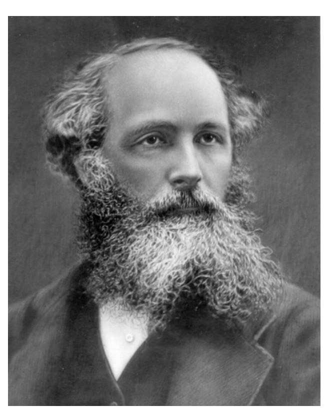
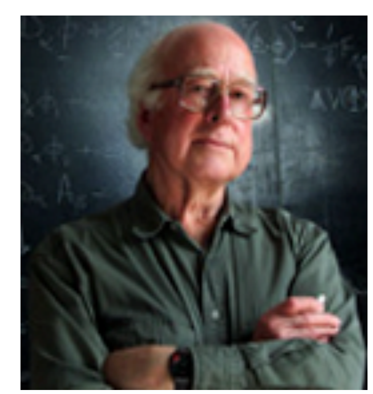
Supersymmetric supergravity backgrounds via Lie superalgebras

José Miguel Figueroa-O'Farrill

29 April 60HE



THE UNIVERSITY OF EDINBURGH



1983-84
(26HE)


© AP Images

3 lectures of CMH @ MIT

Physics	**Relativity Group**	**Geometry**
Newtonian	Galilean	$\mathbb{A}^3 \times \mathbb{A}^1$
Special Relativity	Poincaré	$\mathbb{A}^{3,1}$
General Relativity	Diffeomorphisms	(M, g)

The lesson

Symmetry dictates the geometry.

In this talk we will describe another example of this philosophy.

“From *graded* to *filtered* ”

“From *graded* to *filtered*”

Poincaré algebra is **graded**

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-2} = \mathfrak{so}(V) \oplus V$$

$$[A, B] = AB - BA$$

$$[A, v] = Av$$

$$[v, w] = 0$$

$$A, B \in \mathfrak{so}(V) \quad v, w \in V$$

“From *graded* to *filtered*”

Poincaré algebra is **graded**

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-2} = \mathfrak{so}(V) \oplus V$$

$$[A, B] = AB - BA$$

$$[A, v] = Av$$

$$[v, w] = 0$$

$$A, B \in \mathfrak{so}(V) \quad v, w \in V$$

Isometry algebra is **filtered**

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-2} = \mathfrak{h} \oplus V'$$

$$[A, B] = AB - BA$$

$$[A, v] = Av + \lambda(A, v)$$

$$[v, w] = 0 + \alpha(v, w) + \rho(v, w)$$

$$A, B \in \mathfrak{h} \quad v, w \in V'$$

“From *graded* to *filtered*”

Poincaré algebra is **graded**

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-2} = \mathfrak{so}(V) \oplus V$$

$$[A, B] = AB - BA$$

$$[A, v] = Av$$

$$[v, w] = 0$$

$$A, B \in \mathfrak{so}(V) \quad v, w \in V$$

Isometry algebra is **filtered**

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-2} = \mathfrak{h} \oplus V'$$

$$[A, B] = AB - BA$$

$$[A, v] = Av + \lambda(A, v)$$

$$[v, w] = 0 + \alpha(v, w) + \rho(v, w)$$

$$A, B \in \mathfrak{h} \quad v, w \in V'$$

curvature

d=11 SUGRA backgrounds

[Nahm ('79)]
[Cremmer+Julia+Scherk ('79)]

(M, g) lorentzian, spin, 11-dimensional

$\$ \to M$ spinor bundle, real, rank 32, symplectic

$F \in \Omega^4(M) \quad dF = 0$

$$\left.\begin{aligned} d \star F &= -\tfrac{1}{2} F \wedge F \\ \mathrm{Ric} - \tfrac{1}{2} R g &= T(g, F) \end{aligned}\right\}$$

(bosonic)
field equations

Supersymmetric backgrounds

A connection on spinors

$$D_X = \nabla_X - \frac{1}{24}(X \cdot F - 3F \cdot X)$$

A **supersymmetric** background admits **Killing spinors**

$$D\varepsilon = 0$$

$$\nu := \frac{\dim\{\varepsilon \in \$ \mid D\varepsilon = 0\}}{\text{rank } \$} \in \{0, \tfrac{1}{32}, \tfrac{1}{16}, \tfrac{3}{32}, \dots, \tfrac{31}{32}, 1\}$$

Can we classify them?

Gap phenomenon

Gap phenomenon

$\nu = 1$

Gap phenomenon

$\nu = 1$

$\nu = \frac{31}{32}$

Gap phenomenon

$\nu = 1$

$\nu \cancel{=} \frac{31}{32}$

Gap phenomenon

$\nu = 1$

$\nu = \frac{31}{32}$

$\nu = \frac{15}{16}$

Gap phenomenon

$$\nu = 1$$

$$\nu \not= \frac{31}{32}$$

$$\nu \not= \frac{15}{16}$$

Gap phenomenon

$\nu = 1$

$\nu \neq \frac{31}{32}$

$\nu \neq \frac{15}{16}$

?

