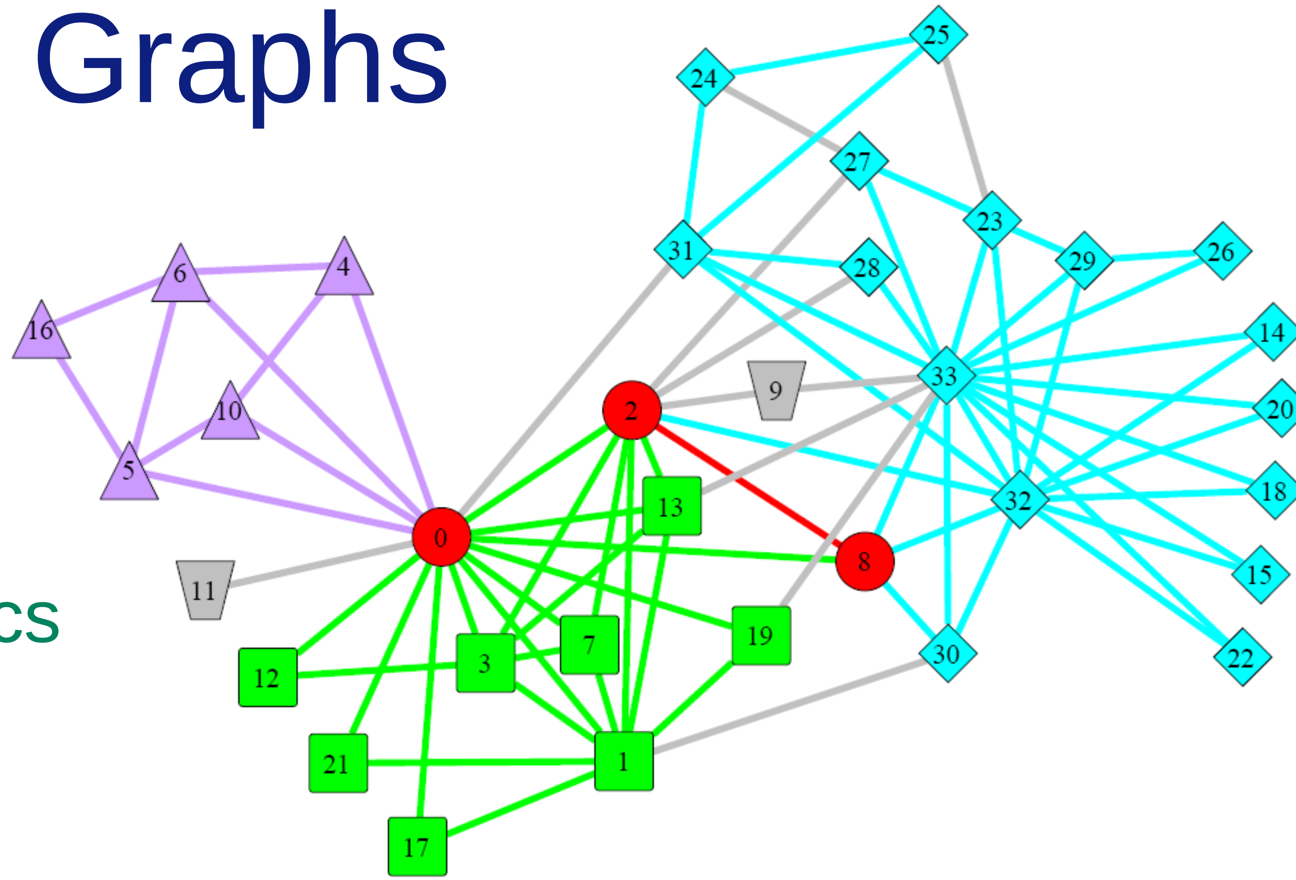
Imperial College London

Clique Graphs



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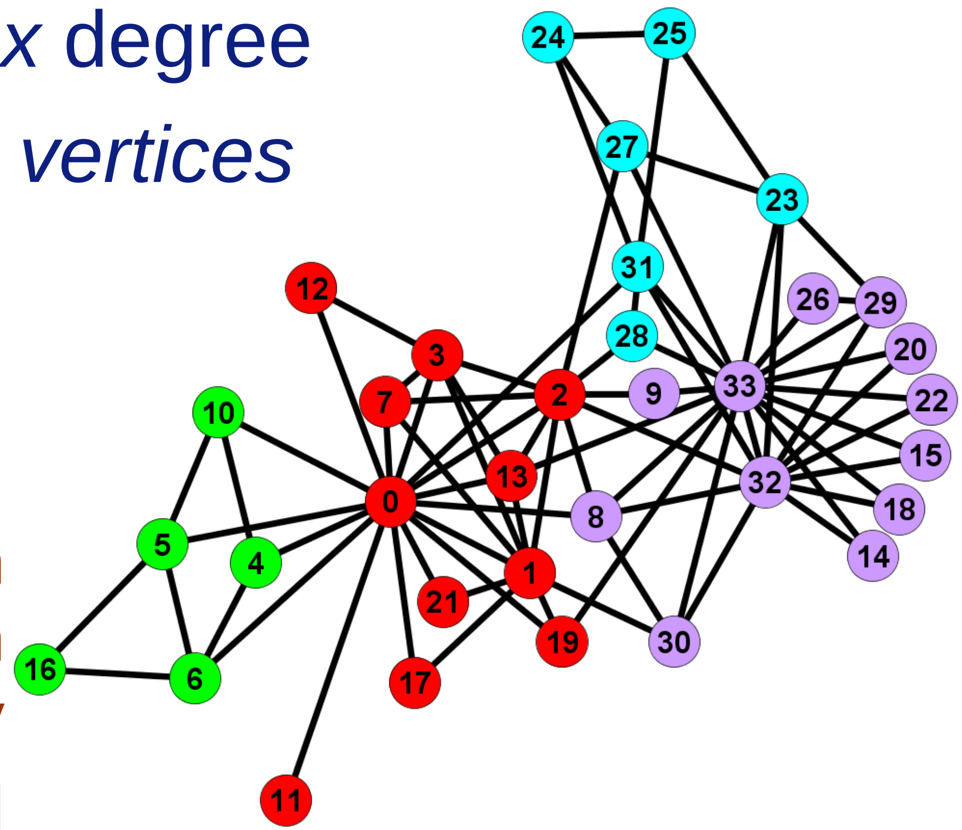
Vertex Centric Viewpoint

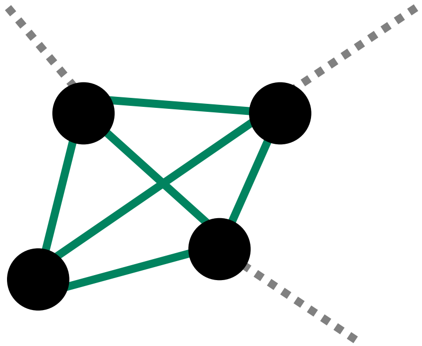
The focus in the literature is often on the vertices:-

- Distributions of *vertex* degree
- Cluster coefficient of *vertices*
- *Vertex* partitions as communities
- …..



Vertex partition of Karate club graph with optimal modularity [Agarwal & Kempe 2007]

Cliques – complete subgraphs



