Witten-Nester Energy in TMG

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ArXive:09.0903.3779

- Why study 3D gravity?
- Some of the issues involved
- Search for a positive energy theorem

Duff Fest, April 1-2, 2009, London

Why 3D?

Quantum gravity in 3D is much simpler than in 4D. One may try to solve the theory exactly and understand the black hole physics, dynamics and all host of issues.

Which model in 3D?

$\int \sqrt{-g} R$?   Witten 1998

Exploited its being a CS theory. But there are problems in exact solubility due to singular configurations $g_{\mu\nu} = 0$, etc

$\int \sqrt{-g}(R - 2\Lambda)$?

Witten July'07, Maloney & Witten, Dec'07

Old problems avoided but the partition function fails to admit a physical interpretation (lack of holomorphic factorization, etc)

Most promising model:

Deser, Jackiw & Templeton'82 & Deser'84

$$\frac{1}{16\pi G}\int d^3x\left(e(R-2\Lambda)+\frac{1}{2\mu}\mathcal{L}_{CS}\right)$$

where $\mathcal{L}_{CS} = \mathrm{tr}(R\wedge\omega+\frac{2}{3}\omega\wedge\omega\wedge\omega)$

- Has propagating single helicity 2 state
- Admits BTZ black hole solution

- For $\mu\ell = 1$, has been conjectured by Li, Song & Strominger to be dual to a particular CFT with holomorphic partition function.

- Central charge at the boundary CFT:

$$c_L = \frac{3}{2G}\left(1 - \frac{\mu}{\ell}\right) \quad , \qquad c_L = \frac{3}{2G}\left(1 + \frac{\mu}{\ell}\right)$$

Nonnegative central charge $\Rightarrow$ $\boxed{\mu\ell \geq 1}$.

At chiral point $\boxed{c_L = 0 \ , c_R = \frac{3}{G}}$

- For $G > 0$, BTZ black hole mass > 0, but graviton energy < 0 unless $\mu\ell = 1$ where bulk graviton ceases to propagate in bulk.

- For $G < 0$, the graviton energy > 0, but BTZ black hole mass < 0

Deser & collaborators:

Take $G < 0$, show that there is a superselection sector in which the BTZ black holes are excluded.

Strominger & collaborators:

Take $G > 0$, and $\mu\ell = 1$, standard Brown-Henneaux (asymptotically AdS) boundary conditions, show that the energy is always positive, investigate the solution space, explore AdS/CFT.

Abbott-Deser-Tekin charges in TMG

(based on ADM and Abbott & Deser)

Start with the field equations:

$$\mathcal{E}_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} + \mu^{-1} C_{\mu\nu} = 0$$

where

$$\begin{aligned} \mathcal{G}_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R - m^2 g_{\mu\nu} \\ C_{\mu\nu} &= \varepsilon_{\mu}{}^{\rho\sigma} \nabla_\rho (R_{\sigma\nu} - \tfrac{1}{4} g_{\sigma\nu} R) \end{aligned}$$

For a given solution $\bar{g}_{\mu\nu}$, write

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Compute the linearized tensor $\delta\mathcal{E}_{\mu\nu}$. If a background admits a Killing vector ξ^μ, it follows from the Bianchi identity that $\delta\mathcal{E}^{\mu\nu}\xi_\nu$ is covariantly conserved. Thus we can write

$$\delta\mathcal{E}^{\mu\nu}\xi_\nu = \bar{\nabla}_\nu \mathcal{F}^{\mu\nu}$$

and define the conserved charge associated with Killing vectors ξ^μ as

$$Q(\xi) = \frac{1}{8\pi G}\int_\Sigma \delta\mathcal{E}^{\mu\nu}\xi_\nu \, d\Sigma_\mu = \frac{1}{8\pi G}\int_{\partial\Sigma} \mathcal{F}^{\mu\nu} d\Sigma_{\mu\nu}$$

Result for the boundary integral:

