On interactions between non-threshold bound states in string theory

In celebrating Mike's 60th birthday

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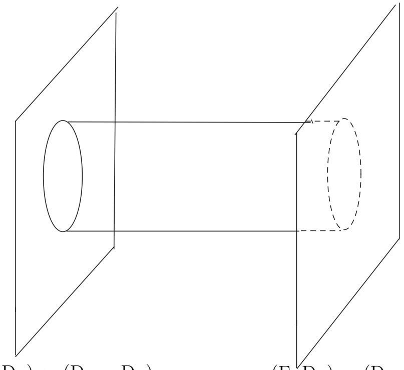
Outline

- Motivation and Intention
- Preliminaries for the stringy force calculations
- The Class I interactions
  (Two fluxes with at least one common index)
- The Class II interactions
  (Two fluxes with no common index)

Motivation & Intention

- The interaction between two (F, D$_p$) or two (D$_{p-2}$, D$_p$) or one (F, D$_p$) and one (D$_{p-2}$, D$_p$) at string level.



(F, Dp) or (D$_{p-2}$, Dp) (F, Dp) or (D$_{p-2}$, Dp)

Motivation & Intention

- Unlike the simple brane case, the force structure here is richer and more interesting to explore
- Examples include the onset of various instabilities and the open string pair production
- The boundary state representation of such bound states will be employed for the computations.
- The long range force can be calculated independently from the low energy description of bulk massless fields and the bound states (but will not present here).

The boundary state

In the closed string operator formalism, 1/2 BPS D-branes of Type II theories can be described by means of boundary states. The GSO projected boundary state in either NSNS or RR sector is given

$$|B\rangle_{\rm NS} = \frac{1}{2}\left[|B,+\rangle_{\rm NS} - |B,-\rangle_{\rm NS}\right], \tag{2.1}$$

$$|B\rangle_{\rm R} = \frac{1}{2}\left[|B,+\rangle_{\rm R} + |B,-\rangle_{\rm R}\right], \tag{2.2}$$

where

$$|B,\eta\rangle = \frac{c_p}{2}|B_{\rm mat},\eta\rangle|B_{\rm g},\eta\rangle, \tag{2.3}$$

with $\eta = \pm$, $c_p = \sqrt{\pi}\left(2\pi\sqrt{\alpha'}\right)^{3-p}$ and

$$|B_{\rm mat},\eta\rangle = |B_X\rangle|B_\psi,\eta\rangle, \qquad |B_{\rm g},\eta\rangle = |B_{gh}\rangle|B_{sgh},\eta\rangle. \tag{2.4}$$

The boundary state

$$|B_X\rangle = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\cdot S\cdot\tilde{\alpha}_{-n}\right]|B_X\rangle^{(0)}, \qquad (2.5)$$

$$|B_\psi,\eta\rangle_{\rm NS} = -\mathrm{i}\,\exp\left[i\eta\sum_{m=1/2}^{\infty}\psi_{-m}\cdot S\cdot\tilde{\psi}_{-m}\right]|0\rangle, \qquad (2.6)$$

$$|B_\psi,\eta\rangle_{\rm R} = -\exp\left[i\eta\sum_{m=1}^{\infty}\psi_{-m}\cdot S\cdot\tilde{\psi}_{-m}\right]|B,\eta\rangle_{\rm R}^{(0)} \qquad (2.7)$$

The boundary state

$$S = \left(\left[(\eta - \hat{F})(\eta + \hat{F})^{-1} \right]_{\alpha\beta}, -\delta_{ij} \right), \qquad (2.8)$$

$$|B_X\rangle^{(0)} = \sqrt{-\det\left(\eta + \hat{F}\right)}\, \delta^{9-p}(q^i - y^i) \prod_{\mu=0}^{9} |k^\mu = 0\rangle, \qquad (2.9)$$

$$|B_\psi, \eta\rangle_{\mathrm{R}}^{(0)} = \left(C\Gamma^0\Gamma^1 \cdots \Gamma^p \frac{1 + \mathrm{i}\eta\Gamma_{11}}{1 + \mathrm{i}\eta} U \right)_{AB} |A\rangle|\tilde{B}\rangle, \qquad (2.10)$$

with $\hat{F} = 2\pi\alpha' F$ and

$$U = \frac{1}{\sqrt{-\det(\eta + \hat{F})}}; \exp\left(-\frac{1}{2}\hat{F}_{\alpha\beta}\Gamma^\alpha\Gamma^\beta \right); \qquad (2.11)$$