- Name originates from representation of cliques of people in social science
  [Luce and Perry ‘49]
- Theory of Triadic Closure
- Maximal Clique and Clique Cover problems
  [Bron-Kerbosch algorithm, ‘73]
- Community detection
  [Palla et al, 05; Yan & Gregory ‘10]
- Structure Analysis
  [Samudrala & Moult, 98; Takemote et al. 07]

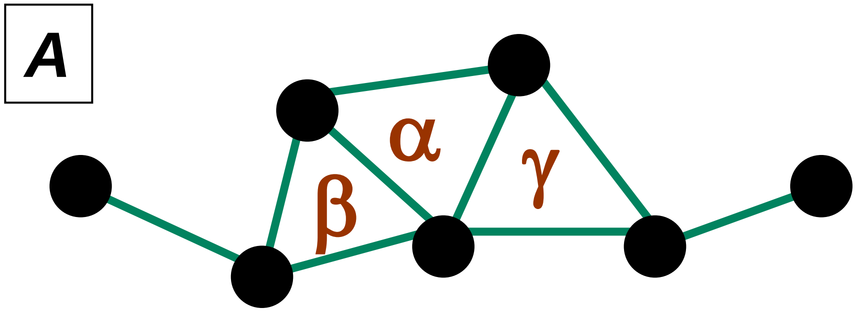
Give Cliques a Chance – a clique centric viewpoint

Can we shift out view point from vertices to cliques?

Use a **Clique Graph** to represent the way that ***n***-cliques overlap.