Deser & Tekin, Bouchareb & Clement

$$Q_{ADT}[\xi] = \frac{1}{8\pi G}\int_{\partial\Sigma}\left(\mathcal{F}_E^{\mu\nu}(\xi) + \frac{1}{\mu}\mathcal{F}_E^{\mu\nu}(\Xi) + \frac{1}{\mu}f_C^{\mu\nu}(\xi)\right) d\Sigma_{\mu\nu}$$

where

$$\mathcal{F}_E^{\mu\nu}(\xi) = \frac{1}{2}\left(\xi^\nu\bar{\nabla}_\lambda h^{\lambda\mu} + \xi_\lambda\bar{\nabla}^\mu h^{\lambda\nu} + \xi^\mu\bar{\nabla}^\nu h + \cdots\right)$$

$$f_C^{\mu\nu}(\xi) = \bar{\varepsilon}^{\mu\nu\rho}\delta\mathcal{G}_{\rho\sigma}\xi^\sigma$$

$$\Xi^\mu \equiv \frac{1}{2}\bar{\varepsilon}^{\mu\nu\rho}\bar{\nabla}_\nu\xi_\rho$$

$Q_{ADT}[\xi]$ for a timelike Killing vector is the energy. Is it positive for any solution with fixed boundary condition?

Consider the theory as the bosonic sector of an appropriate supergravity theory. Exploit the fact that formally

$$H = \frac{1}{\hbar} \sum_{\alpha} Q_{\alpha}^{2}$$

Deser & Teitelboim, 1977, Grisaru, 1978

Quantum theory and classical limit not under control (Witten'81), and this argument does not guarantee the positivity.

However, motivated by this, consider the Noether supercurrent associated with local supersymmetry, and study its susy variation in turn.

How do we construct the Noether supercurrent for which the associated conserved charges agree with $Q_{ADT}[\xi]$?

Start with $N=(1,0)$ supergravity.

$$
\begin{aligned}
e^{-1}\mathcal{L} \; = \; & R - 2\varepsilon^{\mu\nu\rho}\bar{\psi}_\mu D_\nu(\omega)\psi_\rho \\
& +2m^2 - m\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu \\
& -\frac{1}{4}\mu^{-1}\,\varepsilon^{\mu\nu\rho}\left(R_{\mu\nu}{}^{ab}\omega_{\rho ab} + \tfrac{2}{3}\omega_\mu^{ab}\omega_{\nu b}{}^{c}\omega_{\rho ca}\right) \\
& -\mu^{-1}\bar{R}^\mu\gamma_\nu\gamma_\mu R^\nu
\end{aligned}
$$

- Overall factor $1/16\pi G$ is set to one.

- Definition: $R^\mu = \varepsilon^{\mu\nu\rho}D_\nu(\omega)\psi_\rho$

Supersymmetry:

$$\delta e^a_\mu = \bar{\epsilon}\gamma^a\psi_\mu$$
$$\delta\psi_\mu = D_\mu(\omega)\epsilon - \frac{1}{2}m\gamma_\mu\epsilon \equiv \widehat{D}^L_\mu\epsilon$$

We have constructed the 1st order formulation of this model, and used the procedure developed by Silva, Henneaux & Julia, Regge & Teitelboim to construct the Noether supercurrent.

At the end we find:

$$\boxed{J^{\mu L}_\epsilon = \nabla_\nu U^{\mu\nu L}_\epsilon}$$

$$\boxed{U^{\mu\nu L}_\epsilon = 4\left(1+\frac{m}{2\mu}\right)\varepsilon^{\mu\nu\rho}\bar{\epsilon}\psi_\rho + \frac{2}{\mu}\bar{\epsilon}\gamma_\rho\gamma^{\mu\nu}R^\rho(\omega)}$$

L refers to $N=(1,0)$ susy leading to $SO(2,1)$ charges. We also have R charges from $N=(0,1)$ susy for which $m \to -m$ leads to $SO(2,1)_R$ charges.