(for example, see Di Vecchia & Licciardo hep-th/9912161)

The stringy interaction

The interaction can be calculated via the vacuum amplitude

$$\Gamma = \Gamma_{\rm NS} + \Gamma_{\rm R} \tag{2.12}$$

where

$$\Gamma_{\rm NS/R} = {}_{NS/R}\langle B^1|D|B^2\rangle_{NS/R}, \tag{2.13}$$

with the propagator

$$D = \frac{\alpha'}{4\pi}\int_{|z|\le 1}\frac{d^2z}{|z|^2} z^{L_0}\bar{z}^{\tilde{L}_0}, \tag{2.14}$$

for example, $L_0 = L_0^X + L_0^\psi + L_0^{gh} + L_0^{sgh}$.

The stringy interaction

To calculate $\Gamma_{\mathrm{NS/R}}$, need to calculate the following first

$$\begin{aligned}\Gamma(\eta', \eta) &= \langle B^1, \eta'|D|B^2, \eta\rangle \\ &= \frac{n_1 n_2 c_p^2}{4}\frac{\alpha'}{4\pi}\int_{|z|\leq 1}\frac{d^2 z}{|z|^2} A^X\, A^{bc}\, A^{\psi}(\eta', \eta)\, A^{\beta\gamma}(\eta', \eta),\end{aligned} \tag{2.15}$$

where we have replaced c_p by nc_p to count the multiplicity of D$_p$ branes in the bound state and $\eta\,\eta' = \pm$. The above respective matrix elements, considering $\tilde{L}_0|B\rangle = L_0|B\rangle$, are

$$\begin{aligned} A^X &= \langle B^1_X||z|^{2L_0^X}|B^2_X\rangle, & A^{\psi}(\eta', \eta) &= \langle B^1_{\psi}, \eta'||z|^{2L_0^{\psi}}|B^2_{\psi}, \eta\rangle, \\ A^{bc} &= \langle B^1_{gh}||z|^{2L_0^{gh}}|B^2_{gh}\rangle, & A^{\beta\gamma}(\eta', \eta) &= \langle B^1_{sgh}, \eta'||z|^{2L_0^{sgh}}|B^2_{sgh}, \eta\rangle.\end{aligned} \tag{2.16}$$

The stringy interaction

Let us denote the boundary states $\langle B^1, \eta'|$ and $|B^2, \eta\rangle$ above as BS1 and BS2, respectively. Without loss of generality, we can always choose the external flux $\hat{F}_1$ associated with BS1 the following way for simplicity. When this boundary state is the type of (F, D$_p$), we choose $\hat{F}_1$ as

$$\hat{F}_1 = \begin{pmatrix} 0 & -f_1 & & & & \\ f_1 & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 0 \end{pmatrix}_{(p+1)\times(p+1)} . \qquad (2.17)$$

The stringy interaction

The corresponding longitudinal part of the S matrix is now

$$S_{1\alpha\beta} = \begin{pmatrix} -\frac{1+f_1^2}{1-f_1^2} & \frac{2f_1}{1-f_1^2} & & & \\ -\frac{2f_1}{1-f_2^2} & \frac{1+f_1^2}{1-f_1^2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}_{(p+1)\times(p+1)} . \quad (2.18)$$

The stringy interaction

While for the boundary state being (D_{p-2}, D_p), we choose the $\hat{F}_1$ as

$$\hat{F}_1 = \begin{pmatrix} 0 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & & \\ & & & & 0 & -f_1 \\ & & & & f_1 & 0 \end{pmatrix}_{(p+1)\times(p+1)} . \qquad (2.19)$$

