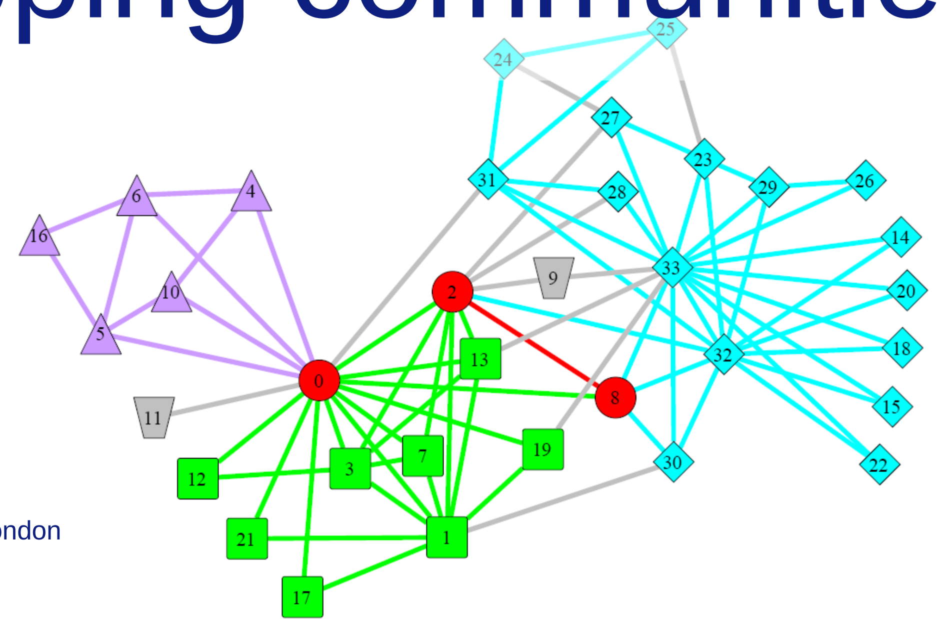
Imperial College London

Making Sense of Data - Theory and Practice
NCAF Summer meeting
Guildford, 12th July 2010

Tim Evans
Theoretical Physics

Give cliques a chance: line graphs, clique graphs and overlapping communities



- Graphs, Data and our Vertex Centric Viewpoint
- Clique Graphs
- Overlapping Communities
- Line Graphs and Overlapping Communities
- Conclusions

Network/Graph Notation

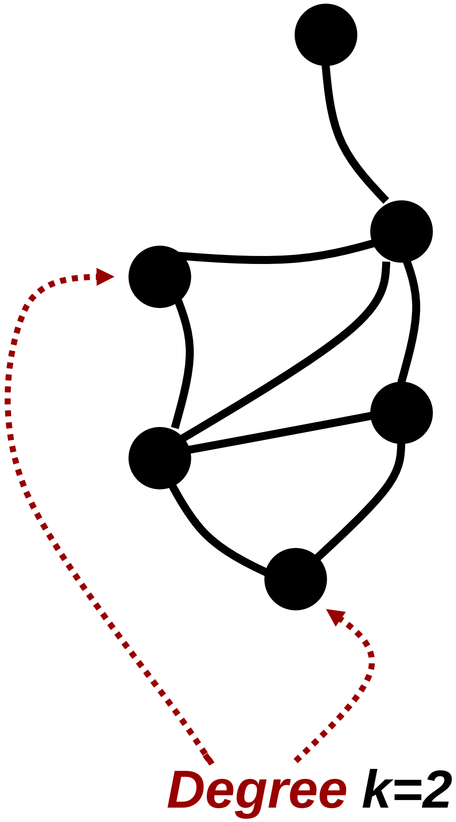
I will focus on **Simple Graphs**

(no values or directions on edges, no values for vertices)

- ***N*** = *number of vertices in graph*
- ***E*** = *number of edges in graph*
- ***k*** = *degree of a vertex*
- ***<k>*** = *average degree* = ***2E/N***
- Degree Distribution

  n(k) = *number of vertices with degree* ***k***

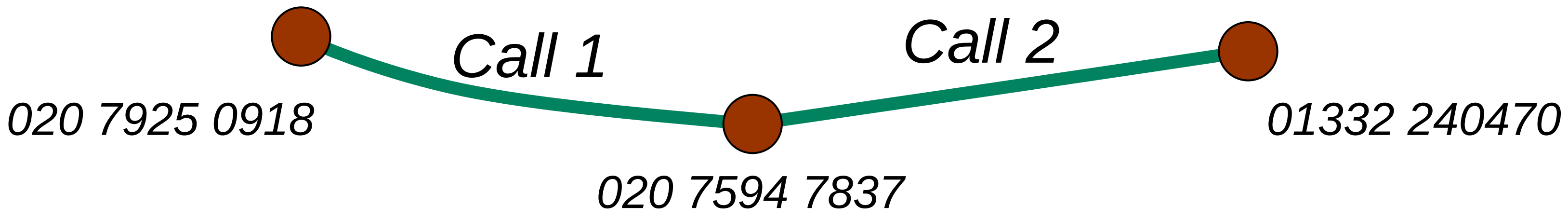
  p(k) = ***n(k)/N*** = *normalised distribution*



Graphs and "Making Sense of Data"

- Many data sets can be represented by a graph
  e.g. vertex = distinct value in a database field
  edge = two values in same record

Originator	Receiver	Time	Duration
020 7594 7837	020 7925 0918	2010.07.09:11.10	360
020 7594 7837	01332 240470	2010.07.09:11.17	415



Graphs as Data Reduction

Disadvantages

- Loss of Information
  - User Account Information, …
- No unique graph representation
  - Edge directions, edge weights, …

Advantages

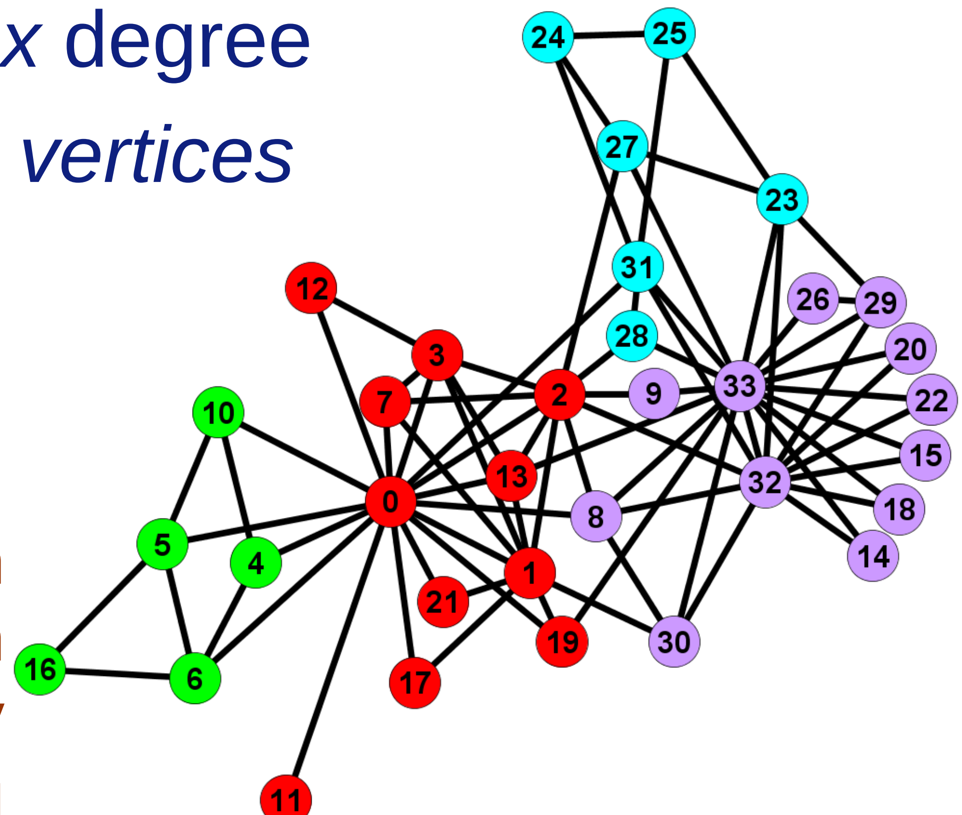
- Graphs are simpler
- Universal representations
  - Comparison to other networks, standard libraries, …