The Witten-Nester charges

As a bulk integral:

$$Q^L_{WN} = \int_\Sigma \left(\delta_{\epsilon_2} J^{\mu L}_{\epsilon_1}\right) d\Sigma_\mu$$

As a boundary integral:

$$Q^L_{WN} = \int_{\partial\Sigma} \left(\delta_{\epsilon_2} U^{\mu\nu L}_{\epsilon_1}\right) d\Sigma_{\mu\nu}$$

Strategy:

- Under appropriate boundary conditions show that the boundary integral for Q_{WN} agrees precisely with Q_{ADT}
- Study the bulk integral representation of Q_{WN} to seek positivity property

- Attempt by Gibbons, Pope & E.S. uses the μ independent part of the supercurrent, and treats the CS term as a source.

- Deser approach: It is not clear if the supercurrents, and consequently the definition of the Witten-Nester charges are the same compared to ours.

We have shown that indeed $\boxed{Q_{WN} = Q_{ADT}}$ by studying the boundary integral. More on this slater. For now let us study the bulk integral.

The bulk integral and a bound on Q_{WN}

$$Q^L_{WN} = \int_\Sigma d\Sigma_\mu \Big[4\left(1+\tfrac{m}{\mu}\right)\Big(\widehat{\nabla}^L_\nu\bar{\epsilon}_1\gamma^{\mu\nu\rho}\widehat{\nabla}^L_\rho\epsilon_2 - \frac{1}{2\mu}C^{\mu\nu}\bar{\epsilon}_1\gamma_\nu\epsilon_2\Big) - \frac{2}{\mu^2}\nabla_\nu\left(\varepsilon^{\mu\nu\rho}C_{\rho\sigma}\bar{\epsilon}_1\gamma^\sigma\epsilon_2\right)\Big]$$

Not very instructive. A more useful form obtains by defining

$$\boxed{\widetilde{\nabla}^L_\mu\epsilon := \left(\nabla_\mu - \tfrac{1}{2}m\gamma_\mu - \tfrac{1}{2\mu(\mu+m)}C_{\mu\nu}\gamma^\nu\right)\epsilon}$$

Next, we impose a generalized version of the Witten spinor condition as

$$\gamma^i e_i{}^\mu\widetilde{\nabla}^L_\mu\epsilon = 0\ , \quad i = 1,2$$

Next, we use the identity $\gamma^{0ij} = -\gamma^i\gamma^0\gamma^j - \delta^{ij}\gamma^0$. This gives our key result:

$$Q^L_{WN} \;=\; 4\left(1+\frac{m}{\mu}\right)\int_\Sigma \left(\widetilde{\nabla}^L_i\epsilon_1\right)^\dagger\left(\widetilde{\nabla}^L_i\epsilon_2\right)\, {e_0}^\mu d\Sigma_\mu$$
$$-\frac{2}{\mu^3(\mu+m)}\int_\Sigma da\, X_{\mu\nu}\, u^\mu v^{(+)}$$

Spacelike initial value surface: $d\Sigma_\mu = u_\mu da$
$v^{(+)\nu} := \bar{\epsilon}\gamma^\nu\epsilon$ where ϵ obeys the generalized Witten condition, and approaches Killing spinors from $\widehat{D}^L_\mu\epsilon = 0$.

$$\boxed{X_{\mu\nu} = {C_\mu}^\lambda C_{\lambda\nu} - \tfrac{1}{2}g_{\mu\nu}C^2}$$

Thus we have the bounds:

$\mu \geq m :$ $$\boxed{Q^L_{WN} \geq -\frac{2}{\mu^3(\mu+m)} \int_\Sigma da\, X_{\mu\nu}\, u^\mu v^{(+)\nu}}$$

$\mu > m :$ $$\boxed{Q^R_{WN} \geq -\frac{2}{\mu^3(\mu-m)} \int_\Sigma da\, X_{\mu\nu}\, u^\mu v^{(-)\nu}}$$

$\mu = m :$ $$\boxed{Q^R_{WN} = 0}$$

with standard Brown-Henneaux b.c. assumed in the last case.

