Duff-Fest

Squashed Einstein Metrics and Supergravity

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Mike Duff Superlative Conference:

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Based on work with

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Einstein Metrics in Kaluza-Klein Supergravity

- The Kaluza-Klein idea found its natural home after the discovery of supergravities and then superstrings, which require some form of "spontaneous compactification" in order to make contact with the four-dimensional world.

- The first example was the seven-sphere compactification of eleven-dimensional supergravity, to give maximal $\mathcal{N} = 8$ gauged $SO(8)$ supergravity in four dimensions (Duff, CNP, de Wit, Nicolai, Englert, van Nieuwenhuizen, Nilsson, Warner,...)

- Although we didn't realise it at the time (1982), we had actually stumbled on the first example of a consistent Pauli reduction; i.e. a consistent Kaluza-Klein reduction on a curved manifold that yields a finite set of fields including the gauge bosons of the full isometry group of the internal manifold.

- Pauli first had this idea in 1953, when he envisioned reducing 6-dimensional Einstein gravity on S^2, to give Einstein gravity coupled to $SU(2)$ Yang-Mills. But he realised that it wouldn't work, because of the inconsistency of the reduction.

- (Consistent reduction: Any solution of the lower-dimensional theory lifts, using the reduction ansatz, to give a solution of the higher-dimensional theory.)

- Eleven-dimensional supergravity and the seven-sphere evades the Pauli problem because of remarkable conspiracies.

The Squashed Seven-Sphere

- In the simplest type of compactification of eleven-dimensional supergravity, any compact seven-dimensional Einstein space of positive Ricci curvature can be substituted for the standard seven-sphere. (However, it will no longer give a *consistent* reduction.)

- Mathematicians knew of a second Einstein metric on the seven-sphere; initially, I played around with this just for fun, but then with Mike Duff and Mustafa Awada we realised that it is actually of interest in Kaluza-Klein supergravity.

- The family of "squashed seven-spheres" can be described in a variety of ways, including; The Distance Sphere in HP^2; Single-instanton $SU(2)$ bundle over S^4. There is a "squashing parameter," and for two values, the metric is Einstein. One is the usual round S^7, and the other is the "squashed S^7 Jensen Einstein metric.

- The squashed S^7 is a coset $Sp(2)/Sp(1)$ ($\sim SO(5)/SO(3)$), with isometry group $SO(5) \times SO(3)$. It has one Killing spinor, implying $\mathcal{N} = 1$ supersymmetry.

Simple Construction of Squashed S^7

Define the left-invariant 1-forms of $SO(5)$ to be $L_{AB} = -L_{BA}$, for $1 \le A \le 5$. They satisfy the Cartan-Maurer equations

$$dL_{AB} = L_{AC} \wedge L_{CB}$$

Decompose $A = (a, 5)$, so L_{ab} are left-invariant 1-forms for $SO(4)$ and L_{a5} are in the coset $SO(5)/SO(4)$.

View $SO(4)$ as $SU(2)_L \times SU(2)_R$, with left-invariant 1-forms

$$L_1 = L_{12} - L_{34}\,, \qquad L_2 = L_{23} - L_{14}\,, \qquad L_3 = L_{31} - L_{24}$$
$$R_1 = L_{12} + L_{34}\,, \qquad R_2 = L_{23} + L_{14}\,, \qquad R_3 = L_{31} + L_{24}$$

We then consider metrics on $SO(5)$

$$ds_{10}^2 = x_1\, L_{a5}^2 + x_2\, R_i^2 + x_3\, L_i^2$$

and factor out $SU(2)_L$ (à la Kaluza-Klein, sending $x_3 \to 0$) to get metrics on $S^7 = SO(5)/SU(2)_L$:

$$ds_7^2 = x_1\, L_{a5}^2 + x_2\, R_i^2$$

Taking $x_1 = 1$ w.o.l.o.g. we find two solutions to the Einstein equations $R_{\mu\nu} = \lambda\, g_{\mu\nu}$:

$$(x_1, x_2) \;=\; (1, 1), \qquad \lambda = \frac{3}{2}, \qquad I = \frac{7}{3}$$
$$(x_1, x_2) \;=\; (1, \frac{1}{5}), \qquad \lambda = \frac{27}{10}, \qquad I = \frac{1591}{243}$$

These are the round and the squashed Einstein metrics on S^7.