$\Rightarrow$ Making Sense of Data

Vertex Centric Viewpoint

The focus in the literature is often on the vertices:-

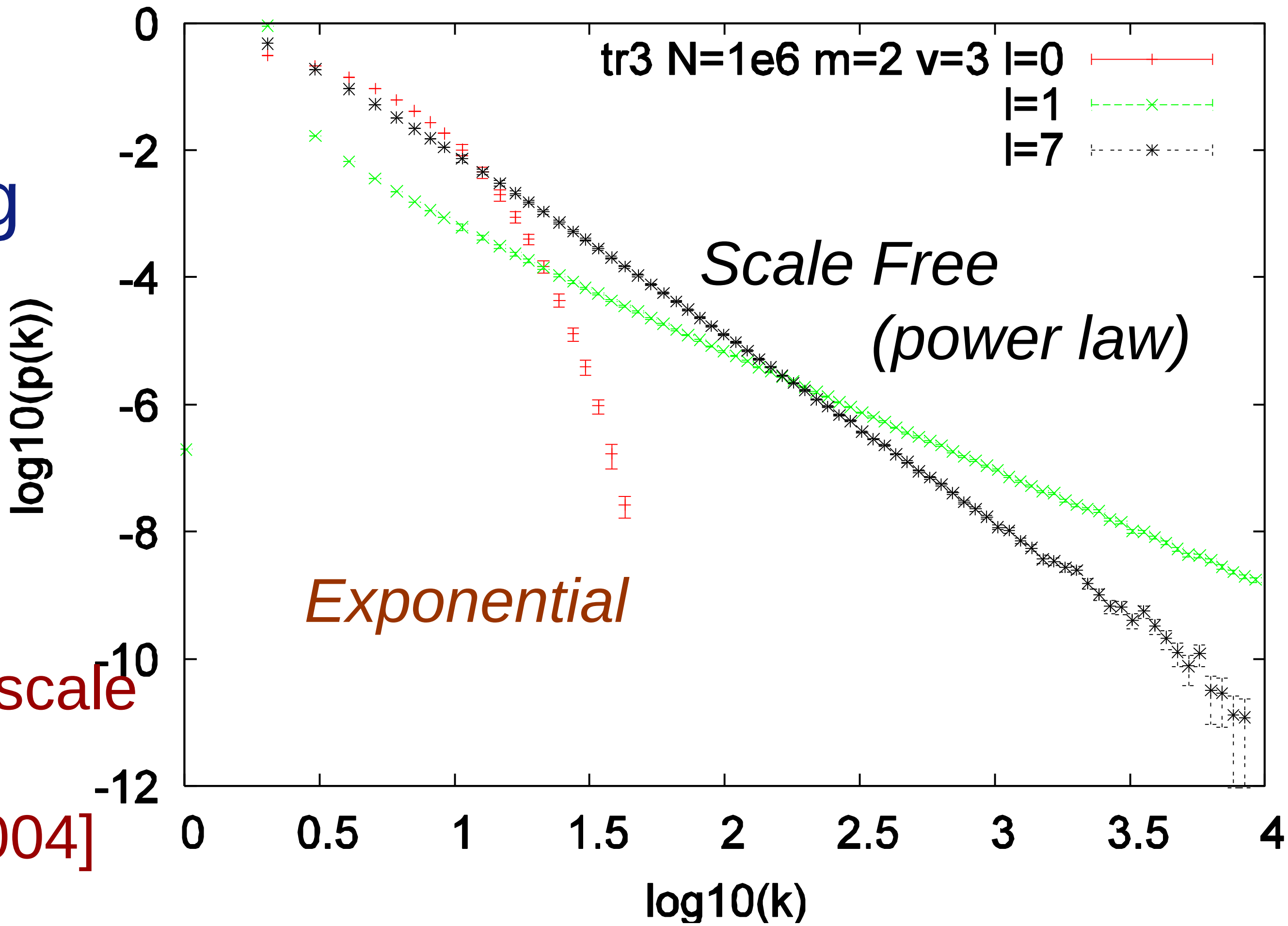
- Distributions of *vertex* degree
- Cluster coefficient of *vertices*
- *Vertex* partitions as communities
- …..



Vertex partition of Karate club graph with optimal modularity [Agarwal & Kempe 2007]

Vertex Centric – Degree Distributions ***p(k)***

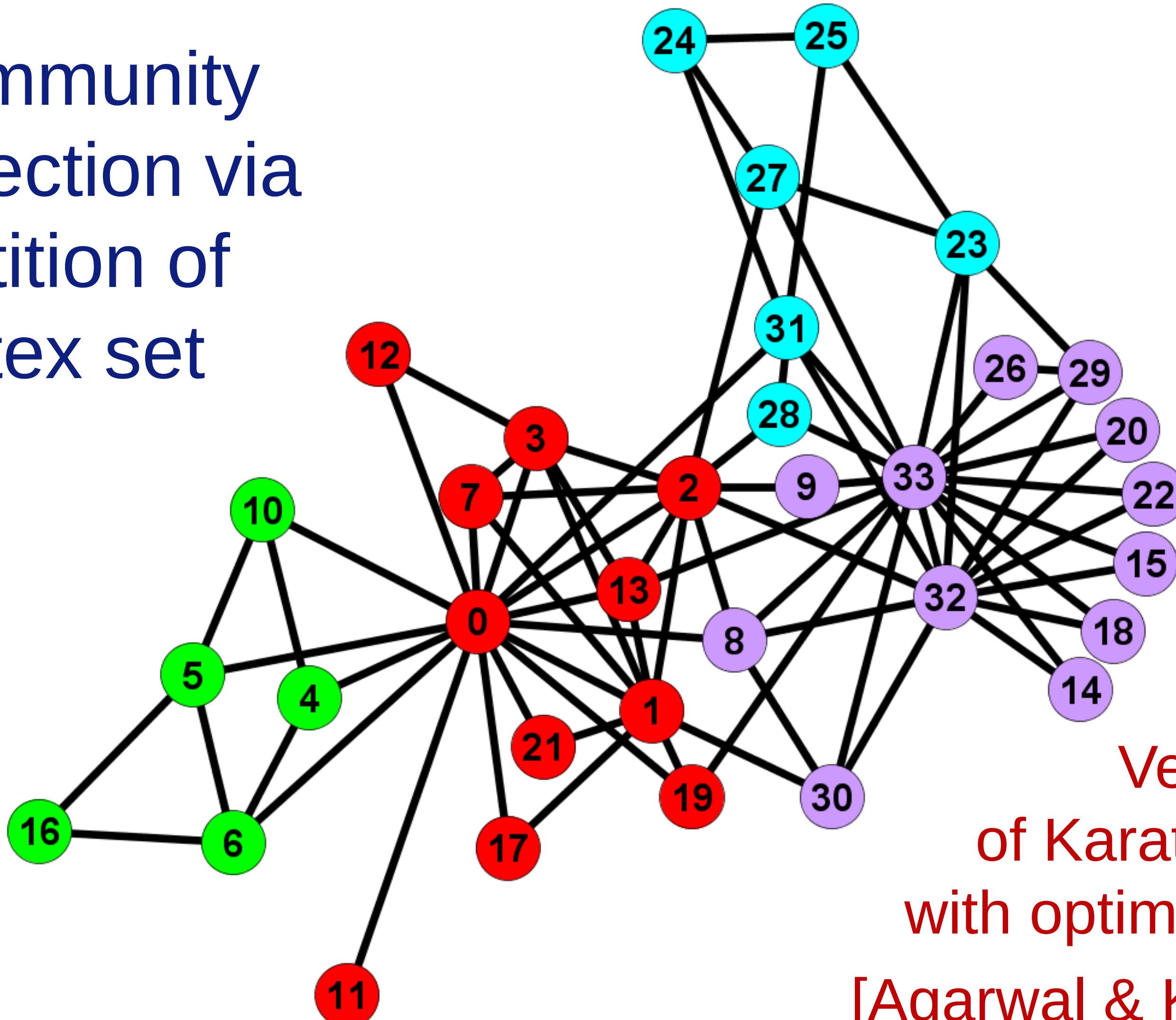
p(k) **=** probability of vertex having degree ***k***



Random walk on vertices produces scale free graphs
[TSE, Saramäki 2004]

Vertex Centric – Vertex Partitions

Community detection via partition of vertex set



Vertex partition of Karate club graph with optimal modularity [Agarwal & Kempe 2007]

Word Count of Network Review [Evans '04]

Stem	Rank	Count	Stem	Rank	Count
network	1	254	number	11	58
vertic	2	107	distanc	12	48
edg	3	86	model	13	47
random	3	86	connect	14	46
graph	5	81	data	15	40
degre	6	78	link	16	38
power	7	68	world	16	38
lattic	8	67	hub	33	25
law	9	65	point	38	23
vertex	10	61	site	40	22

Stop words removed then stemming

Word Count - Approximate Ratios

Network Words	Vertex Words	Edge Words
6	3	2

Word count shows focus on vertices

How can we compensate for vertex bias?

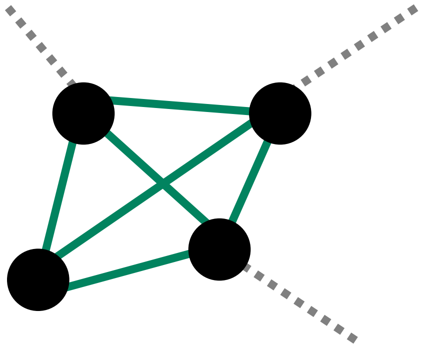
One Answer:-

1. Represent structures in the original graph ***A*** (edges, cliques, motifs, …) as vertices in a new graph
2. Analyse new graph as usual

$\Rightarrow$ Vertex bias on new graph = emphasis on other structures in original graph

- Graphs, Data and our Vertex Centric Viewpoint
- Clique Graphs
- Overlapping Communities
- Line Graphs and Overlapping Communities
- Conclusions

Cliques – complete subgraphs



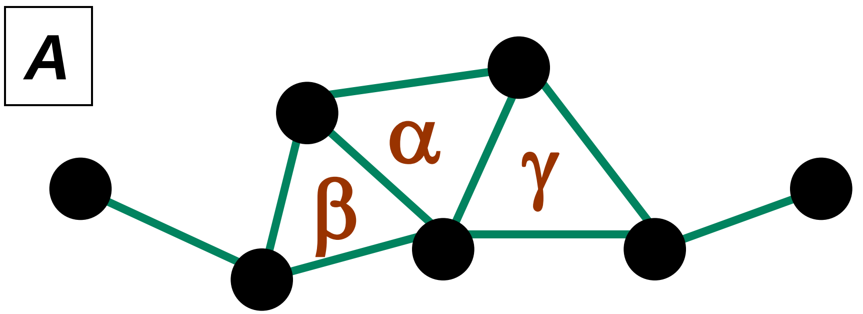
- Name originates from representation of cliques of people in social science
  [Luce and Perry ‘49]
- Theory of Triadic Closure
  [Granovetter, ‘73]
- Maximal Clique and Clique Cover problems
  [Bron-Kerbosch algorithm, ‘73]
- Clustering/Community detection
  [Freeman ’92, ‘96; Palla et al, ‘05; Yan & Gregory ‘10]
- Structure Analysis
  [Samudrala & Moult, ‘98; Takemote et al. ‘07]

Give Cliques a Chance – a clique centric viewpoint

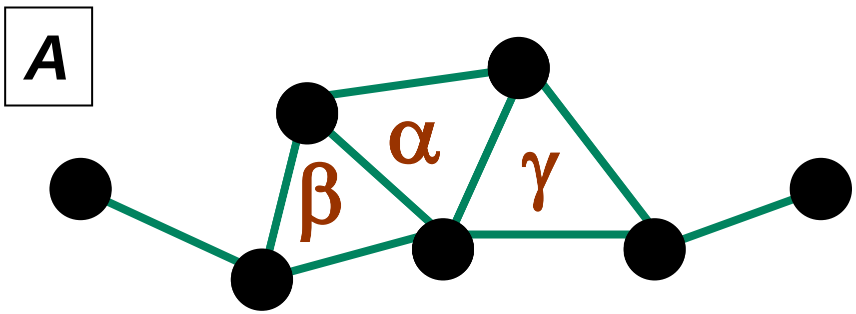
Shift our viewpoint from vertices to cliques using a **Clique Graph**

- New Vertices = original cliques
- New Edges = overlap of cliques

Basic idea also used in social science [Everett and Borgatti, ‘98]



Example



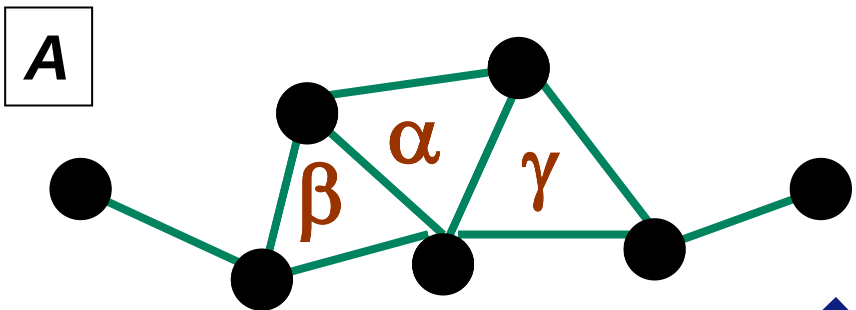
- three cliques of order 3 = triangles, triads
  - labelled α,β,γ.

