n \rangle$ $n$-th moment of degree distribution gets contributions only from eigenfunctions $m \leq n$ only
- $m=0$ eigenfunction number zero constant ($\lambda_0=1$)

  $\Rightarrow$ equilibrium solution
- $m=1$ eigenfunction *never contributes*
- Slowest time dependence comes from $m=2$ eigenfunction setting time scale $\tau_2 = -1/\ln(\lambda_2)$

# Homogeneity Measures $F_n$

## BEST WAY TO STUDY DEGREE DISTRIBUTION

- ***$F_n(t)$* = probability of choosing *n* different individuals connected to the same artifact**
  - ***$F_n = 0$*  if no artifact chosen more than once**
  - ***$F_n = 1$*  if all individuals attached to same artifact**
- Related to *m*-th moments ($m \leq n$) of degree distribution via Stirling numbers but ***$F_n = 0$ if $n>E$***
- *n*-th derivatives of generating function gives homogeneity measure ***$F_n$***

$$F_n(t) := \frac{\Gamma(E+1-n)}{\Gamma(E+1)} \frac{d^n\, G(z,t)}{dz^n}\bigg|_{z=1} = \sum_{k=0}^{E} \frac{k}{E} \frac{(k-1)}{(E-1)} \Lambda \frac{(k-n+1)}{(E-n+1)} n(k)$$

# Exact Solution for $F_2$ Homogeneity Measure

$F_2$ = probability that two different individuals have chosen the same artifact

$$F_2(t) = F_2(\infty) + (\lambda_2)^t \left(F_2(0) - F_2(\infty)\right)$$

$$F_2(\infty) = \frac{1 + p_r(\langle k \rangle - 1)}{1 + p_r(E - 1)}$$

3rd eigenfunction controls all time dependence

$\tau_2 = -1/\ln(\lambda_2)$

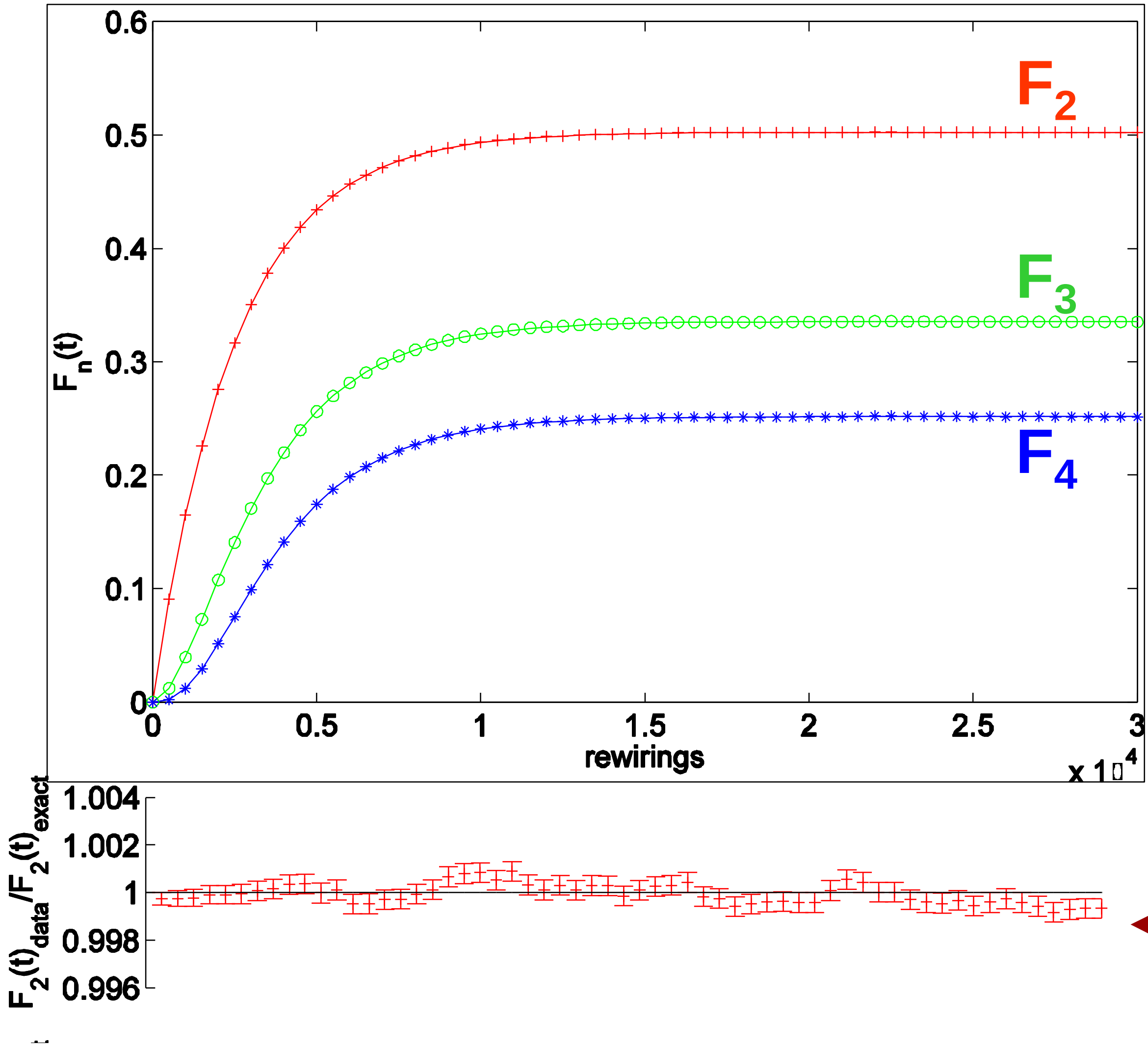
$\approx [2(p_r/E) + 2(1-p_r)/E^2]^{-1}$

Initial values fix $F_2(0)$

e.g. $F_2(0)=0$ if each individual starts attached to unique individual

# $F_n$ numerical results

$E=N=100$, $p_r=0.01 \cong p_*$,
Points: average of $10^5$ simulations
Lines: exact mean field prediction
Start: $n(k)=\delta_{k,1}$