The quantity $I \equiv |\text{Riem}|^2/\lambda^2$ is a dimensionless invariant, which is useful for telling if two Einstein metrics are equivalent or not. If I is different for two metrics, they *must* be inequivalent.

Einstein Metrics on Group Manifolds

We can also look for Einstein metrics on the group manifold $SO(5)$ itself. Or on any group manifold. Suppose we have left-invariant 1-forms σ^a defined on a compact simple group manifold G. They will satisfy the Cartan-Maurer equations

$$d\sigma^a = -\tfrac{1}{2} f^a{}_{bc} \sigma^b \wedge \sigma^c$$

where $f^a{}_{bc}$ are the structure constants of the Lie algebra of G. The general G_L-invariant metric on G is (for constant x_{ab})

$$ds^2 = x_{ab}\, \sigma^a\, \sigma^b$$

In principle, finding Einstein metrics is just a mechanical exercise, of calculating R_{ab}, and solving the non-linear algebraic equations for x_{ab} that follow from $R_{ab} = \lambda g_{ab}$. In practice, this is intractably complicated. For example, for $SO(5)$ there are $\frac{1}{2} \cdot 10 \cdot 11 = 55$ constants x_{ab}; too many unknowns.

One solution is $x_{ab} = c\, \delta_{ab}$; this is the standard metric $\mathrm{tr}(g^{-1} dg)^2$, invariant under $G_L \times G_R$. What about squashed Einstein metrics?

By a theorem of D'Atri and Ziller, every compact simple group manifold except $SO(3)$ or $SU(2)$ admits a second, inequivalent, Einstein metric. Are there any more?

Einstein Metrics on Low-Dimensional Group Manifolds

The first non-trivial case is $SU(3)$, in eight dimensions. Introducing Hermitean left-invariant 1-forms ${L_A}^B$, $A = 1, 2, 3$, satisfying $d{L_A}^B = \mathrm{i}\, {L_A}^C \wedge {L_C}^B$, the $SO(3)_{\rm max}$ subgroup has left-invariant 1-forms H_i, and the coset $SU(3)/SO(3)_{\rm max}$ has 1-forms K_i:

$$
\begin{aligned}
H_1 &= \mathrm{i}\,({L_2}^3 - {L_3}^2)\,, \quad H_2 = \mathrm{i}\,({L_3}^1 - {L_1}^3)\,, \quad H_3 = \mathrm{i}\,({L_1}^2 - {L_2}^1)\,,\\
K_1 &= {L_2}^3 + {L_3}^2\,, \quad K_2 = {L_3}^1 + {L_1}^3\,, \quad K_3 = {L_1}^2 + {L_2}^1\,,\\
K_4 &= {L_1}^1 - {L_2}^2\,, \qquad K_5 = \frac{1}{\sqrt{3}}({L_1}^1 + {L_2}^2 - 2{L_3}^3)
\end{aligned}
$$

The family of $SU(3)$-invariant metrics

$$
ds_8^2 = x_1\,(K_1^2 + K_2^2 + K_3^2) + x_2\,(K_4^2 + K_5^2) + x_3\,(H_1^2 + H_2^2 + H_3^2)
$$

is Einstein, $R_{ab} = \lambda g_{ab}$, if

$$
\begin{aligned}
x_i = (1, 1, 1):& \quad |\mathrm{Riem}|^2/\lambda^2 = 8\\
x_i = (11, 11, 1):& \quad |\mathrm{Riem}|^2/\lambda^2 = \frac{764}{63}
\end{aligned}
$$

There is also a pseudo-Riemannian Einstein metric, with signature $(2, 6)$, with

$$x_i = \left(x, -\frac{(1-x)(1-5x)}{5x}, 1\right), \qquad 85x^3 - 29x^2 + 27x - 3 = 0$$

and x is the real root of the cubic; $x \approx 0.12130$.