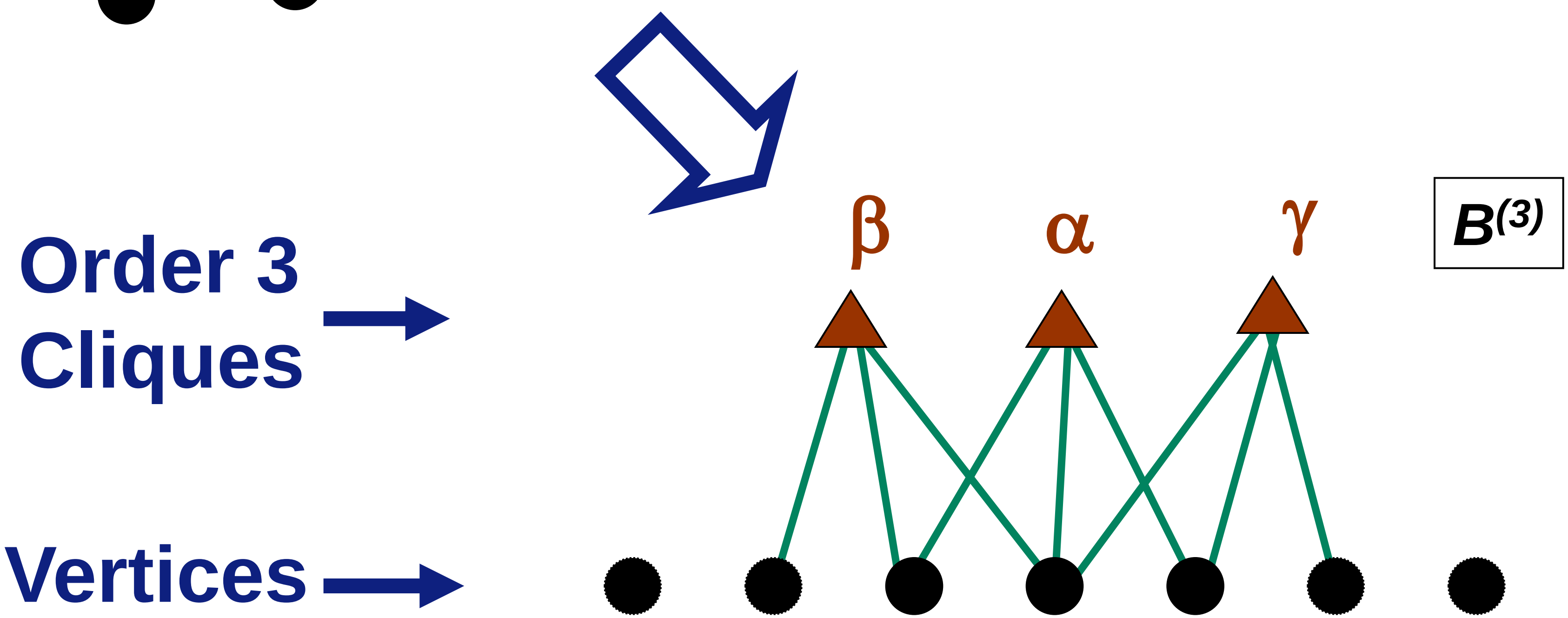
- nine cliques of order 2 = edges

I will only use cliques of one order unlike most Social Science work.

Clique-Vertex Bipartite Graph Construction

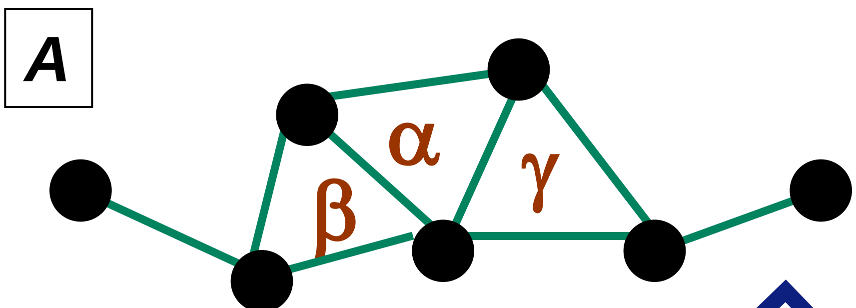
Adjacency matrix $\boldsymbol{B^{(n)}_{i\alpha} = 1}$
if vertex $\boldsymbol{i}$ is in clique $\boldsymbol{\alpha}$
of fixed order $\boldsymbol{n}$

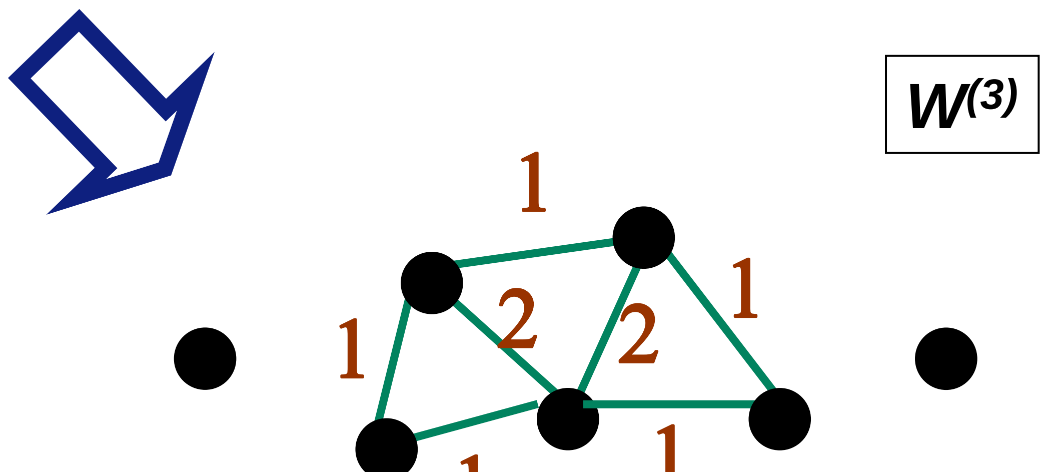




Usual Clique Overlap Graph $W^{(n)}$

- Vertices unchanged
- Edge weights equal to the number of ***n***-cliques containing that edge in in the original graph ***A*.**





Unweighted $W^{(3)}$ is isomorphic to an *n*-regular hypergraph

Clique Overlap Graph $W^{(n)}$

- In Social Science clique overlap is usually studied with cliques of several orders (e.g. subset of maximal cliques)

$$W_{ij} = \Sigma_n W^{(n)}_{ij}$$

- $W^{(n)}$ without weights is isomorphic to an ***n***-uniform hypergraph

Drawback of $W^{(3)}$ is that the vertices remain the same

$\Rightarrow$ same old vertex centric viewpoint

Definition of the Basic Clique Graph $C^{(n)}$

Project bipartite graph onto clique vertices.

⇒Weight of an edge in the basic Clique Graph $C^{(n)}$ records the number of vertices common to two ***n***-cliques in in the original graph ***A*.**

$$C_{\alpha\beta}^{(n)} = \sum_{i} B_{i\alpha}^{(n)} B_{i\beta}^{(n)} \left(1 - \delta_{\alpha\beta}\right)$$