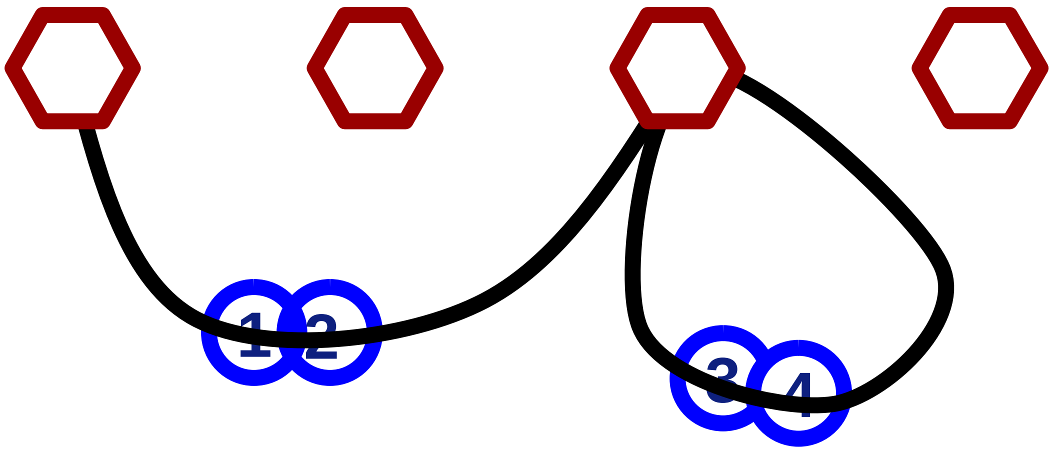



**F increases as homogeneity increases with time**

**Time dependence of averages predicted very accurately, deviations less than 1%**

# Phase transitions in real time

- Bipartite graph can be projected onto a unipartite graph of the artifact vertices
- Artifact degree distribution p(k) is the degree distribution for a random graph




**N artifacts**

**(E/2) edges**

A Molloy-Reed [1995] projection

# Graph Transition in Real Time

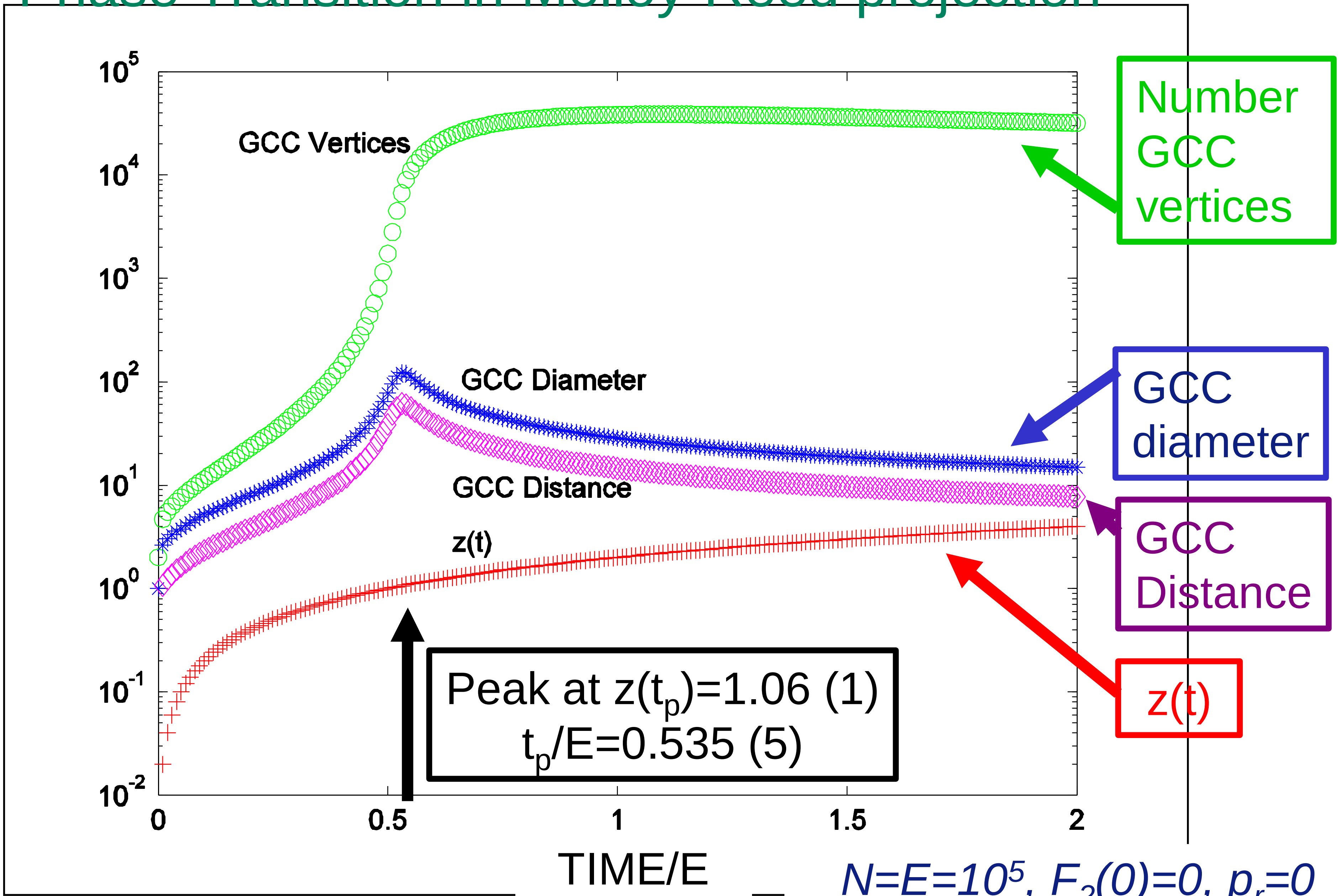
Infinite Random Graphs (given *p(k)* but otherwise completely randomised) have a phase transition (e.g. appearance of **GCC** - Giant Connected Component) at [Fronczak et al 2005, etc]

$$\boldsymbol{z(t)=1}$$

where

$$z(t) = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1 = (E-1)F_2(t)$$

# Phase Transition in Molloy-Reed projection




$N=E=10^5$, $F_2(0)=0$, $p_r=0$

# Phase Transition in Molloy-Reed projection

For $N=E=10^5$, $p_r=0$, initial $F_2(0)=0$

- $z(t)=1$ at $t=0.50000$ (2) as predicted
- Transition at $t/E = 0.535$ (5)
- At transition $z(t)=1.06$ (1) not $z(t)=1$
- Average distance and diameter of GCC maximum at this point and second derivative of number of vertices in GCC zero at this point (within errors)