The stringy interaction

We then have the longitudinal part of the S matrix as

$$S_{1\alpha\beta} = \begin{pmatrix} -1 & & & & & & \\ & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \frac{1-f_1^2}{1+f_1^2} & \frac{2f_1}{1+f_1^2} \\ & & & & & -\frac{2f_1}{1+f_1^2} & \frac{1-f_1^2}{1+f_1^2} \end{pmatrix}_{(p+1)\times(p+1)} . \quad (2.20)$$

The stringy interaction

With the above choice for $\hat{F}_1$, we can choose the non-vanishing components of $\hat{F}_2$ for BS2 so all possible vacuum amplitudes can be evaluated. These can be grouped as the following three cases:

- The two bound states are both of the type (F, D$_p$) and $\hat{F}_{20a} = -\hat{F}_{2a0} = -f_2$ with $a = 1 \,\mathrm{or}\, a \neq 1$,
- The two bound states are both of the type (D$_{p-2}$, D$_p$) and $\hat{F}_{2bc} = -\hat{F}_{2cb} = -f_2$ with $c > b$ and $b = p - 1 \,\mathrm{or}\, b \neq p - 1$,
- BS1 and BS2 are of different type, for definiteness choosing BS1 to be of (F, D$_p$), then $\hat{F}_{2bc} = -\hat{F}_{2cb} = -f_2$ with $c > b$ and $b = 1 \,\mathrm{or}\, b \neq 1$.

The stringy interaction

The various matrix elements mentioned above and the amplitudes can be expressed uniformly when the two fluxes share at least one common index or share no common index. The latter class is more general and includes the former class as a special case. For this reason, we will perform the calculations according to whether the two fluxes share at least one common index or not as follows:

- Class I: The two fluxes share at least one common index,
- Class II: The two fluxes share no common index

The stringy interaction

Before proceeding, we would like to point out one key trick in evaluating the matrix A^X or $A^\psi(\eta', \eta)$. The important property for the S matrix is

$$S^T{}_\mu{}^\rho S_\rho{}^\nu = \delta_\mu{}^\nu, \tag{2.21}$$

with T denoting the transpose. Denote the S-matrix for BS1 as S_1 and for BS2 as S_2. Then the above property enables to perform unitary transformations of the respective operators in the boundary states such that the S matrix appearing, for example, in BS1 completely disappears, while leaving BS2 with a new S matrix as $S = S_2 S_1^T$, in the course of evaluating the respective A^X or A^ψ. This new S matrix shares the same property as the original S_1 and S_2 do but its determinant is always equal to one. Therefore this S matrix under consideration can always be diagonalized to give the structure we are going to present next.

Class I

We are now ready to present the explicit matrix elements and the corresponding amplitude for this class. The matrix elements are

$$
\begin{aligned}
A^X &= C_F\, V_{p+1}\, e^{-\frac{Y^2}{2\pi\alpha' t}} \left(2\pi^2\alpha'\, t\right)^{-\frac{9-p}{2}} \\
&\quad \times \prod_{n=1}^{\infty} \frac{1}{(1-\lambda|z|^{2n})(1-\lambda^{-1}|z|^{2n})(1-|z|^{2n})^8}, \\
A^{bc} &= |z|^{-2} \prod_{n=1}^{\infty} (1-|z|^{2n})^2, \qquad (3.1)
\end{aligned}
$$

$$
A^{\beta\gamma}_{\rm NS}(\eta', \eta) = |z| \prod_{n=1}^{\infty} \frac{1}{(1+\eta'\eta\,|z|^{2n-1})^2},
$$

$$
A^{\psi}_{\rm NS} = \prod_{n=1}^{\infty} (1+\eta'\eta\,\lambda|z|^{2n-1})(1+\eta'\eta\,\lambda^{-1}\,|z|^{2n-1})(1+\eta'\eta\,|z|^{2n-1})^8, \qquad (3.2)
$$

Class I

$$A_{\rm R}^{\beta\gamma}(\eta',\eta)A_{\rm R}^{\psi}(\eta',\eta) = -2^4\,|z|^2\,D_F\,\delta_{\eta'\eta,+}$$
$$\times\prod_{n=1}^{\infty}(1+\lambda\,|z|^{2n})(1+\lambda^{-1}\,|z|^{2n})(1+|z|^{2n})^6. \quad (3.3)$$