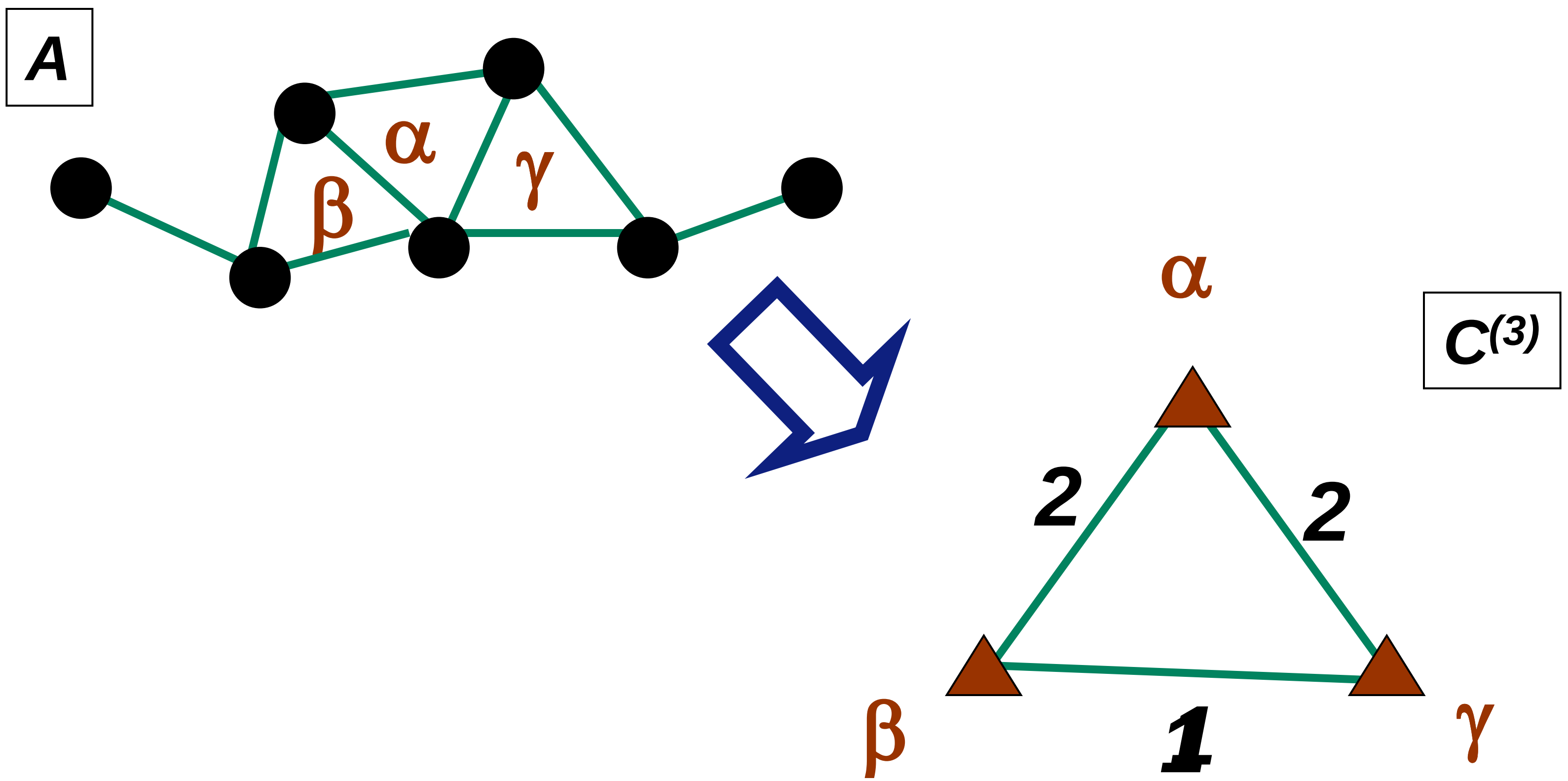
i = vertex ***A***

α,β* = *n-cliques in ***A***

(Rarely used in the Social Science literature)

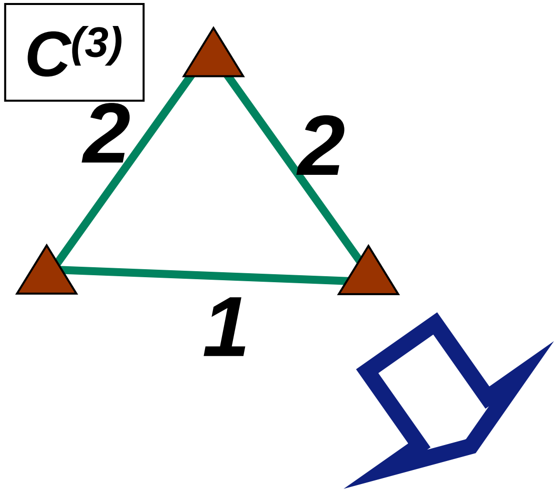
Edge Weights in Basic Clique Graph

Weight of an edge in the basic Clique Graph ***C*** records the number of vertices common to two ***n***-cliques in in the original graph ***A*.**



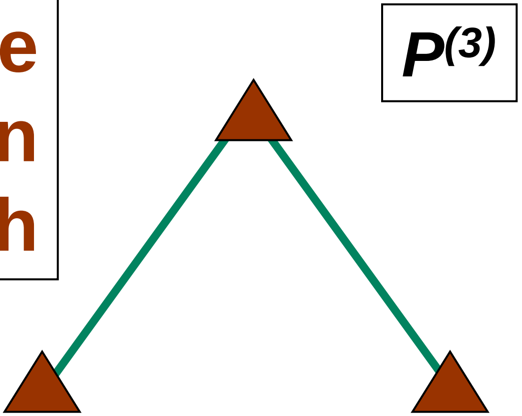
Clique Percolation Graph Construction

Threshold edges of clique Graph with *(n-1)*



i.e. only retain cliques which in original graph ***A*** share all but one vertex with another clique

Clique Percolation Graph



Connected Components = Communities [Palla et al, '05]

$$P_{\alpha\beta} = \Theta\left(C_{\alpha\beta} - n + 1\right)$$

Basic Clique Graph Weights – A problem

- The weight of basic Clique Graph $C^{(3)}$ records the number of vertices common to two n-cliques in in the original graph A.

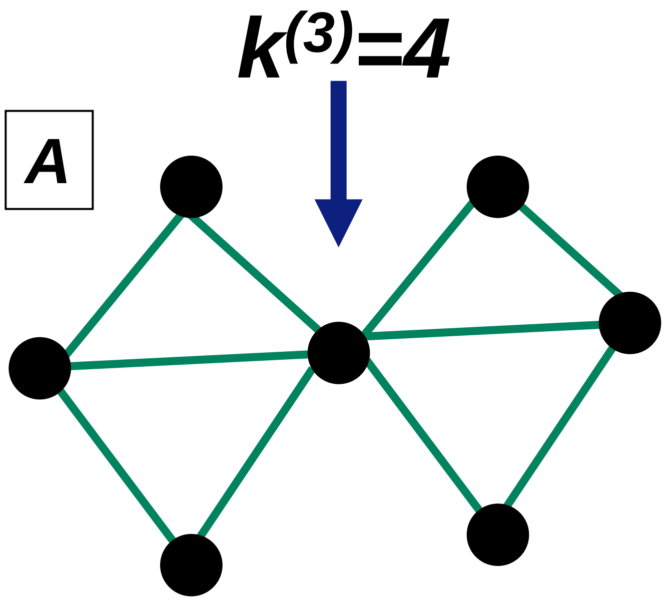
- **Thus a vertex which is in $k^{(n)}$ cliques will contribute a total weight of**

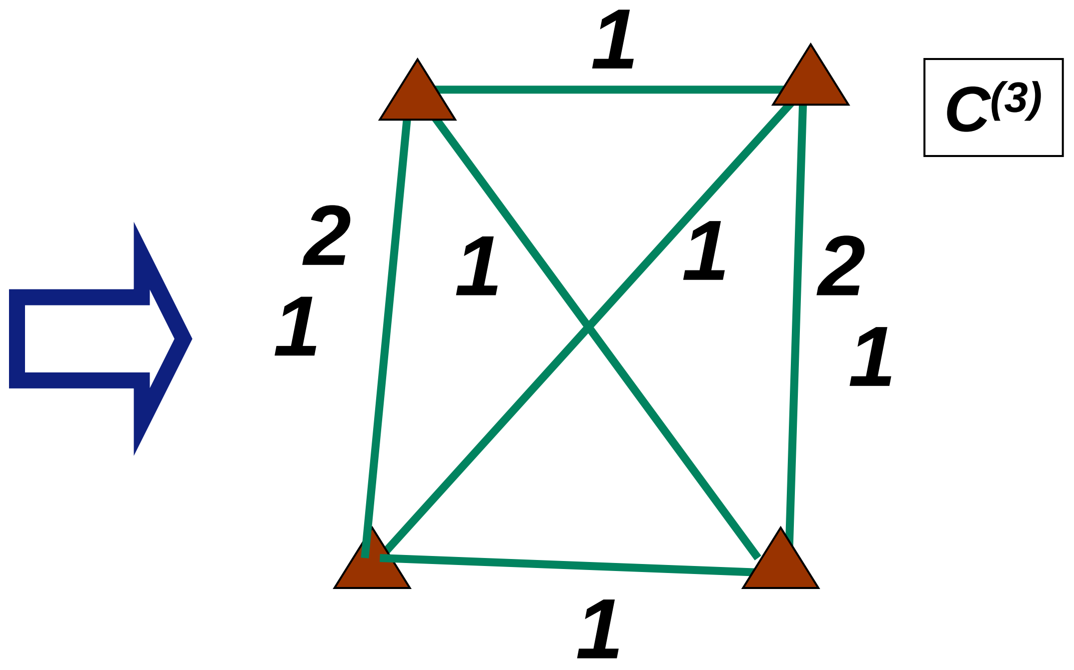
$$k^{(n)} (k^{(n)} -1)/2$$

to the basic clique graph $C^{(3)}$.

Basic Clique Graph Weights – A problem

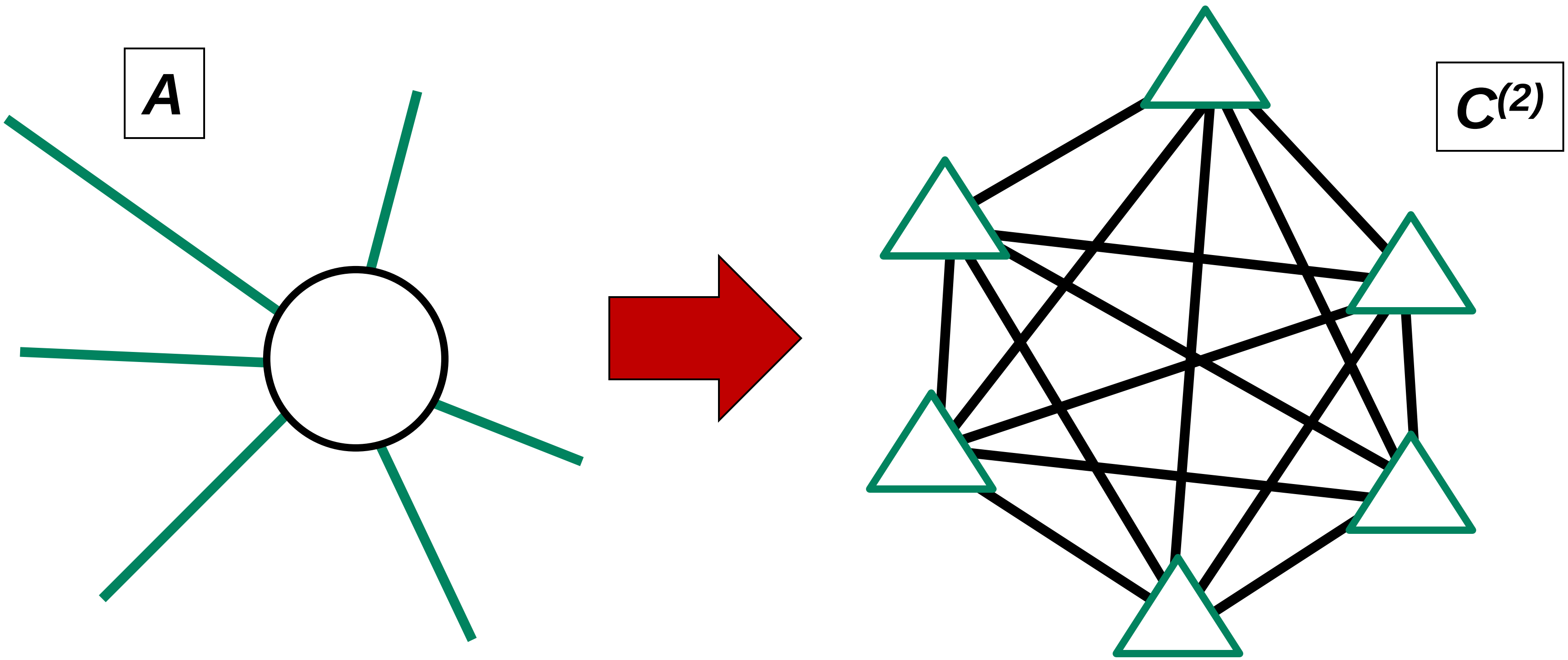
- **A vertex in a large number of cliques contributes too much weight to the clique graph *C*.**





The Problem for ***n*=2** Edges/2-cliques

High degree vertices in original graph ***G*** over represented by edges in Line Graph $\boldsymbol{C^{(2)}(A)}$.