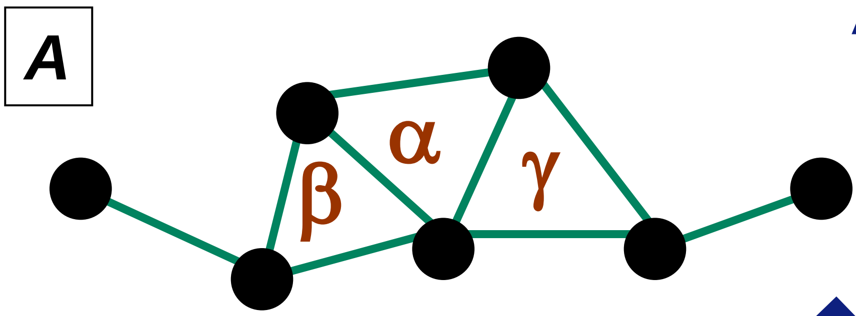
Example

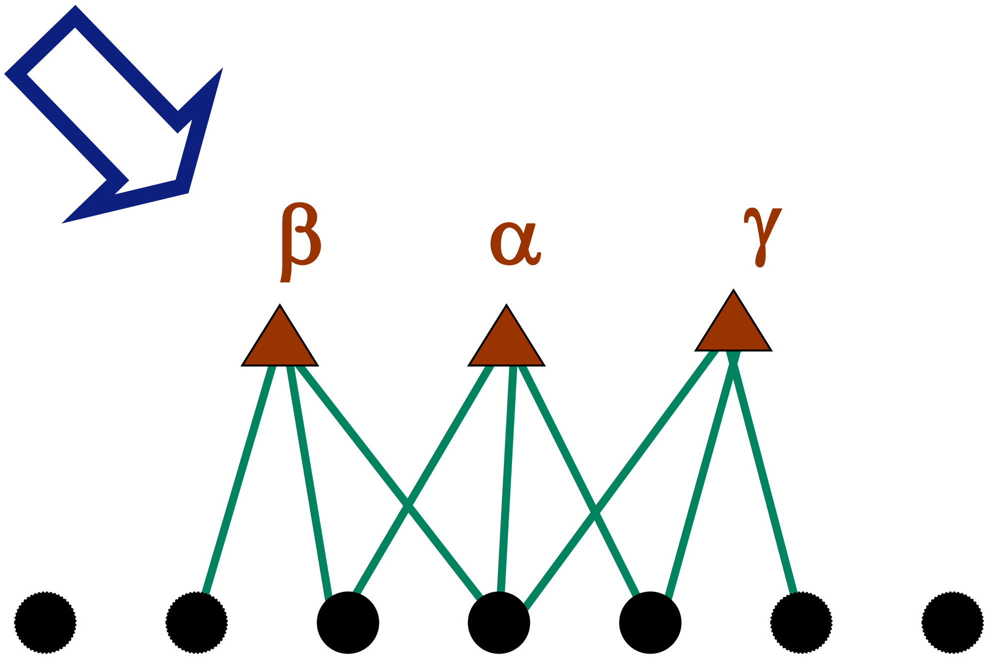
Graph ***A*** has three 3-cliques = triangles



Clique-Vertex Bipartite Graph Construction

Adjacency matrix $B_{i\alpha} = 1$
if vertex i is in clique α





3-Cliques →

Vertices →

Definition of the Basic Clique Graph ***C***

Project bipartite graph onto clique vertices.

⇒Weight of an edge in the basic Clique Graph ***C*** records the number of vertices common to two ***n***-cliques in in the original graph ***A*.**

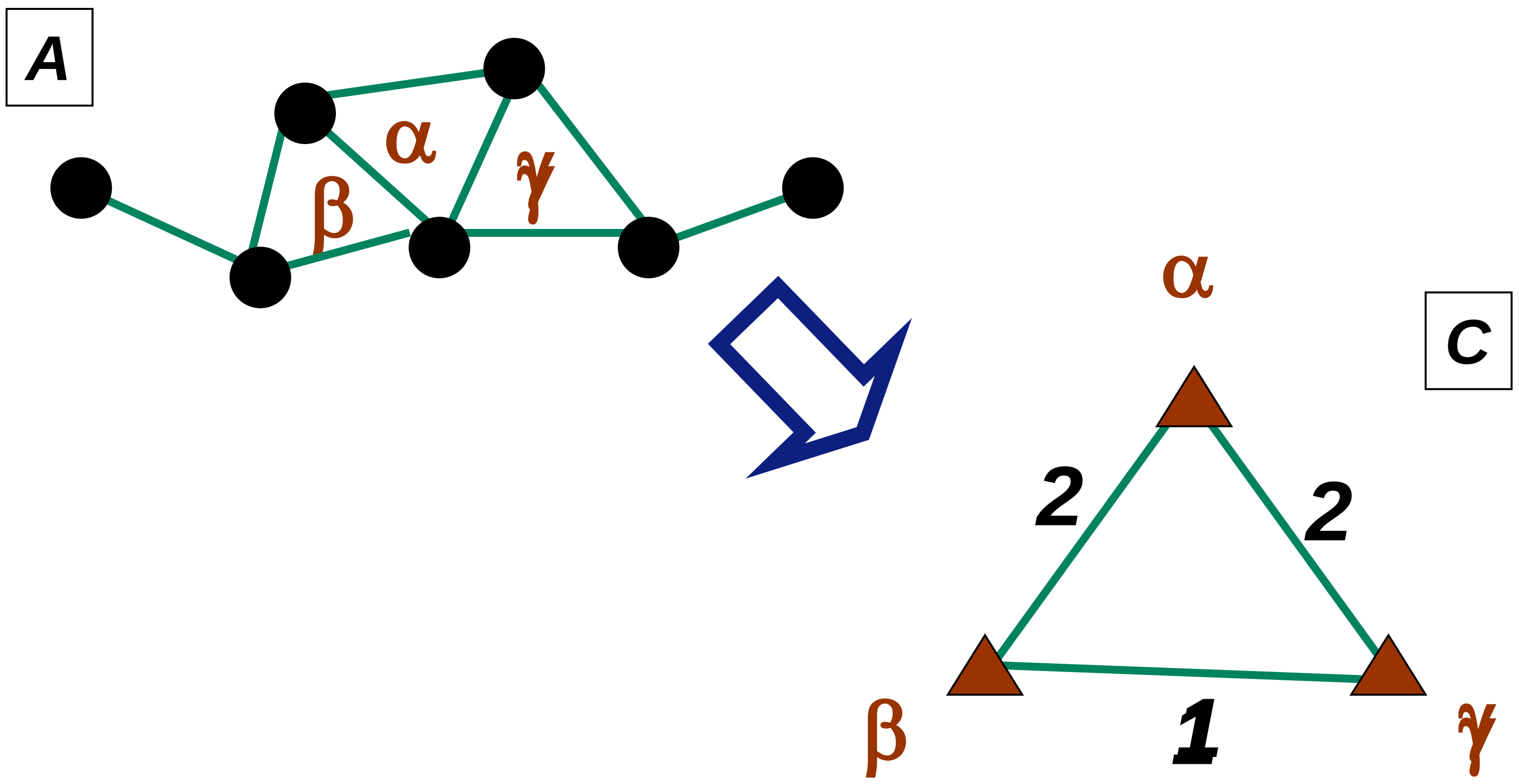
$$C_{\alpha\beta} = \sum_i B_{i\alpha} B_{i\beta} \left(1 - \delta_{\alpha\beta}\right)$$