⇒ **Finite size effects clearly present**

⇒ **Can follow a system through a phase transition in time *exactly***

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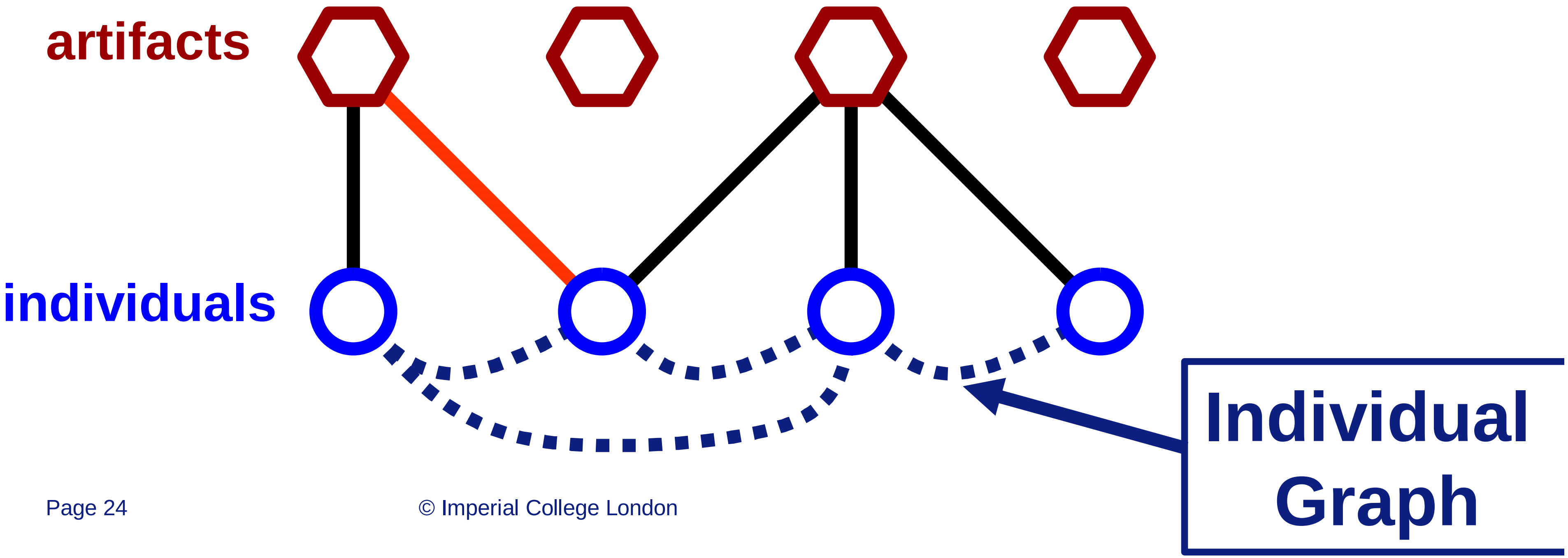
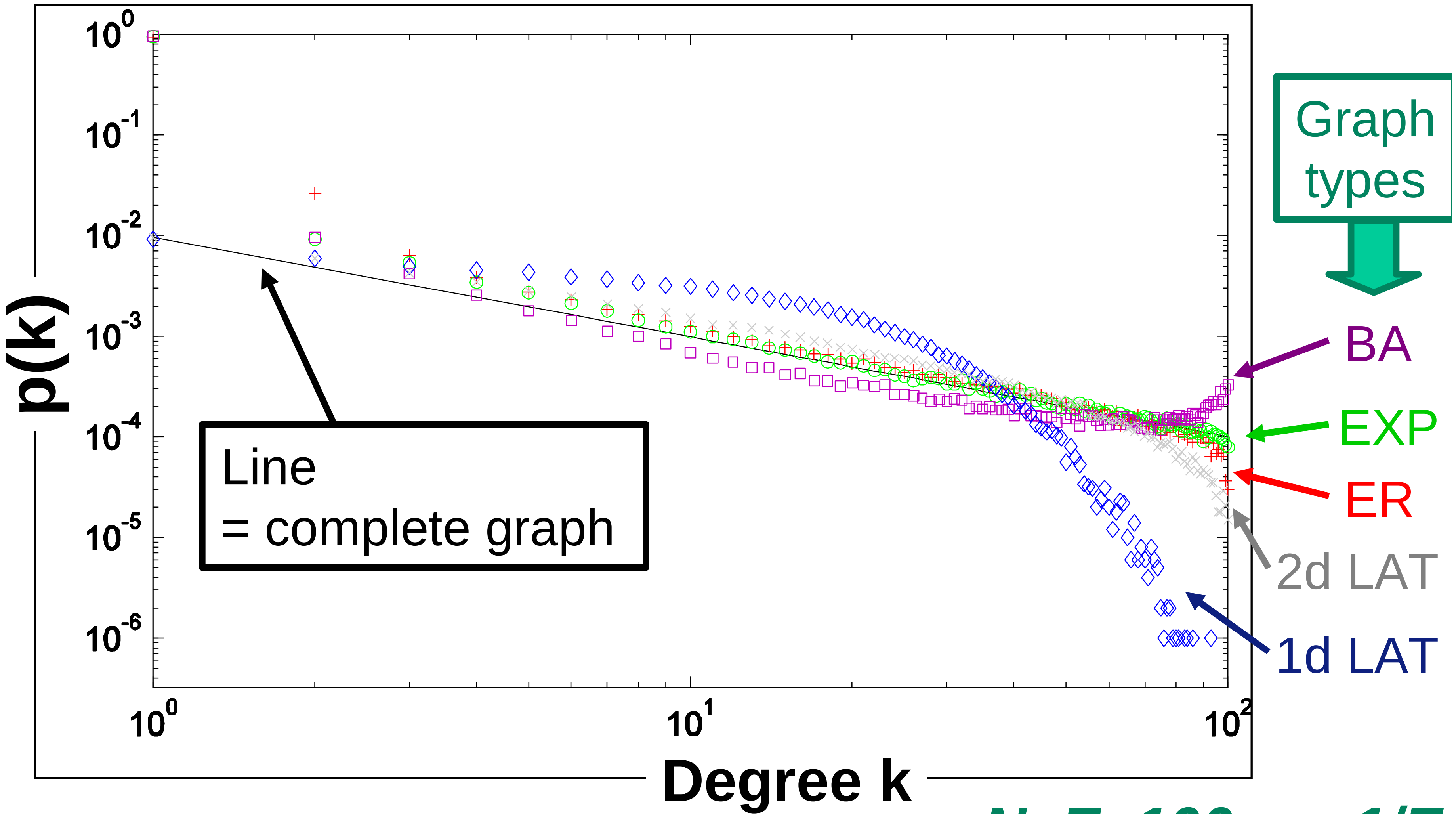
## Adding a Network of Individuals