Degree *k* vertex

k(k-1)/2 edges

Better Clique Graph Weight Construction

A vertex common to a pair of cliques contributes a weight equal to

$$1/(k^{(n)}-1)$$

to an edge in the new weighted Clique Graph $D^{(n)}$.

$$D_{\alpha\beta}^{(n)} = \sum_i \frac{B_{i\beta}^{(n)} B_{i\beta}^{(n)}}{k^{(n)}-1}\left(1-\delta_{\alpha\beta}\right)$$

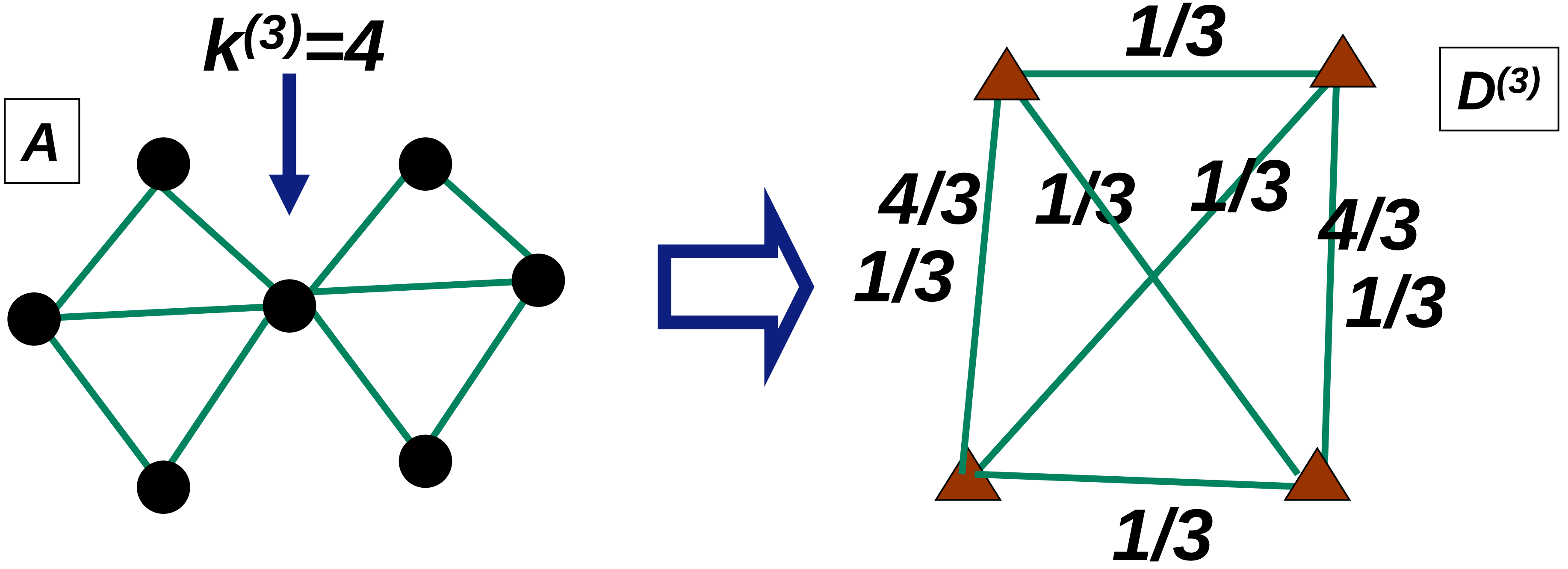
(vertex is in $k^{(n)}$ n-cliques)

Better Clique Graph Weight Construction

A vertex common to a pair of cliques contributes a weight equal to

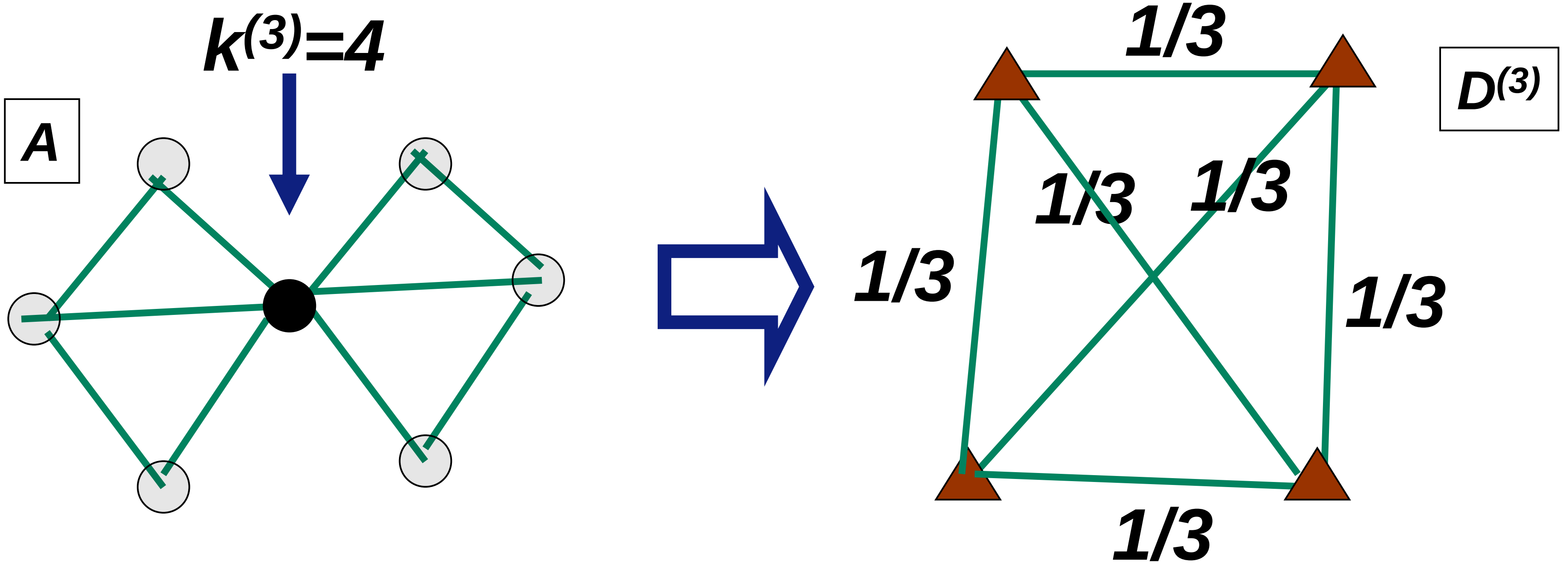
$$1/(k^{(n)} - 1)$$

to an edge in the new weighted Clique Graph $D^{(3)}$. (vertex is in $k^{(n)}$ n-cliques)



Better Clique Graph Weight Construction

Now each vertex *i* in the original graph A contributes a weight to the new weighted Clique Graph $D^{(n)}$ proportional to the number of cliques containing that vertex, $k^{(n)}$.



Applications

Now apply ***ANY*** standard vertex based algorithm to vertices of clique graph $\boldsymbol{C^{(n)}}$ or $\boldsymbol{D^{(n)}}$ or other weighted version

- Degree distribution ⇒ clique overlap distribution
- Cluster coefficient ⇒ overlap of order ***n*** cliques
- Community detection ⇒ clique communities
- etc, etc

- Graphs, Data and our Vertex Centric Viewpoint
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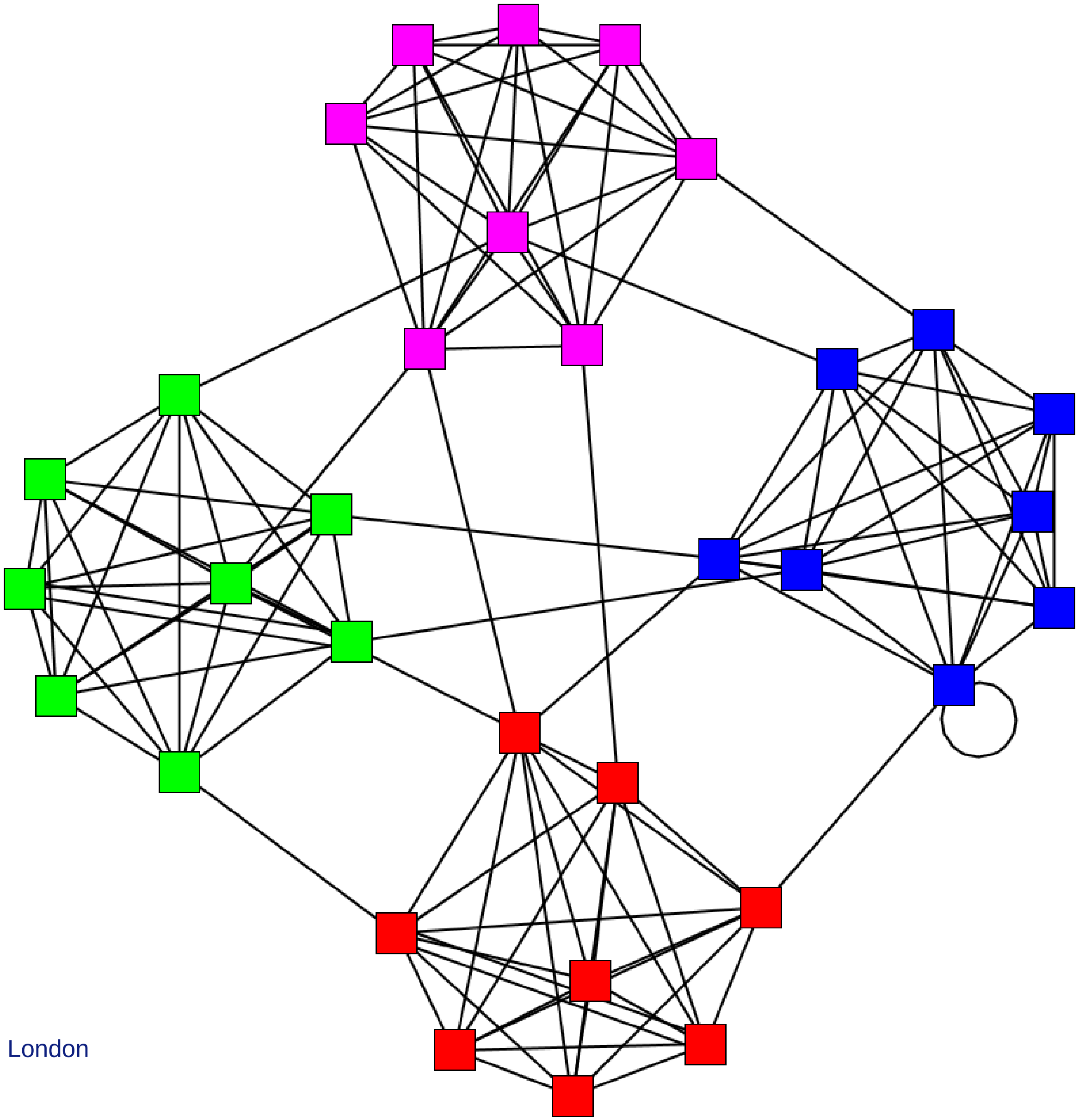
Communities in Networks