i = vertex ***A***

α,β* = *n-cliques in ***A***

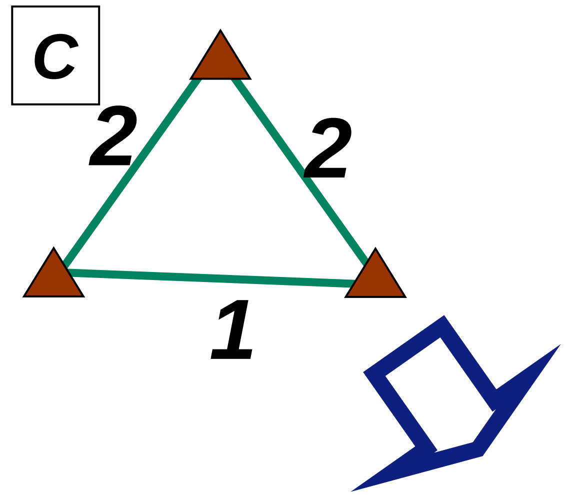
Edge Weights in Basic Clique Graph

Weight of an edge in the basic Clique Graph ***C*** records the number of vertices common to two ***n***-cliques in in the original graph ***A*****.**



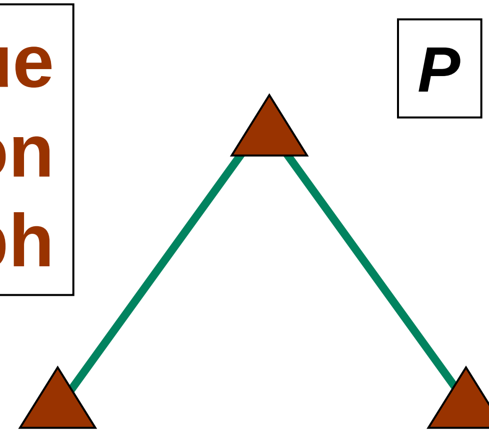
Clique Percolation Graph Construction

Threshold edges of clique Graph with *(n-1)*



i.e. only retain cliques which in original graph ***A*** share all but one vertex with another clique

Clique Percolation Graph



Connected Components = Communities [Palla et al, '05]

$$P_{\alpha\beta} = \Theta\left(C_{\alpha\beta} - n + 1\right)$$

Basic Clique Graph Weights – A problem

- The weight of basic Clique Graph ***C*** records the number of vertices common to two ***n***-cliques in in the original graph ***A***.