- **Removal**: Choose random individual as before
- **Attachment**:
  With probability $(1-p_r)$ the individual **copies** the existing choice of any **neighbouring** individual.
  With probability $(p_r)$ the individual **innovates**



# Equilibrium with a Network of Individuals

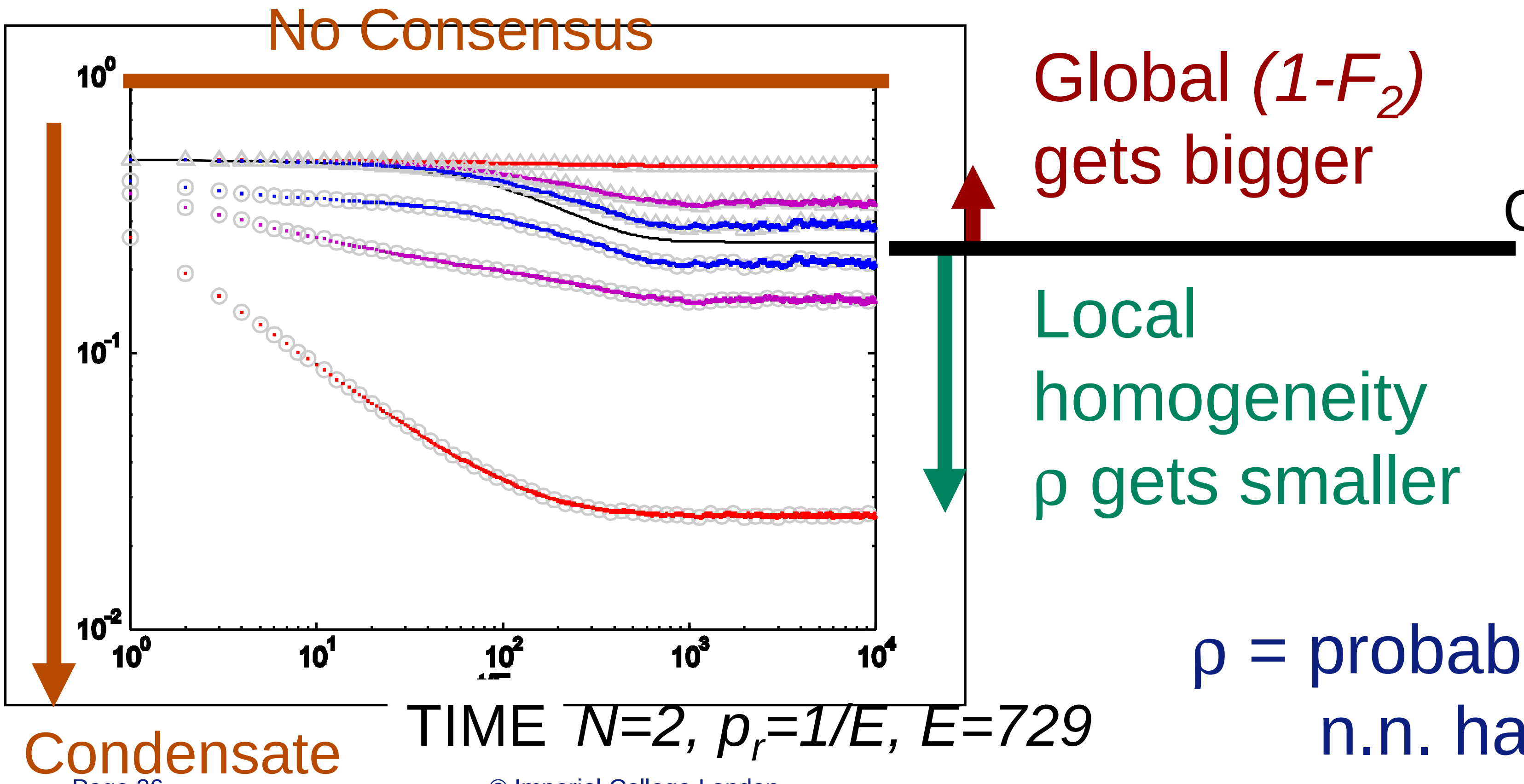
## Qualitative behaviour largely unchanged except for 1d Lattice




***N=E=100, $p_r$=1/E***

# Approach to Equilibrium for different Individual networks

- Results move away from complete graph as move from 3d -> 1d lattice




ρ = probability that n.n. has made different choice

## Voter Model [Liggett 1999; Sood & Redner 2005]

- At each time step an individual is chosen randomly who copies the choice of a neighbour in an individual network
- Equivalent to $\boldsymbol{N=2, p_r=0}$ limit here
- Study time scales to come to complete **consensus** = condensation
- Used for models of language [Stauffer et al. 2006]

⇒We find approach to complete consensus is slow but a little randomness ($\boldsymbol{p_r > 0}$) can speed this up while leaving a fairly complete condensation