In the above, C_F, D_F and the eigenvalues λ and λ^{-1} can be given explicitly in terms of the fluxes f_1 and f_2 for each case in this class. A few sample cases are

$$C_F = \begin{cases} \sqrt{(1-f_1^2)(1-f_2^2)}, & (\hat F_1)_{01} = -f_1,\ (\hat F_2)_{01} = -f_2 \\ \sqrt{(1-f_1^2)(1-f_2^2)}, & (\hat F_1)_{01} = -f_1,\ (\hat F_2)_{02} = -f_2 \\ \sqrt{(1-f_1^2)(1+f_2^2)}, & (\hat F_1)_{01} = -f_1,\ (\hat F_2)_{12} = -f_2 \\ \sqrt{(1+f_1^2)(1+f_2^2)}, & (\hat F_1)_{(p-1)p} = -f_1,\ (\hat F_2)_{(p-1)p} = -f_2 \end{cases} \quad (3.4)$$

Class I

$$D_F = \begin{cases} \frac{1-f_1 f_2}{\sqrt{(1-f_1^2)(1-f_2^2)}}, & (\hat{F}_1)_{01} = -f_1,\ (\hat{F}_2)_{01} = -f_2 \\ \frac{1}{\sqrt{(1-f_1^2)(1-f_2^2)}}, & (\hat{F}_1)_{01} = -f_1,\ (\hat{F}_2)_{02} = -f_2 \\ \frac{1}{\sqrt{(1-f_1^2)(1+f_2^2)}}, & (\hat{F}_1)_{01} = -f_1,\ (\hat{F}_2)_{12} = -f_2 \\ \frac{1+f_1 f_2}{\sqrt{(1+f_1^2)(1+f_2^2)}}, & (\hat{F}_1)_{(p-1)p} = -f_1,\ (\hat{F}_2)_{(p-1)p} = -f_2 \end{cases} \tag{3.5}$$

Class I

$$\lambda+\lambda^{-1} = \begin{cases} 2\frac{(1+f_1^2)(1+f_2^2)-4f_1f_2}{(1-f_1^2)(1-f_2^2)}, & (\hat{F}_1)_{01} = -f_1,\ (\hat{F}_2)_{01} = -f_2 \\ 2\frac{1+f_1^2+f_2^2-f_1^2f_2^2}{(1-f_1^2)(1-f_2^2)} & (\hat{F}_1)_{01} = -f_1,\ (\hat{F}_2)_{02} = -f_2 \\ 2\frac{1+f_1^2-f_2^2+f_1^2f_2^2}{(1-f_1^2)(1+f_2^2)} & (\hat{F}_1)_{01} = -f_1,\ (\hat{F}_2)_{12} = -f_2 \\ 2\frac{(1-f_1^2)(1-f_2^2)+4f_1f_2}{(1+f_1^2)(1+f_2^2)}, & (\hat{F}_1)_{(p-1)p} = -f_1,\ (\hat{F}_2)_{(p-1)p} = -f_2 \end{cases} \tag{3.6}$$

Class I

The interaction amplitude can be cast in a very simple and universal form as

$$\begin{aligned}\Gamma &= \frac{2\, n_1 n_2\, V_{p+1}\, C_F\, \sin \pi\nu}{(8\pi^2\alpha')^{\frac{1+p}{2}}} \int_0^\infty \frac{dt}{t}\, e^{-\frac{Y^2}{2\pi\alpha' t}}\, t^{-\frac{7-p}{2}}\, \frac{1}{\eta^9(it)}\, \frac{\theta_1^4(\frac{\nu}{2}|it)}{\theta_1(\nu|it)} \\ &= \frac{2^4\, n_1 n_2\, V_{p+1} C_F\, \sin^4 \frac{\pi\nu}{2}}{(8\pi^2\alpha')^{\frac{1+p}{2}}} \int_0^\infty \frac{dt}{t}\, e^{-\frac{Y^2}{2\pi\alpha' t}}\, t^{-\frac{7-p}{2}} \\ &\quad \times \prod_{n=1}^{\infty} \frac{\left(1 - e^{i\pi\nu}|z|^{2n}\right)^4 \left(1 - e^{-i\pi\nu}|z|^{2n}\right)^4}{(1-|z|^{2n})^6\, \left(1 - e^{2i\pi\nu}|z|^{2n}\right)\, \left(1 - e^{-2i\pi\nu}|z|^{2n}\right)} \end{aligned} \tag{3.7}$$