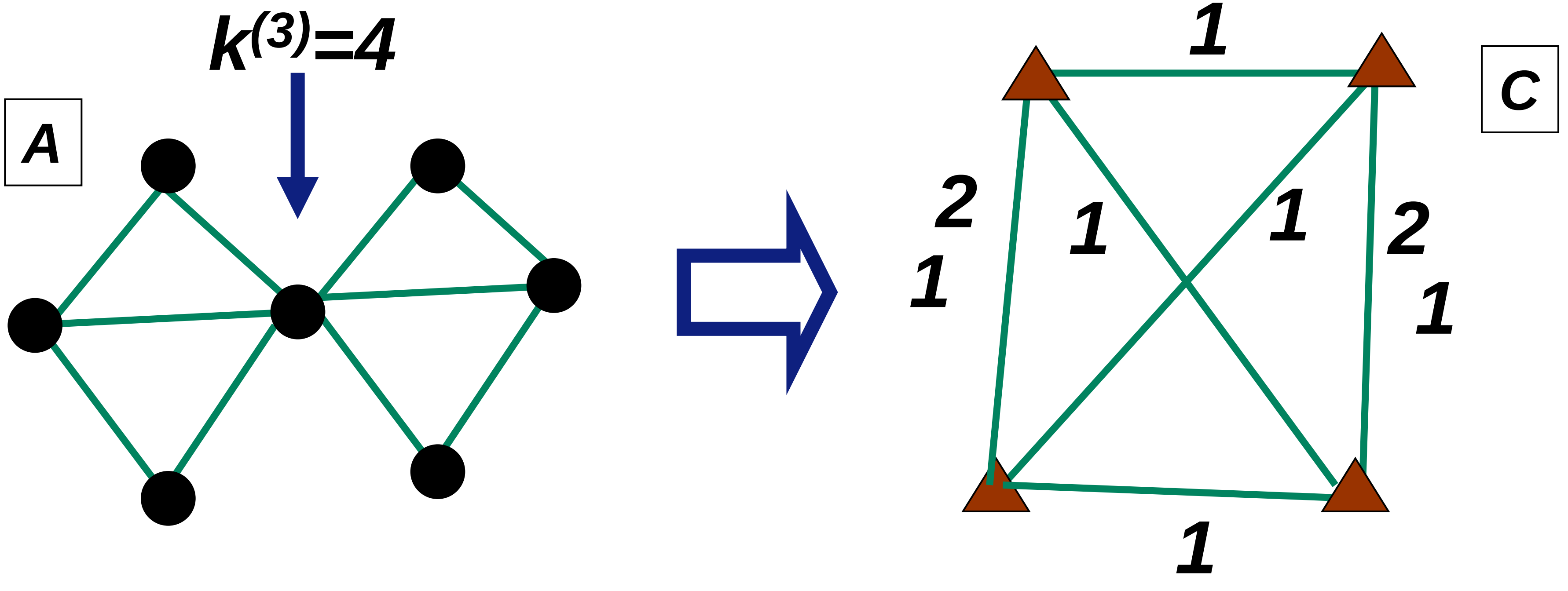
- **Thus a vertex which is in $k^{(n)}$ cliques will contribute a total weight of**

$$k^{(n)}\,(k^{(n)}-1)/2$$

to the basic clique graph *C*.

Basic Clique Graph Weights – A problem

- **A vertex in a large number of cliques contributes too much weight to the clique graph *C*.**



Better Clique Graph Weight Construction

A vertex common to a pair of cliques contributes a weight equal to

$$1/(k^{(n)} - 1)$$

to an edge in the new weighted Clique Graph *D*.

$$D_{\alpha\beta} = \sum_i \frac{B_{i\alpha} B_{i\beta}}{k^{(n)} - 1} \left(1 - \delta_{\alpha\beta}\right)$$