Gap phenomenon

$\nu = 1$

~~$\nu = \frac{31}{32}$~~

~~$\nu = \frac{15}{16}$~~

?

$\nu = \frac{13}{16}$

Gap phenomenon

$\nu = 1$

~~$\nu = \frac{31}{32}$~~

~~$\nu = \frac{15}{16}$~~

?

$\nu = \frac{13}{16}$

Geometric analogy

Gap phenomenon

$\nu = 1$

~~$\nu = \frac{31}{32}$~~

~~$\nu = \frac{15}{16}$~~

?

$\nu = \frac{13}{16}$

Geometric analogy

$$\dim \mathfrak{isom}(M^n, g) \leq \frac{n(n+1)}{2}$$

Gap phenomenon

$\nu = 1$

~~$\nu = \frac{31}{32}$~~

~~$\nu = \frac{15}{16}$~~

?

$\nu = \frac{13}{16}$

Geometric analogy

$$\dim \mathfrak{isom}(M^n, g) \leq \frac{n(n+1)}{2}$$

What is the “submaximal” dimension?

Gap phenomenon

$\nu = 1$

$\nu = \frac{31}{32}$

$\nu = \frac{15}{16}$

?

$\nu = \frac{13}{16}$

Geometric analogy

$$\dim \mathfrak{isom}(M^n, g) \leq \frac{n(n+1)}{2}$$

What is the "submaximal" dimension?

$$\frac{n(n-1)}{2} + 1$$

Kaluza-Klein supergravity

Lots of work in the early-to-mid 1980s, on backgrounds which are metrically a product.

The main tools are homogeneous geometry and *riemannian* holonomy.

Closely related to the classification of homogeneous Einstein manifolds.

[Freund+Rubin ('80)]
[Englert ('82)]
[Pope+Warner ('85)]
[Castellani+Pope+Warner ('85)]
⋮
[Duff+Nilsson+Pope ('86)]

Holonomy (of D)

Trivial holonomy ⇒ classification for $\nu = 1$

[JMF+Papadopoulos ('02)]

Generic holonomy is $\mathrm{SL}(32, \mathbb{R})$ [Hull ('03)]

"Berger" table (if any) huge.

[Duff+Liu ('03)]
[Papadopoulos+Tsimpis ('03)]
[Batrachenko+Duff+Liu+Wen ('03)]

What replaces the torsion-free condition?

Spinorial geometry

General Ansätze for $\nu = \frac{1}{32}$

[Gauntlett+Pakis ('02)]
[Gauntlett+Gutowski+Pakis ('03)]

Rules out $\nu = \frac{31}{32}, \frac{30}{32}$

[Gran+Gutowski+Papadopoulos+Roest ('06)]
[JMF+Gadhia ('07)]
[Gran+Gutowski+Papadopoulos ('10)]

Generalised geometry

Flux compactifications, warped products,… $(\nu \leq \frac{1}{2})$

[Hull ('07)]
[Pacheco+Waldram ('08)]
[Coimbra+Strickland-Constable+Waldram ('11,'14,'16)]
[Coimbra+Strickland-Constable ('16)]

Based on earlier work on type II:

[Graña+Minasian+Petrini+Tomasiello ('04,'05,'06)]

Homogeneity

$\nu > \frac{1}{2}$ backgrounds are (conjecturally) **homogeneous**

[Meessen ('04)]

$\nu > \frac{1}{2}$ backgrounds are (locally) homogeneous

[JMF+Hustler ('12)]

Sharp: ∃ ½-BPS backgrounds which are ***not*** locally homogeneous

[Hull ('84)]
[Duff+Stelle ('91)]

Killing superalgebra

Lie superalgebra generated by the Killing spinors

$$\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$$

$$\mathfrak{k}_{\bar{0}} = \{\xi \in TM \mid \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0\}$$

$$\mathfrak{k}_{\bar{1}} = \{\varepsilon \in \$ \mid D\varepsilon = 0\}$$

⋮

[Acharya+JMF+Hull+Spence ('98)]
[Gauntlett+Myers+Townsend ('98)]
[Townsend ('98)]
[JMF ('99)]
[JMF+Meessen+Philip ('04)]

What kind of Lie superalgebra is it?

The Killing superalgebra is a *filtered* Lie superalgebra.

It is a *filtered deformation* of a graded subalgebra of the Poincaré superalgebra.

[JMF+Santi ('16)]

(*cf*. the isometry algebra is a filtered deformation of a graded subalgebra of the Poincaré algebra.)