Rough definition:-

A *community* is a subgraph which is more tightly connected than average



Community
= cluster
= cohesive subgroup

Advantages of Vertex Partition Communities

- Simplest way of assigning communities across whole graph
- Appropriate for some problems
  e.g. assigning pixels in image analysis
- Vast amount of development of theory and methods
  e.g. free code works on graph of 10^8 vertices
  in 20min
  (Louvain method, Blondel et al. 2008)

Limitations of Vertex Partition Communities

In many applications it is too simplistic to assign one vertex to one community

Examples:

- Friendship networks
- Academic papers

Who am I? Communities in Friendship Networks

Friendship networks have:-

- people as vertices
- edges if friends

We all have different types of friends:-

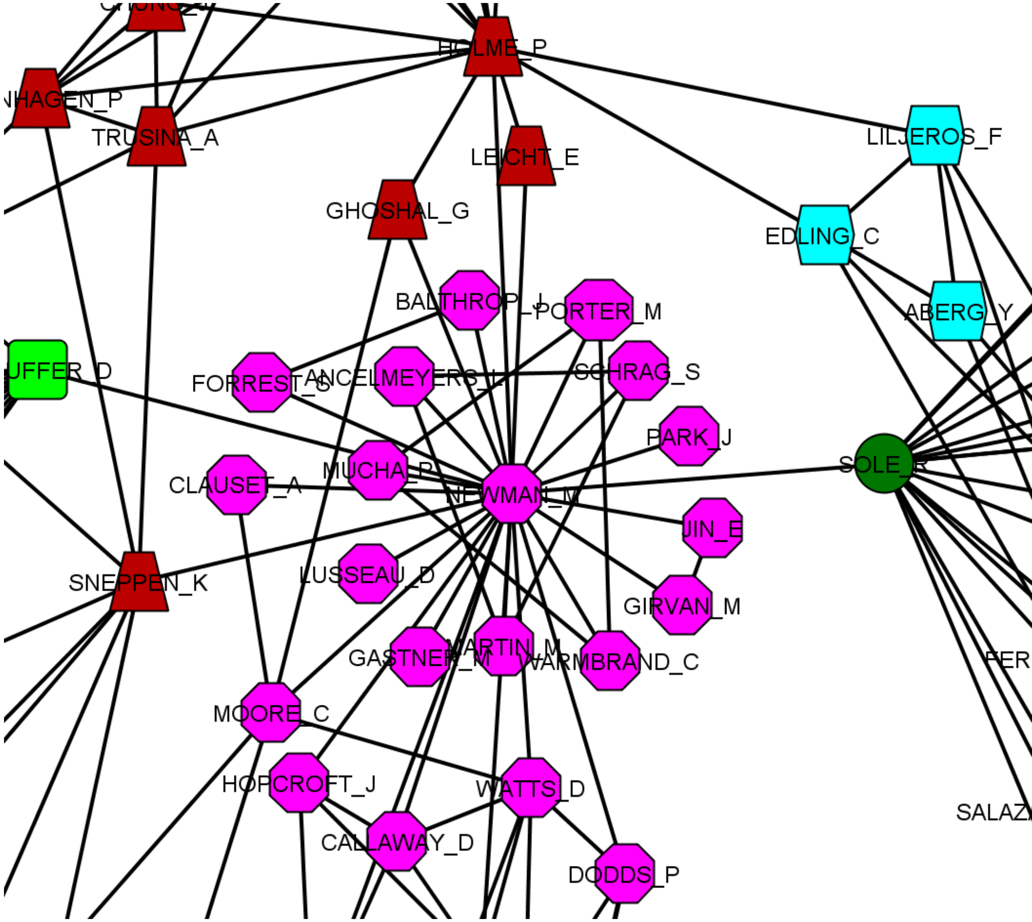
- Family
- Neighbours
- Work Colleagues

I am not just an Evans or an Imperial employee

What am I? Communities from Academic Papers

Coauthorship Networks:-

- authors as vertices
- edges between coauthors of a paper



Academics who work across boundaries always assigned to one community

Clique Graphs for Community Detection

- Produce Clique graph
- Use favourite vertex partition method in clique graph
- Hence assign each clique to a single community
- Deduce community membership of vertices and edges of original graph from clique membership.

⇒Thus vertices and edges maybe part of more than one community.

⇒ Overlapping Communities.

Karate Club [Zachary 1977]

- Community of 34 members of a Karate Club
- Split into two parts during study
  - Group centred on Club Officers
  - Group centred on instructor
- Zachary partitioned vertices into two subsets using Ford-Fulkerson algorithm (source-sink) which matched actual split except for one individual

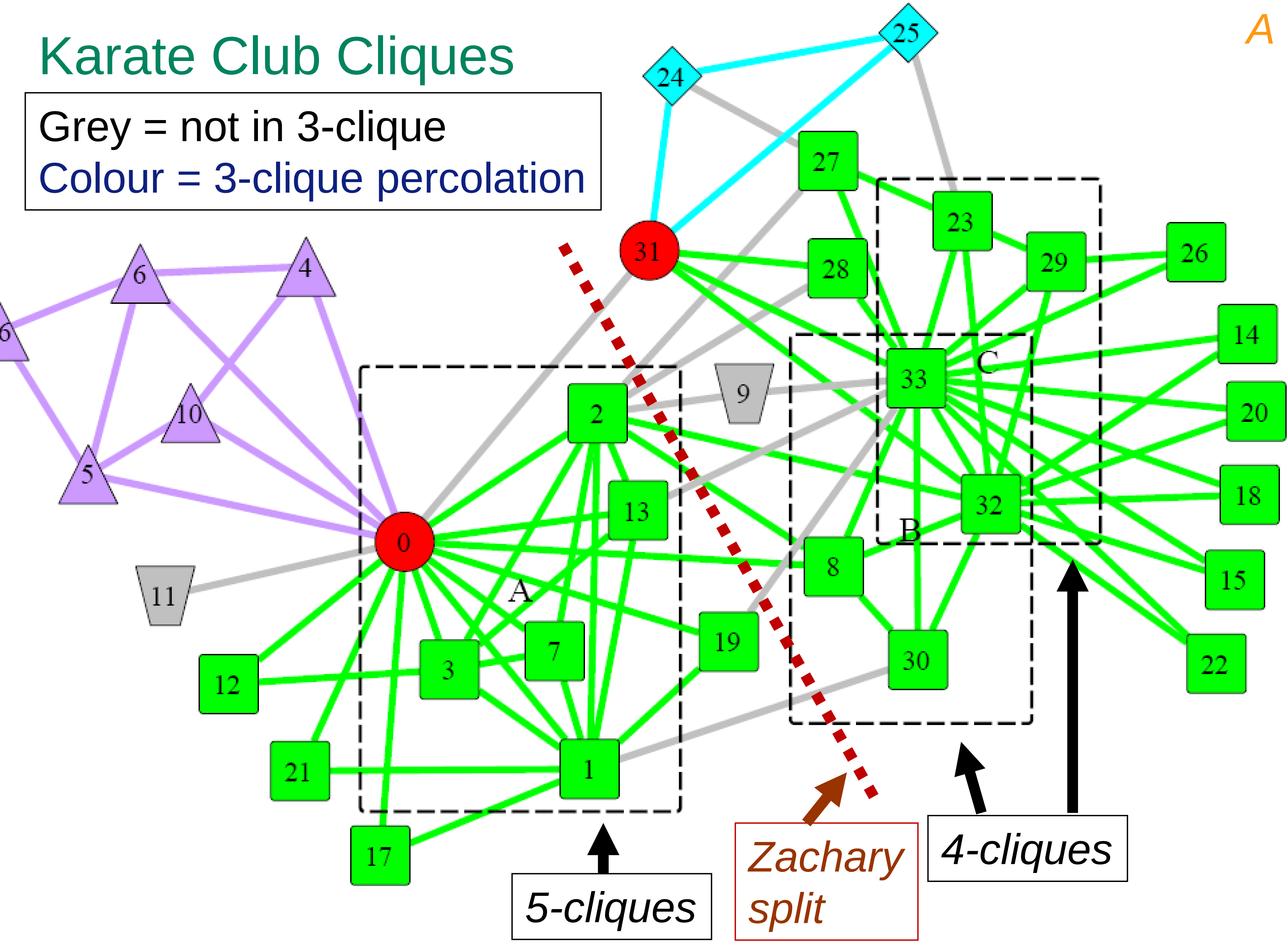
Cliques in Karate Club

- Almost everything is in a 3-clique
  - just 2 vertices and 9 edges not in 3-cliques