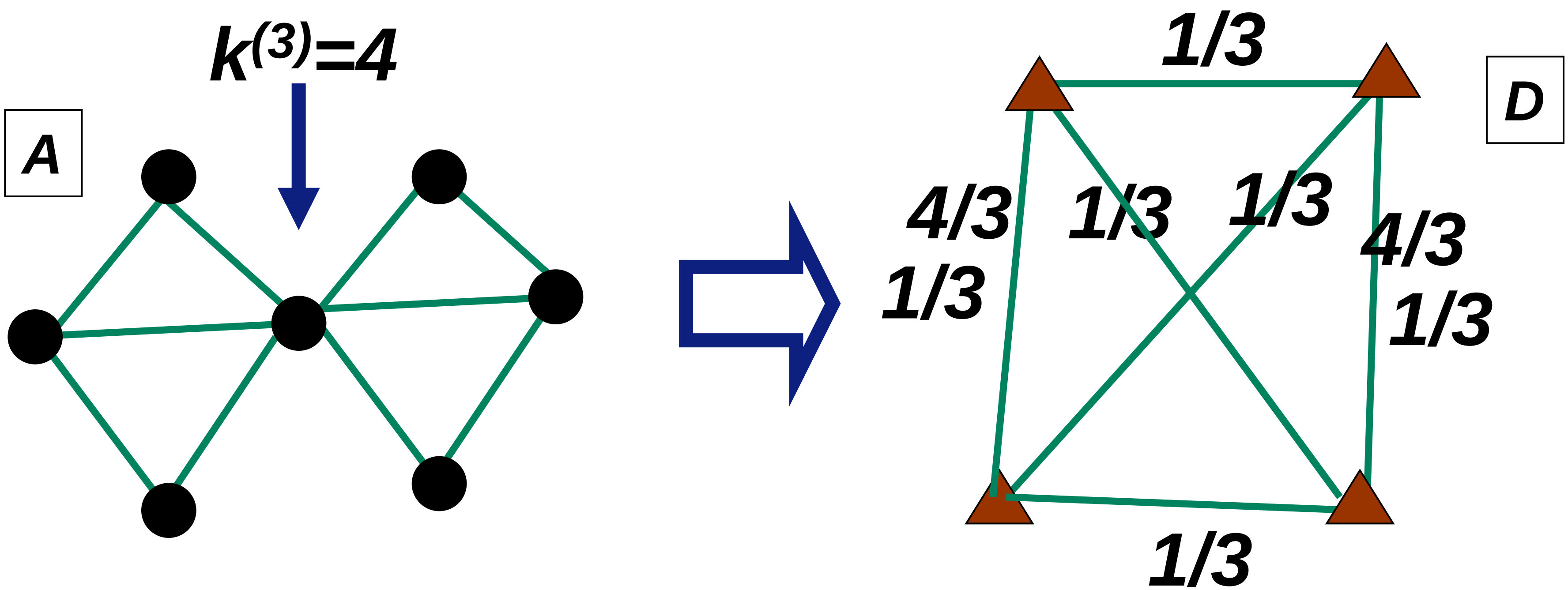
(vertex is in $k^{(n)}$ n-cliques)

Better Clique Graph Weight Construction

A vertex common to a pair of cliques contributes a weight equal to

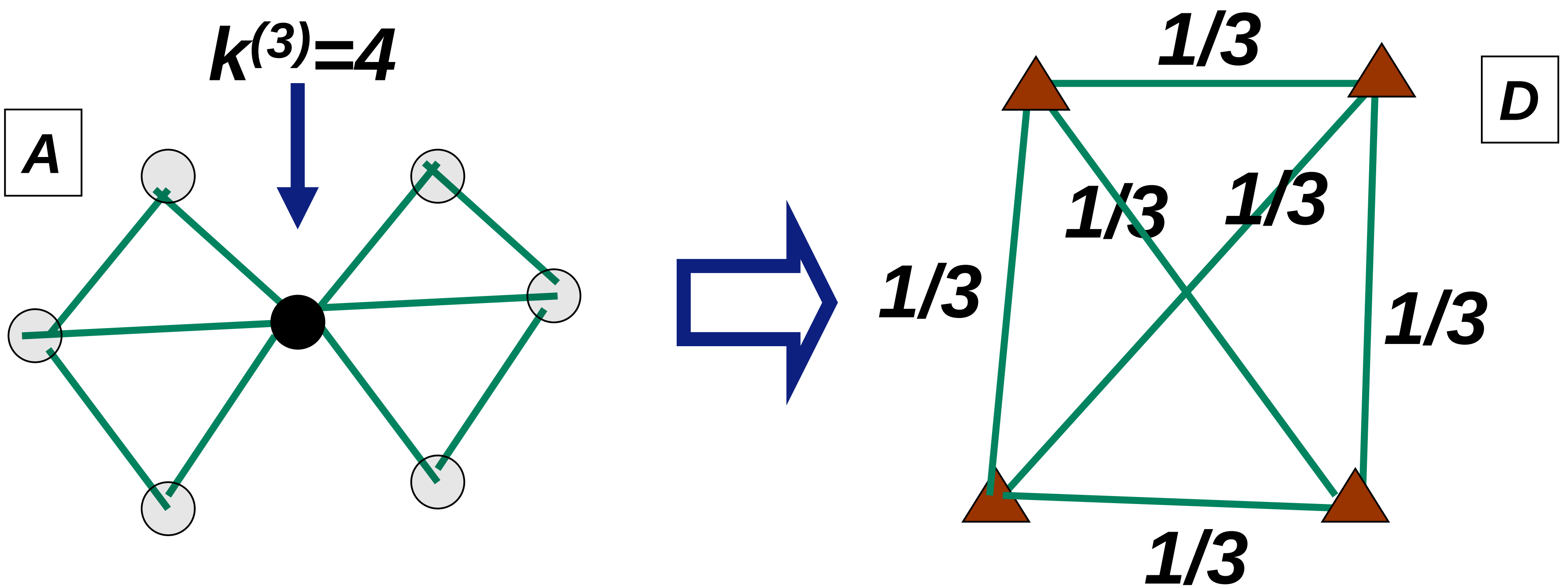
$$1/(k^{(n)}-1)$$

to an edge in the new weighted Clique Graph *D*. (vertex is in $k^{(n)}$ n-cliques)



Better Clique Graph Weight Construction

Now each vertex *i* in the original graph A contributes a weight to the new weighted Clique Graph *D* proportional to the number of cliques containing that vertex, $k^{(n)}$.