Poincaré superalgebra is **graded**

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2}$$

Lorentz $\mathfrak{so}(V)$

supercharges S

translations V

$$[\mathfrak{p}_i, \mathfrak{p}_j] = \mathfrak{p}_{i+j}$$

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2}$$ graded subalgebra of $\mathfrak{p}$

$$\mathfrak{a}_0 \subset \mathfrak{so}(V) \qquad \mathfrak{a}_{-1} \subset S \qquad \mathfrak{a}_{-2} \subset V$$

Modify the Lie brackets by terms of positive degree:

$$[\mathfrak{a}_0, \mathfrak{a}_0] \subset \mathfrak{a}_0$$
$$[\mathfrak{a}_0, \mathfrak{a}_{-1}] \subset \mathfrak{a}_{-1}$$
$$[\mathfrak{a}_0, \mathfrak{a}_{-2}] \subset \mathfrak{a}_{-2} \oplus \mathfrak{a}_0$$
$$[\mathfrak{a}_{-1}, \mathfrak{a}_{-1}] \subset \mathfrak{a}_{-2} \oplus \mathfrak{a}_0$$
$$[\mathfrak{a}_{-1}, \mathfrak{a}_{-2}] \subset \mathfrak{a}_{-1}$$
$$[\mathfrak{a}_{-2}, \mathfrak{a}_{-2}] \subset \mathfrak{a}_0 \oplus \mathfrak{a}_{-2}$$

The Killing superalgebra is **filtered**

Spencer Cohomology

Deformations of algebraic structures are typically governed by a cohomology theory.

e.g., Lie algebra deformations are governed by Chevalley-Eilenberg cohomology.

Filtered deformations of graded Lie superalgebras are governed by generalised Spencer cohomology.

This is a bigraded refinement of Chevalley-Eilenberg cohomology.

Infinitesimal deformations

Infinitesimal deformations of a Lie (super)algebra are classified by the second Chevalley-Eilenberg cohomology:

$$H^2(\mathfrak{g}, \mathfrak{g})$$

Infinitesimal *filtered* deformations of a *graded* Lie (super)algebra are classified by the generalised Spencer cohomology:

$$H^{2,2}(\mathfrak{g}_-, \mathfrak{g})$$

[Cheng+Kac ('98)]

Some calculations

d=11 Poincaré superalgebra

[JMF+Santi ('16)]

d=11 Poincaré superalgebra

[JMF+Santi ('16)]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^4 V$$

d=11 Poincaré superalgebra

[JMF+Santi ('16)]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^4 V$$

We find the supergravity 4-form!

d=11 Poincaré superalgebra

[JMF+Santi ('16)]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^4 V$$

We find the supergravity 4-form!

The cocycle component $\beta : V \otimes S \to S$

of a class $F \in \Lambda^4 V$ is given by

$$\beta(v, s) = \tfrac{1}{24}(v \cdot F - 3F \cdot v) \cdot s$$

d=11 Poincaré superalgebra

[JMF+Santi ('16)]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^4 V$$

We find the supergravity 4-form!

The cocycle component $\beta : V \otimes S \to S$

of a class $F \in \Lambda^4 V$ is given by

$$\beta(v, s) = \frac{1}{24}(v \cdot F - 3F \cdot v) \cdot s$$

We find the gravitino variation!

d=4 Poincaré superalgebra

[de Medeiros+JMF+Santi ('16)]

d=4 Poincaré superalgebra

[de Medeiros+JMF+Santi ('16)]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V$$

d=4 Poincaré superalgebra

[de Medeiros+JMF+Santi ('16)]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V$$

We find the"old" minimal off-shell formulation of supergravity!

d=4 Poincaré superalgebra

[de Medeiros+JMF+Santi ('16)]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V$$

We find the"old" minimal off-shell formulation of supergravity!

(We also find the gravitino variation from the cocycle.)

Reconstruction

Every (geometrically) *realisable* filtered deformation of a graded subalgebra of the Poincaré superalgebra is (contained in) the Killing superalgebra of a >½-BPS supergravity background.

>½-BPS backgrounds can be reconstructed (up to local isometry) from their Killing superalgebra!

In particular, >½-BPS implies the field equations.

[JMF+Santi ('16)]

An Erlangen programme for supergravity?

Felix Klein (1849-1925)

“geometry via symmetry”

Equivalent classifications:

>½-BPS supergravity backgrounds

(up to local isometry)

&

"admissible" filtered deformations of graded subalgebras of the Poincaré superalgebra

(up to isomorphism)

Proofs of concept

(d=11) filtered deformations with 32 supercharges are *precisely* the Killing superalgebras of the maximally supersymmetric backgrounds (and nothing else).

[JMF+Santi ('15)]

(d=4) filtered deformations with 4 supercharges are *precisely* the Killing superalgebras of the Festuccia-Seiberg geometries (and nothing else).

[de Medeiros+JMF+Santi ('16)]

In progress…

- Six-dimensional (1,0) Poincaré: there seems to be ways other than supergravity to generate Lie superalgebras from spinors!

  [de Medeiros+JMF+Santi]

- Superconformal algebras in $d=3$ and $d=4$

  [de Medeiros+JMF]

- $N>1$ $d=4$ Poincaré

  [de Medeiros+JMF]

¡Feliz cumpleaños, Chris!