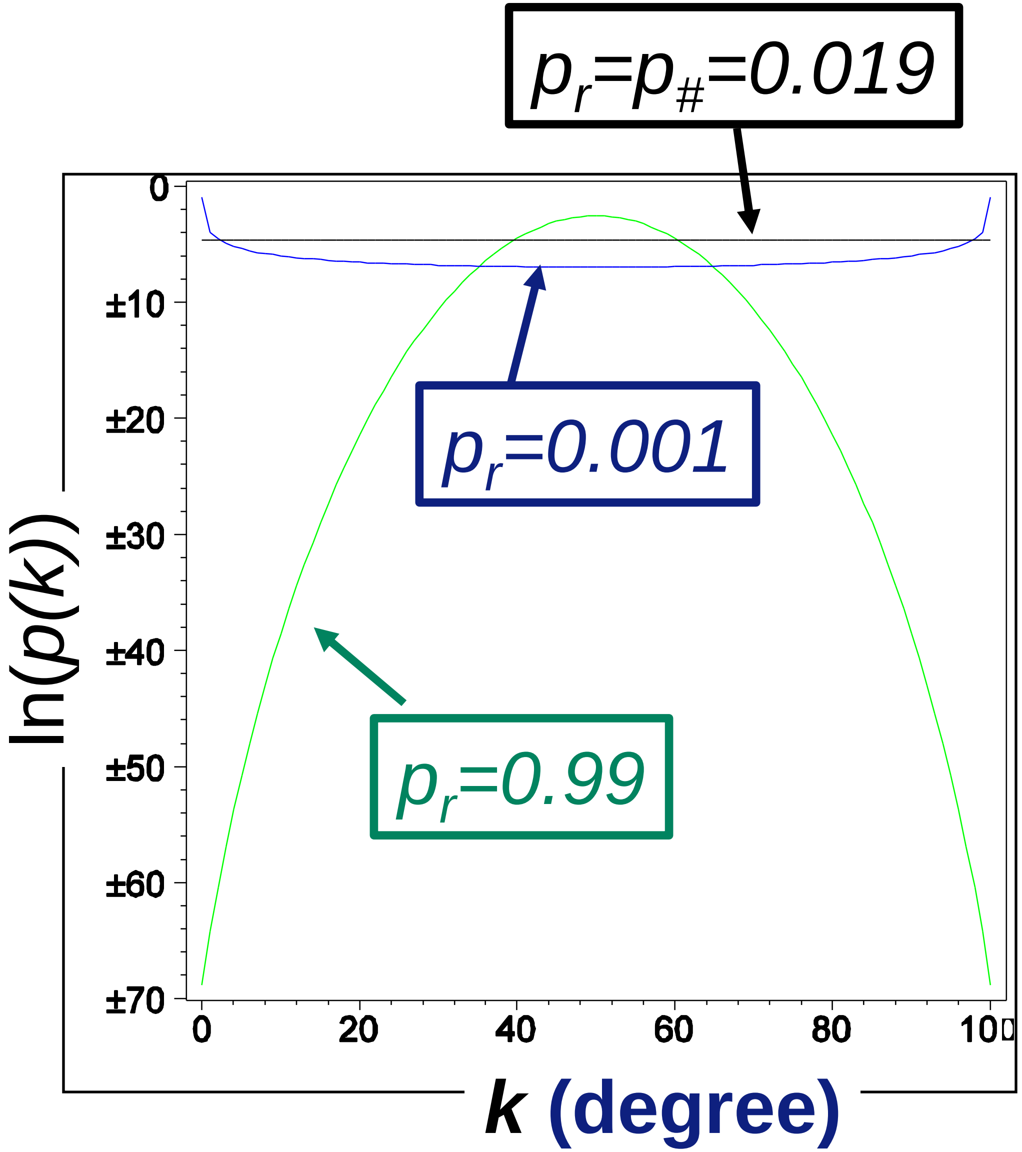
# Phase Transition in the Generalised Voter Model

Here on a complete or random individual network

- $N$=2 so large <$k$> is a special case
- Transition occurs at

$$p_\# = (E+1+\langle k\rangle)^{-1} = (1+(E/2))^{-1}$$

- May be viewed as $Z_2$ symmetry breaking transition



# Minority Game Example - Leaders and Followers

- At each step each individual chooses one or zero
  - the *minority* choice wins
- Choices are made based on one of a large but finite number of strategies using finite history
  - each strategy is a different artifact
- Individuals may follow their own prediction or they may follow the prediction from the most successful nearest neighbour in an ER random graph of individuals
  - i.e. they ***copy*** the strategy of a neighbour

[Anghel et al. PRL 92 (2004) 058701]

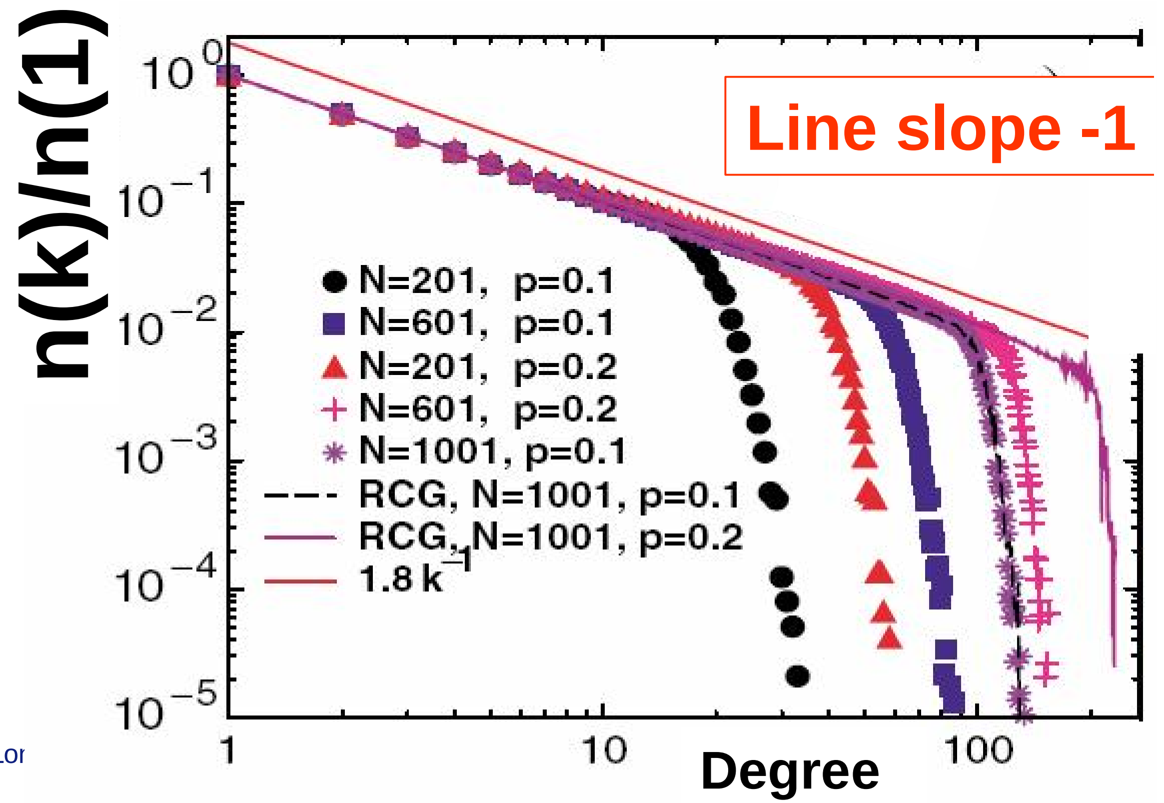
# Minority Game Example – Leaders and Followers [Anghel et al. 2004]