Applications

Now apply ***ANY*** standard vertex based algorithm to vertices of clique graph ***C*** or ***D*** or other weighted version

- Degree distribution $\Rightarrow$ clique overlap distribution
- Cluster coefficient $\Rightarrow$ overlap of 3 cliques
- Community detection $\Rightarrow$ clique communities
- Etc, etc

Clique Graphs for Community Detection

- Produce Clique graph
- Use favourite vertex partition method in clique graph
- Hence assign each clique to a single community
- Deduce community membership of vertices and edges of original graph from clique membership.

$\Rightarrow$ Thus vertices and edges maybe part of more than one community.

$\Rightarrow$ Overlapping Communities.

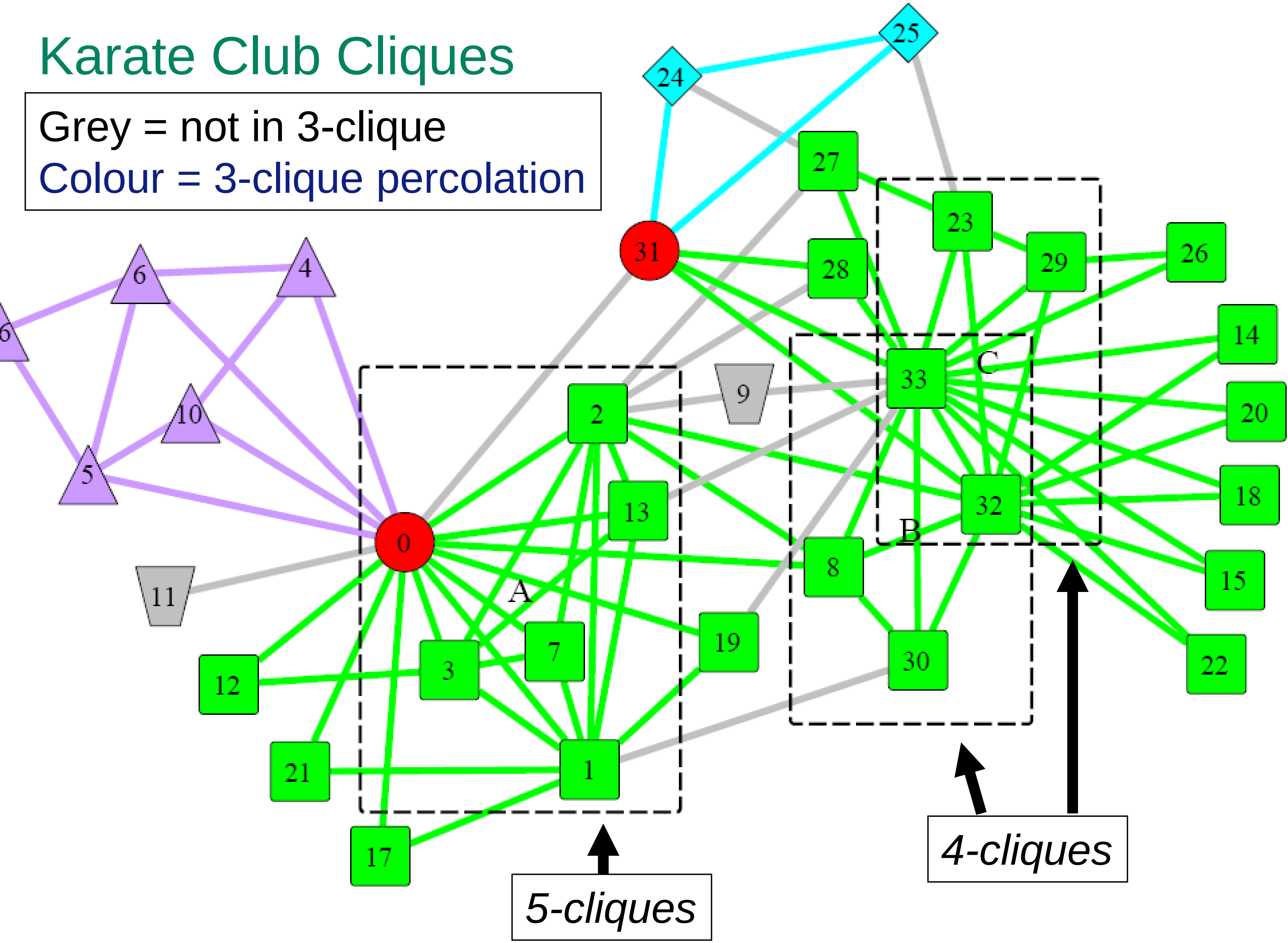
Cliques in Karate Club

- Almost everything is in a 3-clique
  - just 2 vertices and 9 edges not in 3-cliques