The problem of solving the Einstein equations was rendered tractable here by making a very simple ansatz for the coefficients x_{ab} in the general class $ds^2 = x_{ab}\,\sigma^a \sigma^b$.

The ansatz we chose in the $SU(3)$ case was adapted to the embedding of the subgroup $SO(3)_{\rm max}$.

This suggests a general strategy when looking for Einstein metrics on the group manifold G: Make simplified ansätze adapted to the symmetries of the embedding of some subgroup H in G.

For $SU(3)$, we have found only the minimal 2 Riemannian Einstein metrics predicted by D'Atri and Ziller, plus the pseudo-Riemannian Einstein metric of signature $(2, 6)$.

Einstein Metrics on $SO(5)$

We have found a total of four inequivalent Einstein metrics on $SO(5)$, of which 2 appear to be new. We do this by considering two simple metric ansätze, one adapted to the canonical $SO(3)$ subgroup

$$SO(3) \subset SO(4) \subset SO(5)$$

and the other adapted to the $SO(3)_{\rm maximal}$ subgroup.

In the $SO(3)_{\rm canonical}$-adapted basis we write

$$ds_{10}^2 = x_1\, L_{i4}^2 + x_2\, L_{i5}^2 + x_3\, (L_{12}^2 + L_{23}^2 + L_{31}^2) + x_4\, L_{45}^2$$

($i = 1, 2, 3$), and find three Einstein metrics:

$$x_i = (1,1,1,1): \qquad |{\rm Riem}|^2/\lambda^2 = 10$$

$$x_i = (14,14,4,19): \qquad |{\rm Riem}|^2/\lambda^2 = \frac{240}{19}$$

$$x_i = (1,2,1,2): \qquad |{\rm Riem}|^2/\lambda^2 = \frac{98}{9}$$

In the $SO(3)_{\rm maximal}$-adapted basis we find one more Einstein metric, with

$$|{\rm Riem}|^2/\lambda^2 = \frac{16705}{1058}$$

Einstein Metrics on G_2

We have found a total of six inequivalent Einstein metrics on the G_2 group manifold, of which four appear to be new.

We find these by making metric ansätze adapted to the $SU(2)\times SU(2)$ sugbroup; the $SU(2)_{\text{diagonal}}$ sugbroup; and the $SU(2)_{\text{maximal}}$ sugbroup.

These give Einstein metrics with $\equiv |\text{Riem}|^2/\lambda^2$ given by

$$I = 14\,,\quad \frac{19346}{1369}\,,\quad 20.84\,,\quad 19.35\,,\quad 14.30\,,\quad \frac{5719}{260}$$

Further Remarks

- As well as the signature $(2,6)$ Einstein metric on $SU(3)$, we have also found pseudo-Riemannian Einstein metrics of signature $(4,6)$ on $G_2/U(2)$ and signature $(5,6)$ on $G_2/SU(2)$.

- It would be interesting to find Lorentzian signature $(1, D-1)$ Einstein metrics on group manifolds or coset spaces. Or maybe they cannot exist? (Obviously Lorentzian homogeneous metrics can exist, so the question is whether the Einstein condition might rule them out.)

- Since interesting Einstein metrics on simple compact group manifolds only appear in dimensions $D \geq 8$ (and they have positive Ricci tensor), they do not have any obvious application in string theory or M-theory. But interesting mathematical structures have a habit of finding a home in string or M-theory. Maybe these will too?