Plot *n(k)* the average of the number of strategies (of some leader) used by *k* individuals (followers).

Various system sizes and various ER random graphs.

**Result exactly as in our model**

**⇒ Random Copying**



# Minority Game Example - Leaders and Followers

This Minority Game variant again shows how **copying** can arise naturally

c.f. preferential attachment in growing networks

[Saramäki & Kaski 2004, TSE & Saramäki 2005]

# Different Update Methods

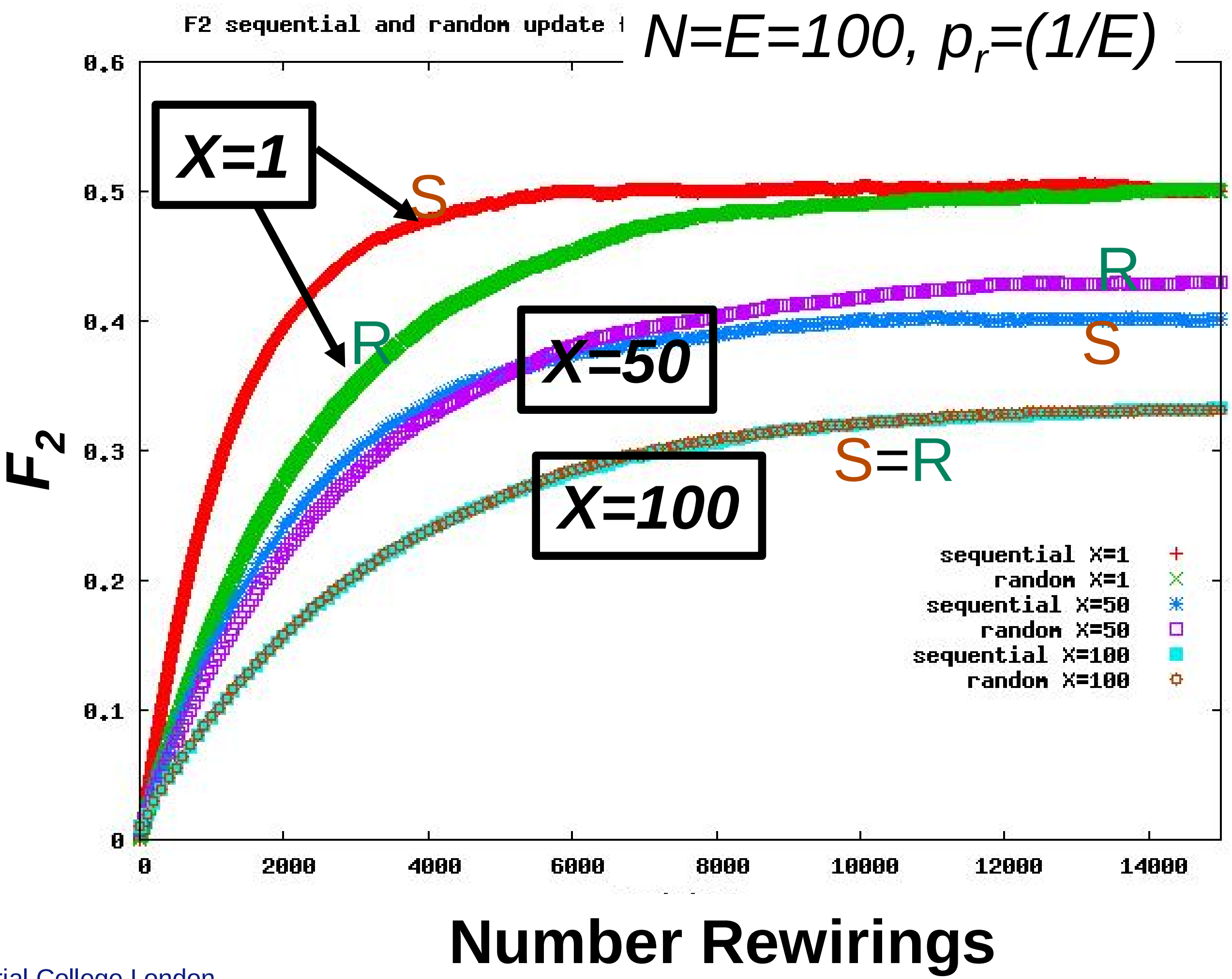
- First select *X* different individuals
  either (R) selected randomly
  or (S) in numerical sequence (1,2,3,…)
- They make their new artifact choices at the same time
- Only now we update the network and repeat

⇒ *X=1* & *random selection* = model so far

⇒ *X=E* & *sequential selection* = models of Bentley et al.