- One group of 6 vertices centred on Instructor have two percolating 5-cliques

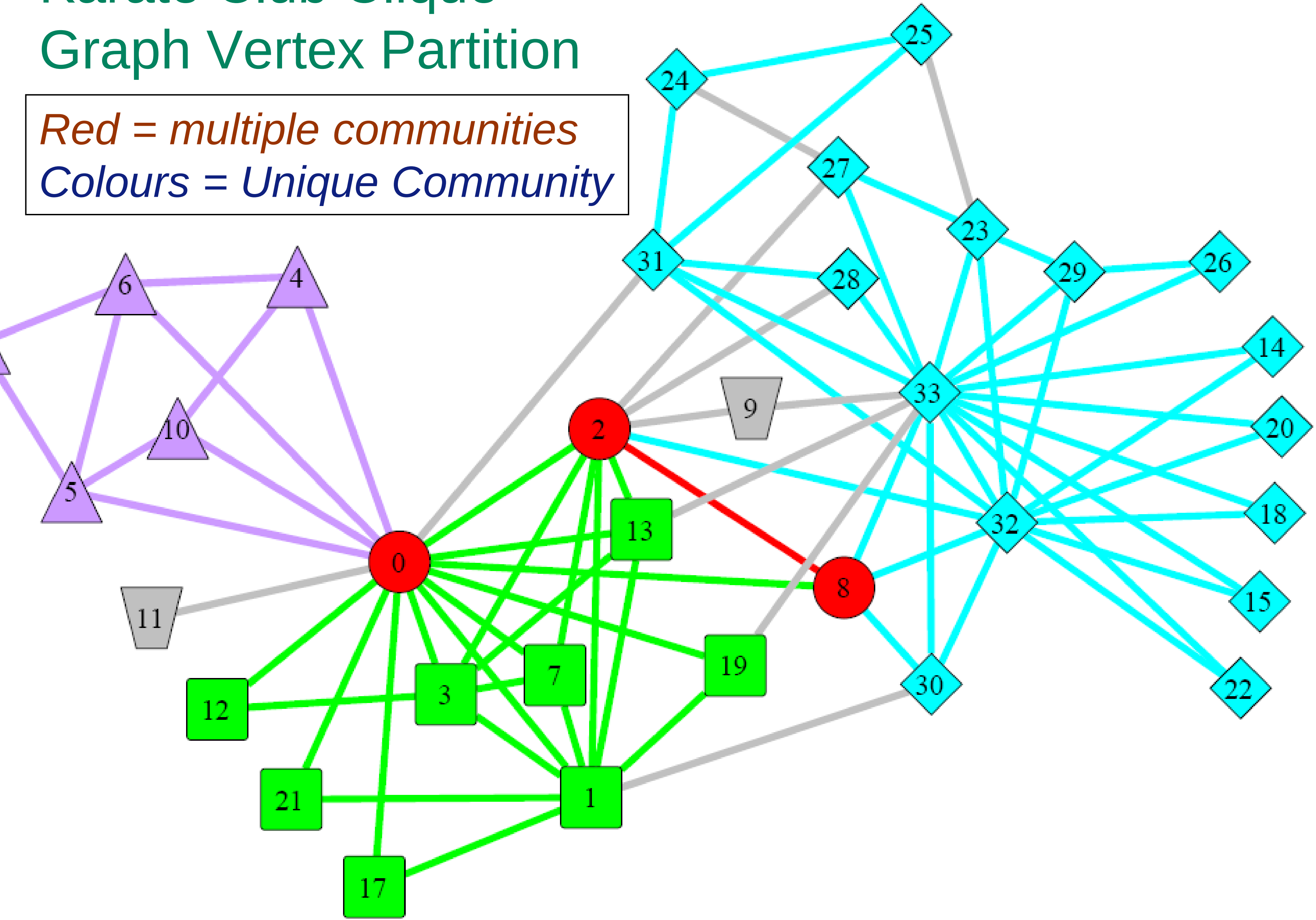
- Two other 4-cliques centred on chief officers, non-percolating

Karate Club Cliques



Karate Club Clique Graph Vertex Partition

Red = multiple communities
Colours = Unique Community



Analysis of Karate Club Results

- Clique percolation doesn't work here
- Despite the fact that almost everything is in a 3-clique, vertex partitioning of a clique graph works extremely well.

Conclusions

- Clique graphs move focus from vertices to cliques with minimal effort
- Altering weights avoids problem of over representation of high degree vertices
- 2-clique graphs are *Weighted Line Graphs* [Evans and Lambiotte, 09]
- Generalisation to motifs straightforward
- Community detection on clique graph produces overlapping vertex communities for original graph