Boundary conditions (brief)

AdS metric: $d\bar{s}^2 = \ell^2 \left[-\cosh^2 \rho \, d\tau^2 + \sinh^2 \rho \, d\phi^2 + d\rho^2 \right]$

Let $h_{ab} = \bar{e}_a{}^\mu \bar{e}_b{}^\nu h_{\mu\nu}$.

$$
|\mu\ell| > 1: \quad
\begin{aligned}
h_{++} &= e^{-2\rho} f_{++} + \cdots \\
h_{+-} &= e^{-2\rho} f_{+-} + \cdots \\
h_{22} &= e^{-2\rho} f_{22} + \cdots \\
h_{--} &= e^{-2\rho} f_{--} + \cdots \\
h_{+2} &= e^{-3\rho} f_{+2} + \cdots \\
h_{-2} &= e^{-3\rho} f_{-2} + \cdots
\end{aligned}
$$

(f's depend only on τ and ϕ). These are the standard Brown-Henneaux b.c.

$$
|\mu\ell| = 1: \quad
\begin{aligned}
h_{++} &= e^{-2\rho} f_{++} + \cdots \\
h_{+-} &= e^{-2\rho} f_{+-} + \cdots \\
h_{22} &= e^{-2\rho} f_{22} + \cdots \\
h_{--} &= \rho e^{-2\rho} \tilde{f}_{--} + e^{-2\rho} f_{--} + \cdots \\
h_{+2} &= e^{-3\rho} f_{+2} + \cdots \\
h_{-2} &= \rho e^{-3\rho} \tilde{f}_{-2} + e^{-3\rho} f_{-2} + \cdots
\end{aligned}
$$

The $\tilde{f}$ terms represent slower fall-off terms.
These are the weak Brown-Henneaux b.c.

Grumiller & Johansson; Henneaux, Martinez & Troncoso

Chiral gravity of Li, Song & Strominger is defined with the standard Brown-Henneaux b.c.

What can we conclude from these results?

- The fact that for $G < 0$ the BTZ black hole has negative energy can already be deduced directly from the ADT formula.

  So, let us focus on $G > 0$, and to have nonnegative boundary CFT central charge $\mu\ell \geq 1$

- For $\mu\ell > 1$ there is negative energy perturbative helicity 2 state, even in case of standard Brown-Henneaux b.c., let us consider the case of $\mu\ell = 1$.

- For $\mu\ell = 1$, there exist negative energy solutions (Deser et al, Grumiller et al) if weak (i.e. logarithmic) Brown-Henneaux b.c. are allowed. So, let us focus on standard Brown-Henneaux b.c.

So, what can we deduce from our result in the case of chiral gravity? (i.e. $\mu\ell = 1$ and standard Brown-Henneaux b.c.) We have:

$$E_{WN} \geq -\frac{\ell^4}{32\pi G} \int_\Sigma \left(C_\mu{}^\lambda C_{\lambda\nu} - \frac{1}{2} g_{\mu\nu} C^2 \right) u^\mu \, v^\nu \, da \, ,$$

So, in particular (existence of regular generalized Witten spinor understood):

- Solutions with $\boxed{C_{\mu\nu} = 0}$, have positive energy. These are all locally, including the BTZ black hole.

- Spacetimes with Cotton tensor of the form $\boxed{C_{\mu\nu} = u_\mu u_\nu}$ where u^μ is a null vector have positive energy.

- If these are the only exact solutions of chiral gravity, then we have a positive energy theorem.

- At present, the only known exact solutions of chiral gravity are locally AdS type.

- Partition function of chiral gravity with based on locally AdS solutions have been computed recently by Maloney, Wei & Strominger who find a result that does have physical interpretation. Encouraging but see their paper for a list of what could still go wrong!