# Numerical Results for Update Variations

- *X=1:* sequential faster than random but equilibrium same
- *X=E/2*: time scales similar but equilibrium $F_2$ lower for sequential
- *X=100:* update all at once and get $F_2$ =1/3 not ½ as we get for *X=1*




*[TSE + You (in progress)]*

# Analytic Results for Update Variations

$$F_2(t) = F_2(0) + (\lambda_2)^t \left( F_2(\infty) - F_2(0) \right)$$

| (R) selected randomly | (S) Sequential Update |
|---|---|
| X=1 | X=100 |
| TSE & Plato | Bentley et al. |

$$F_2(\infty) = \frac{1 + p_r \left( \langle k \rangle - 1 \right)}{1 + p_r \left( E - 1 \right)}$$

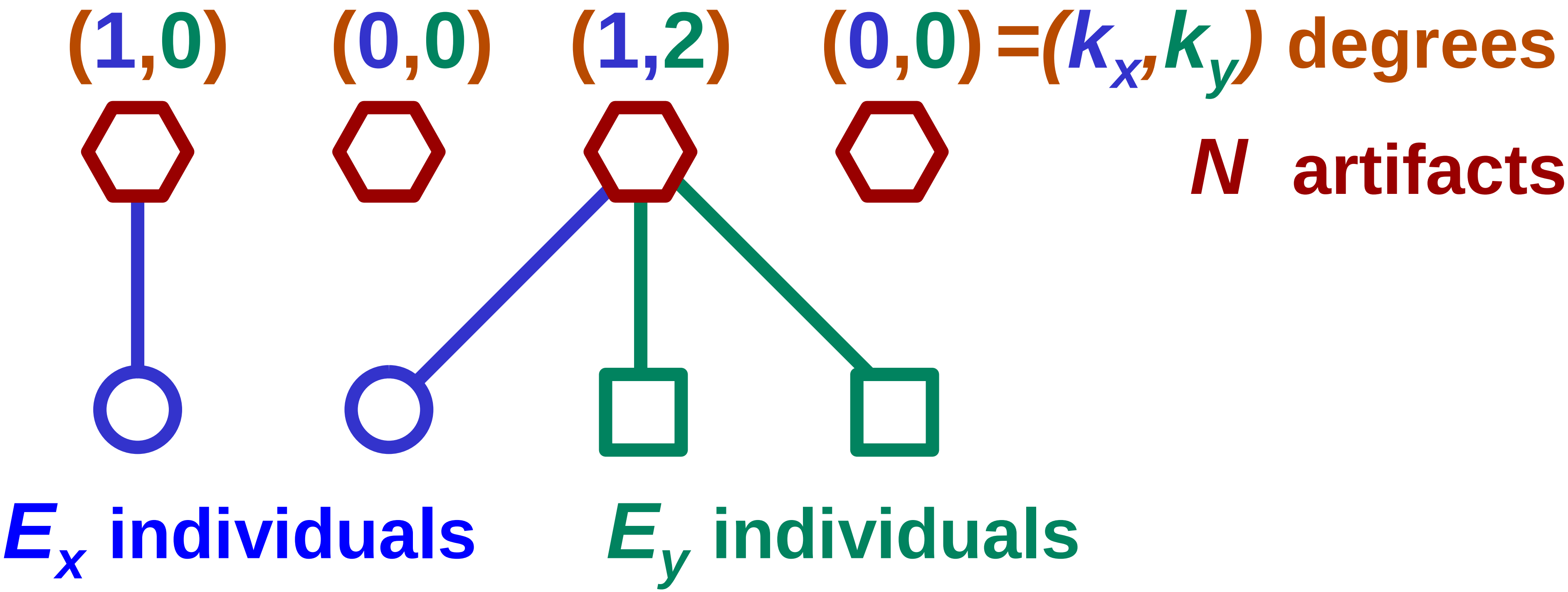
$$F_2(\infty) = \frac{(p_p)^2 + (1-(p_p))^2 \langle k \rangle}{(p_p)^2 + (1-(p_p))^2 E}$$

$$\lambda_2 = 1 - \frac{2 p_r}{E} - \frac{2(1 - p_r)}{E^2}$$

$$\lambda_2 = (1 - p_r)^2 \left(1 - \tfrac{1}{E}\right)$$

# Two Tribes

Change model so there are two types of individual, each type chooses new artifacts with their own probabilities for:- (A) copying from same type, (B) copying from different type, (C) innovation



# Two Tribes

- Exact solutions for inhomogeneity measures $F_{2ab}(t)$ $[a,b \in \{X,Y\}]$ still possible
  - solutions of three-dimensional matrix

- 8 free parameters
  - difficult to draw general conclusions

- Might relate to *Freakonomics* type explanation for baby names in terms of different socioeconomic groups

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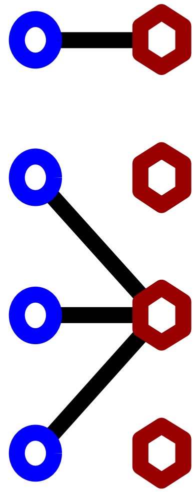
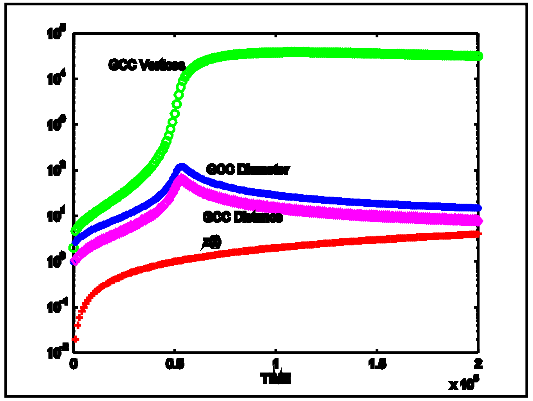
## Summary

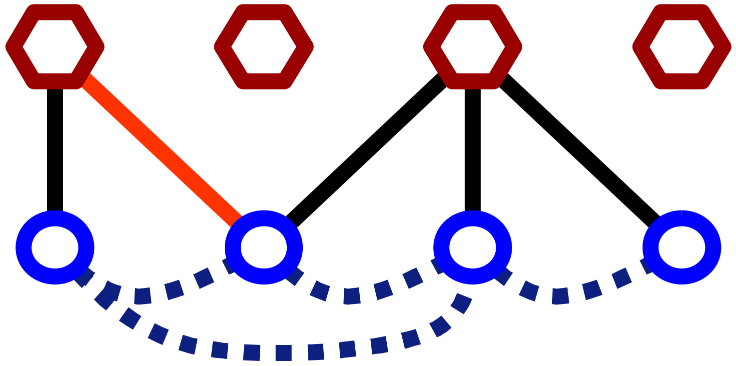
- Made connections between rewiring of bipartite network and many other network, statistical physics and social science models.
  Some connections made in some existing papers.



- Exact mean field equation.
  Only now is behaviour at boundary k=E correct.
- Exact equilibrium solutions.
  Previous results for large degree k, large systems N,E.
- Exact solutions for all times in terms of standard functions – *phase transitions in time*
  I know of no other network solutions for arbitrary time and arbitrary size.