when the following Fundamental Jacobian identity is used

$$2\,\theta_1^4(\nu|\tau) = \theta_3(2\nu|\tau)\,\theta_3^3(0|\tau) - \theta_4(2\nu|\tau)\,\theta_4^3(0|\tau) - \theta_2(2\nu|\tau)\,\theta_2^3(0|\tau) \tag{3.8}$$

Class I

In the above, $|z| = e^{-\pi t}$ and we have also set $\lambda = e^{2i\pi\nu}$. Note that $\cos\pi\nu = D_F$, $\nu = i\nu_0$ with $0 \leq \nu_0 < \infty$ for electric or electrically dominant case and $\nu = \nu_0$ with $0 \leq \nu_0 < 1/2$ for magnetic or magnetically dominant case.

$\Gamma > 0$ (attractive) for magnetic case for which $\nu = \nu_0$ is real while this remains true only for large separation for electric case for which $\nu = i\nu_0$ is imaginary. This can be seen from the sign of the following

$$\left(1 - e^{2i\pi\nu}|z|^{2n}\right)\left(1 - e^{-2i\pi\nu}|z|^{2n}\right) = \left(1 - 2\cos 2\pi\nu\,|z|^{2n} + |z|^{4n}\right), \tag{3.9}$$

which can be negative for small t when ν is imaginary.

Class I

The above implies that the sign of small separation Γ remains uncertain and so expect some interesting physics to appear in this case for small t.

To explore the small t behavior of amplitude, i.e., the physics for small Y, we need to make a Jacobi transformation by sending $t \to t' = 1/t$, converting the cylinder-diagram to an annulus diagram for which the open string description is relevant.

Class I

Before proceeding, we pause to ask: can the cylinder-diagram amplitude vanish? SUSY preservation? Since $C_F \neq 0$,

$$\sin^4 \frac{\pi\nu}{2} = \frac{1}{4}(D_F - 1)^2 = 0 \Rightarrow \Gamma = 0. \tag{3.10}$$

So $D_F = 1$ along with $n_1 n_2 > 0$ is a SUSY preserving condition. This can occur in the following cases:

- $(\hat{F}_1)_{0a} = (\hat{F}_2)_{0a}$ or $(\hat{F}_1)_{ab} = (\hat{F}_2)_{ab}$, preserve 1/2 SUSY
- $(\hat{F}_1)_{01} = -f_1$, $(\hat{F}_2)_{1c} = -f_2$ with

$$f_2 = \pm \frac{f_1}{\sqrt{1 - f_1^2}}, \tag{3.11}$$

preserve only 1/4 SUSY.

Class I

Using the identities

$$\eta(\tau) = \frac{1}{(-i\tau)^{1/2}}\eta\left(-\frac{1}{\tau}\right),$$
$$\theta_1(\nu|\tau) = i\frac{e^{-i\pi\nu^2/\tau}}{(-i\tau)^{1/2}}\theta_1\left(\frac{\nu}{\tau}\Big| -\frac{1}{\tau}\right), \qquad (3.12)$$

the annulus-diagram amplitude is ($|z| = e^{-\pi t'}$)

$$i\,\Gamma = \frac{2n_1n_2V_{p+1}\,C_F\;\sin\pi\nu}{(8\pi^2\alpha')^{\frac{1+p}{2}}}\int_0^\infty \frac{dt'}{t'}\,e^{-\frac{Y^2t'}{2\pi\alpha'}}\,t'^{\frac{1-p}{2}}\frac{1}{\eta^9(it')}\frac{\theta_1^4(\frac{-i\nu t'}{2}|it')}{\theta_1(-i\nu t'|it')},$$