- One group of 6 vertices centred on Instructor have two 5-cliques with 4 common vertices (percolating)

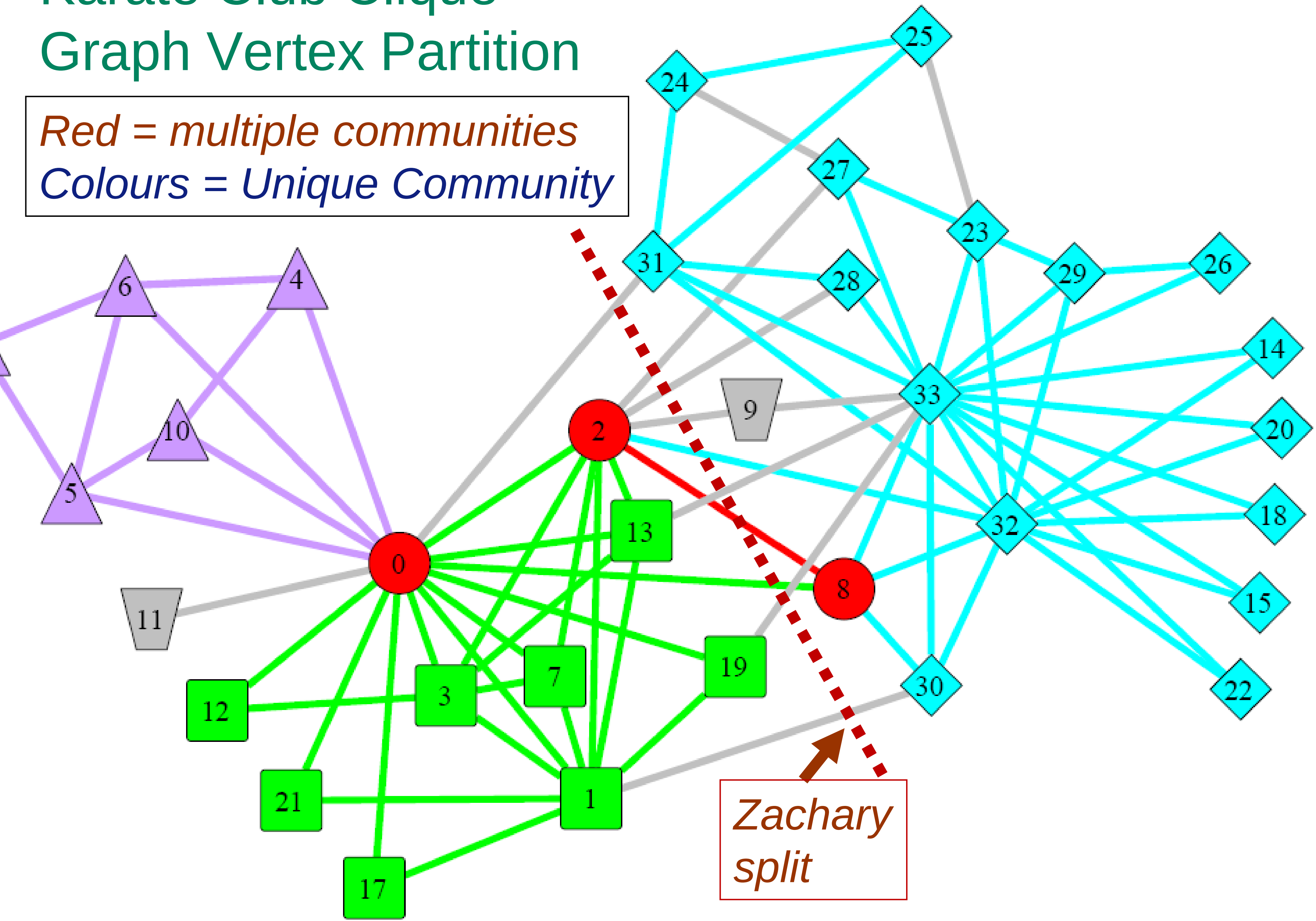
- Two other 4-cliques centred on chief officers, only 2 common vertices (non-percolating)

Karate Club Cliques



Karate Club Clique
Graph Vertex Partition

Red = multiple communities
Colours = Unique Community



Analysis of Karate Club Results for Triangles

- Clique percolation doesn't work here

- Despite the fact that almost everything is in a 3-clique, vertex partitioning of a 3-clique graph $D^{(3)}$ works extremely well.

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2-Clique Graph = Line Graph

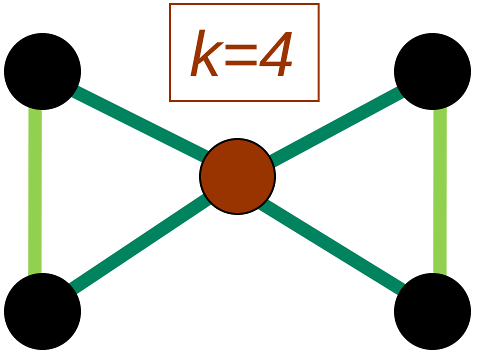
- The 2-clique (=edge) graph $C^{(2)}$ is an unweighted graph called a **Line Graph** ***L(A)***
  – Long history [Whitney, '32; Krausz '43; Harary & Norman '60]
  – Much work in mathematical literature

- The 2-clique (=edge) graph $D^{(2)}$ is a **Weighted Line Graph** ***WL(A)***

[TSE+Lambiotte '09]

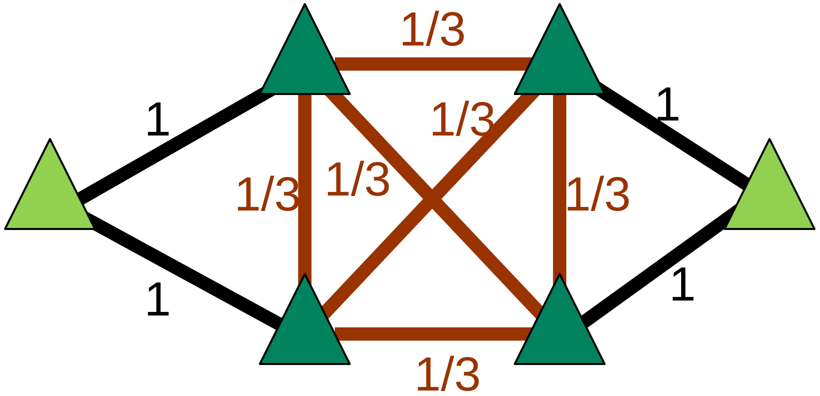
Example – Bow Tie Graph

Graph ***A***



Weighted Line Graph

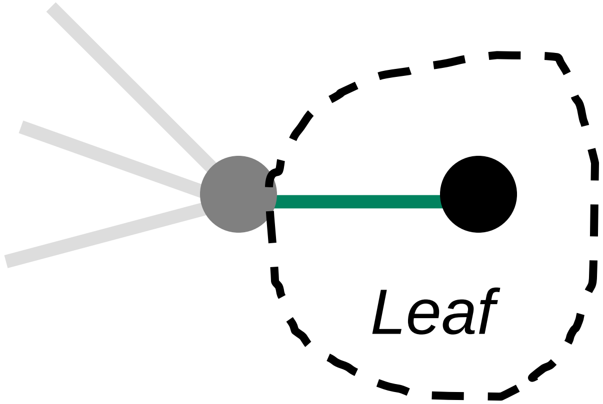
$\boldsymbol{WL(A) = D^{(2)}}$

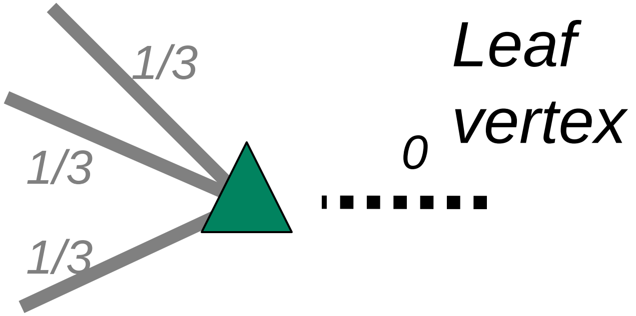


Edges 6
Strength 2

Nice Property of Weighted Line Graph

- Strength of each vertex in ***WL(A)*** is usually ***2*** since every edge in ***A*** has two ends
- Exception for edges of "leaves" in ***A*** which produce strength ***1*** vertices in ***WL(A)***.





From a Vertex to an Edge Centric Viewpoint

- Take your graph ***A*** with ***N*** vertices and ***<k>*** edges

- Make a Weighted Line Graph ***WL(A)*** with ***N<k>*** vertices and ***O(N<k²>)*** edges

- Run any vertex based algorithm on ***WL(A)*** and you are running it on the edges of ***G***.

Overlapping Communities

Use weighted line graphs to study edge colourings and hence to deduce overlapping communities

- Karate Club
- South Florida Words Association Data
- Words from Titles of Scientific Papers

Karate Club [Zachary 1977]