## Summary



# Many variations of model

- Individual Networks

  Only 1d lattice seems to make a big difference to equilibrium

- Generalisation of Voter models

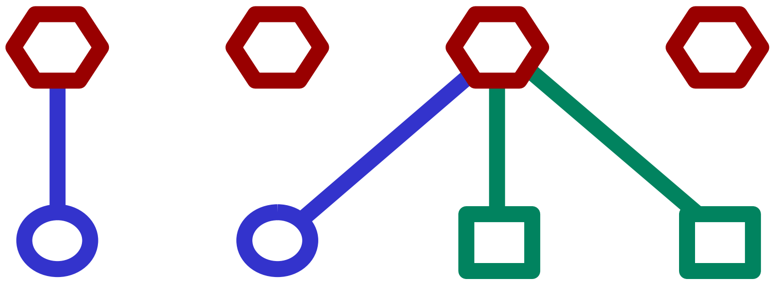
  $p_r$ can speed process up without significantly upsetting consensus

- Two Tribes

  exact solutions for some aspects possible with two types of individual

- Update Variations

  Some exact results possible



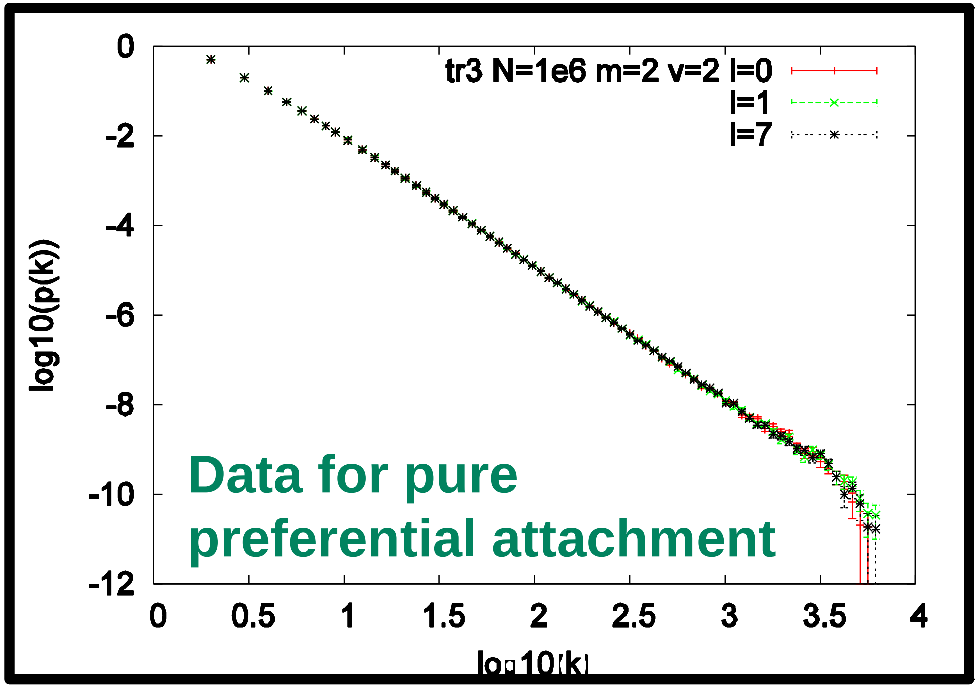
# Finite Size Effects for pure preferential attachment

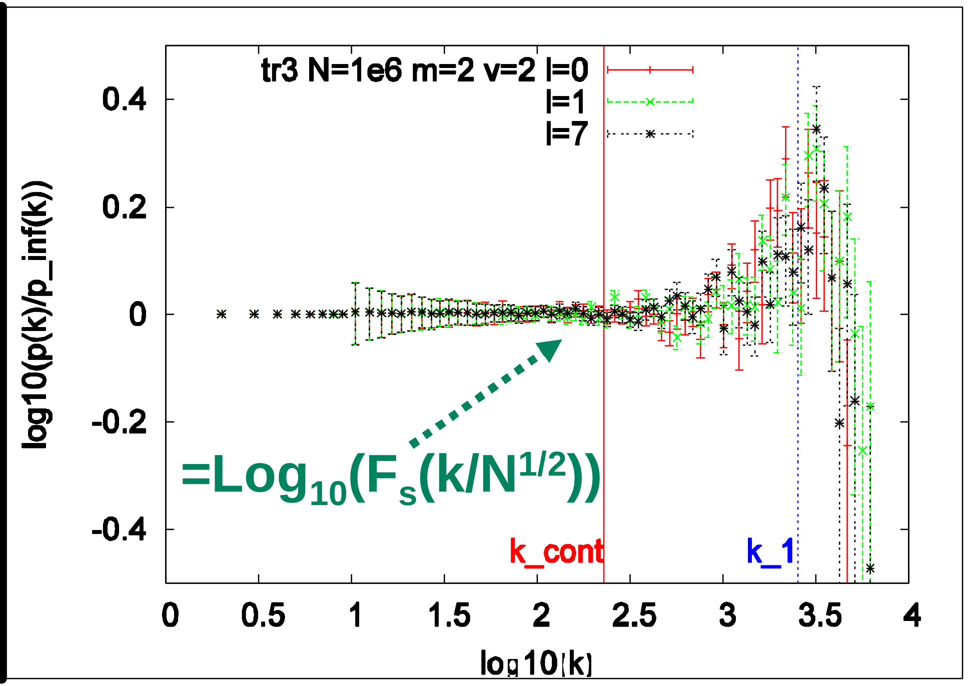
$$p(k) = p_\infty(k).F_S\left(\frac{k}{N^{1/2}}\right), \qquad p_\infty(k) = \frac{\langle k\rangle(\langle k\rangle+2)}{2k(k+1)(k+2)}$$

$$\to \frac{1}{k^3}$$

## Scaling Function $F_s$

$$F_S(x) \approx 1 \quad if \quad x < 1$$







**100 runs to get enough data near $k_1$**

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# Generalisations of Model

- Add a graph to the individual vertices
  -choose who to copy using individual's network

- Add a graph to the artifact vertices
  -mutations/innovations limited by metric in an artifact space

- Different types of individual
  -update their choice and copy/innovate at different rates

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