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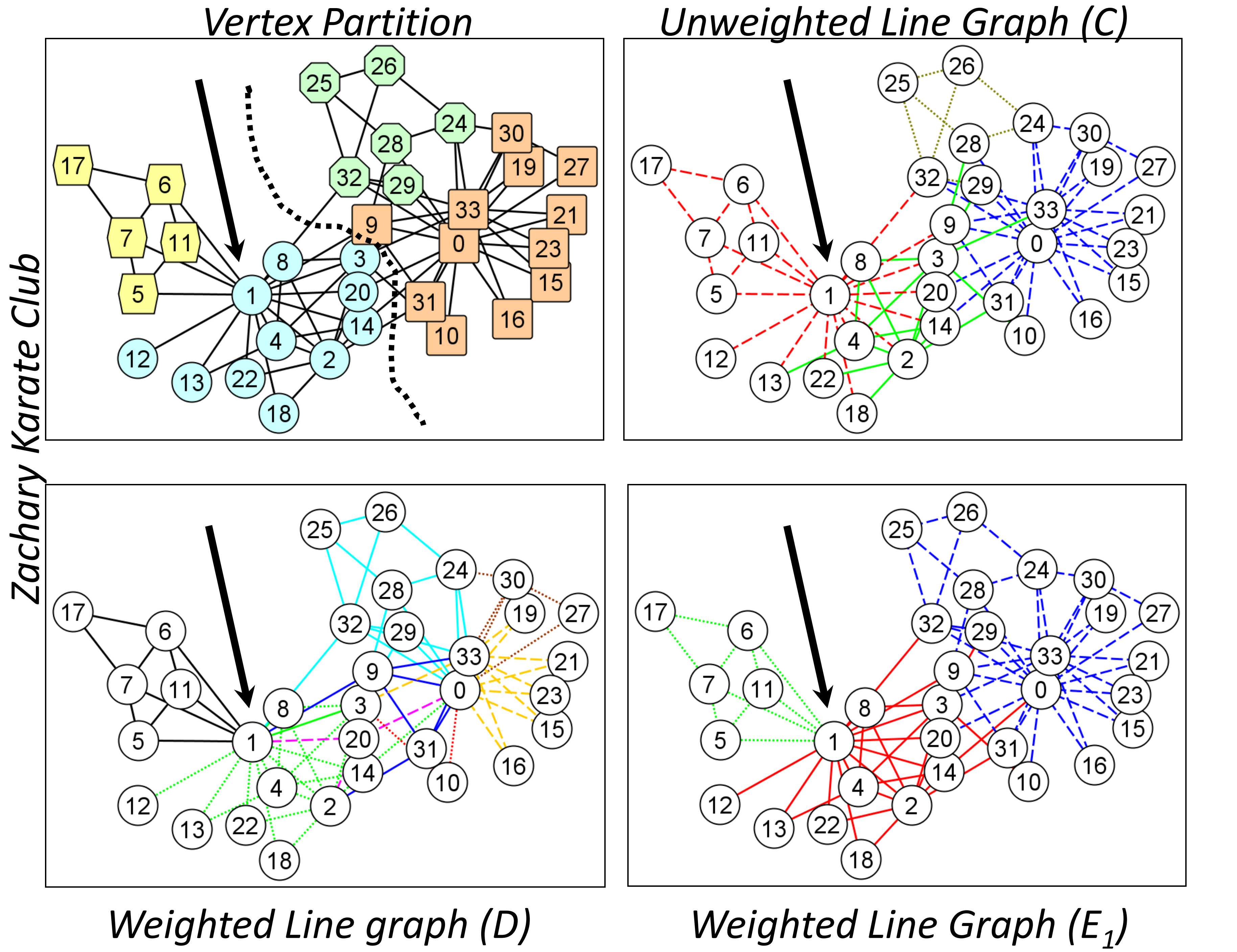
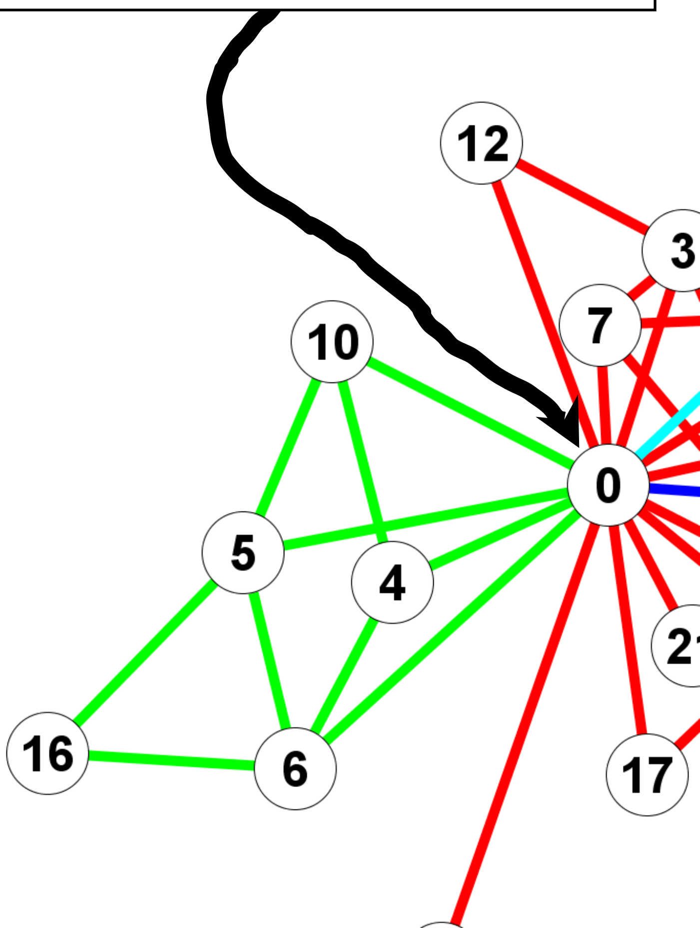

Vertex Partition
Unweighted Line Graph (C)
Zachary Karate Club
Weighted Line graph (D)
Weighted Line Graph (E1)

Karate Club Analysis

Vertices in One Edge Community

#	k	Fraction k In Green C
5	4	100%
6	4	100%
10	3	100%
4	3	100%
16	2	100%
0 (Mr_Hi)	16	25%

Mr Hi (the Instructor) bridges several groups



Karate Club Edge Partition

Vertices can be members of many communities

An overlapping community structure for vertices

Name	Community	Total k	k in C
0 Mr Hi	0	16	10
	1		4
	2		1
	3		1
33 John A	3	17	12
	0		3
	2		2

South Florida Word Association Data

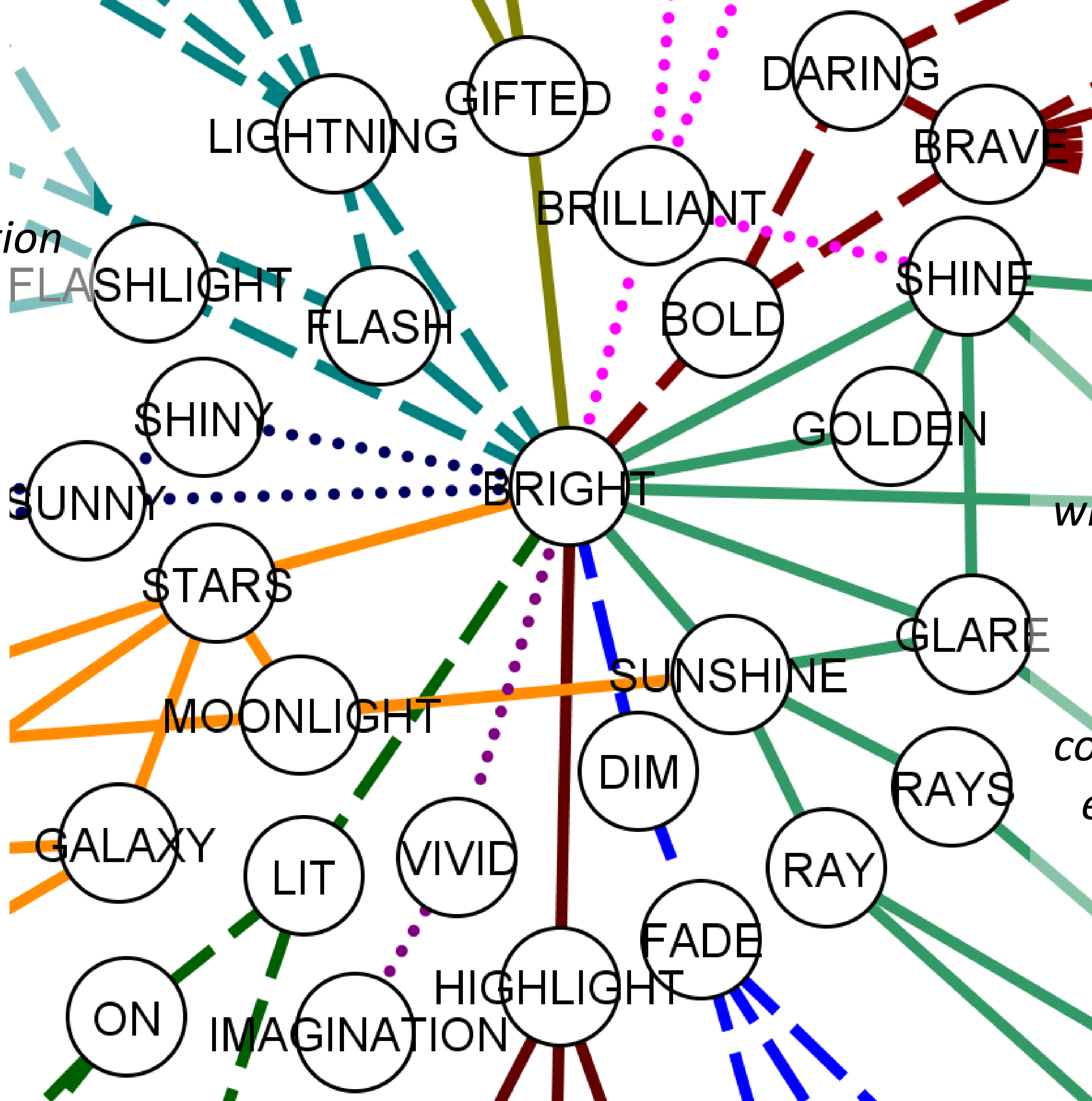
Data from 6,000 participants, nearly three-quarters of a million responses to 5,019 stimulus words.

Original graph A has

- Stimulus words as vertices
- Edges connecting words if paired in data more than a specified threshold.

http://w3.usf.edu/FreeAssociation/

South Florida Word Association Data



Only showing vertices which have 90% of edges in one edge community except for BRIGHT

Title of Papers Data

Data based on collection of science papers from a single institution over several years. Forms a bipartite graph of:-

- 26255 vertices representing the papers
- 17761 vertices representing terms - stemmed words from titles after stop words removed
- 210229 edges

Edge Partition of Terms in Paper Titles

- Some words have all edges in one partition
  – they define these communities e.g. cassini
- Other words have edges in several communities
  – stop words e.g. signature

Stem	Total k	k in C
interplanetari	78	78
cassini	62	62
heliospher	59	59
magnetopaus	53	53
spacecraft	52	52
signatur	91	32
solitari	30	10
radar	21	7
mhd	18	6

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Conclusions

- Clique graphs move focus from vertices to cliques with minimal effort
- Altering weights avoids problem of over representation of high degree vertices
- 2-clique graphs are *Weighted Line Graphs* [Evans and Lambiotte, 09]
- Generalisation to motifs straightforward
- Community detection on clique graph produces overlapping vertex communities for original graph