$$= \frac{2^4n_1n_2V_{p+1}\,C_F\;\sin\pi\nu}{(8\pi^2\alpha')^{\frac{1+p}{2}}}\int_0^\infty \frac{dt'}{t'}\,e^{-\frac{Y^2t'}{2\pi\alpha'}}\,t'^{\frac{1-p}{2}}\frac{\sin^4\left(\frac{-i\pi\nu t'}{2}\right)}{\sin\left(-i\pi\nu t'\right)}$$

$$\times\prod_{n=1}^{\infty}\frac{\left(1-e^{\pi\nu t'}|z|^{2n}\right)^4\left(1-e^{-\pi\nu t'}|z|^{2n}\right)^4}{(1-|z|^{2n})^6\,(1-e^{2\pi\nu t'}|z|^{2n})\,(1-e^{-2\pi\nu t'}|z|^{2n})}, \qquad (3.13)$$

Class I

For magnetic case, $0 < \nu = \nu_0 < 1/2$ and $\Gamma > 0$ and has no singularity unless $Y \leq \pi\sqrt{2\nu\alpha'}$, i.e., on the order of string scale, for which the integrand is dominated by, in the short cylinder limit $t' \to \infty$,

$$\begin{aligned} \lim_{t'\to\infty} \frac{e^{-\frac{Y^2 t'}{2\pi\alpha'}}\theta_1(-i\pi\nu t'/2|it')}{i\,\eta(it')\theta_1(-i\pi\nu t'|it')} &\sim \lim_{t'\to\infty} \frac{e^{-\frac{Y^2 t'}{2\pi\alpha'}}\sin^4(-i\pi\nu t'/2)}{i\,\sin(-i\pi\nu t')} \\ &\sim \lim_{t'\to\infty} e^{-\frac{t'}{2\pi\alpha'}(Y^2 - 2\pi^2\nu\alpha')}, \quad (3.14) \end{aligned}$$

signalling the onset of tachyonic instability such that the system can relax itself to form new non-threshold bound state.

Class I

For electric case, $\nu = i\nu_0$ with $0 < \nu_0 < \infty$ and the integrand has now an infinite number of simple poles on the positive real t'-axis at $t' = (2k+1)/\nu_0$ with $k = 0, 1, \cdots$.

This leads to an imaginary part of the amplitude, given as sum of residues of the simple poles (See, Bachas& Porrati, PLB296 (1992) 77). This gives rise to the rate of pair production of open strings per unit worldvolume as

$$\begin{aligned} \mathcal{W} &\equiv -\frac{2\mathrm{Im}\Gamma}{V_{p+1}}, \\ &= \frac{32 n_1 n_2 \tanh \pi\nu_0}{\nu_0 (8\pi^2\alpha')^{\frac{1+p}{2}}} \sum_{k=0}^{\infty} \left(\frac{\nu_0}{2k+1}\right)^{\frac{1+p}{2}} e^{-\frac{(2k+1)Y^2}{2\pi\nu_0\alpha'}} \\ &\quad \times \prod_{n=1}^{\infty} \left(\frac{1+e^{-2n(2k+1)\pi/\nu_0}}{1-e^{-2n(2k+1)\pi/\nu_0}}\right)^8 . \end{aligned} \tag{3.15}$$

Class I

Critical field limit, i.e, $\nu_0 \to \infty$, gives that each term in the sum diverges and so does the rate, therefore signalling the onset of a new singularity. Weak field implies small ν_0 and the rate is now given by the leading (k = 0) term in the sum as

$$\mathcal{W} \approx 32 n_1 n_2 \, \pi \left(\frac{\nu_0}{8\pi^2 \alpha'} \right)^{\frac{1+p}{2}} e^{-\frac{Y^2}{2\pi\nu_0\alpha'}}, \tag{3.16}$$

very tiny as expected.

Class II

We have only two distinct cases now, namely,

- Case I: $(\hat{F}_1)_{01} = -f_1$, $(\hat{F}_2)_{23} = -f_2$ ($p \geq 3$) or
- Case II: $(\hat{F}_1)_{12} = -f_1$, $(\hat{F}_2)_{34} = -f_2$ ($p \geq 4$)

Class II

Following the same steps as in Class I, we have the amplitude

$$
\begin{aligned}
\Gamma &= \frac{4\, n_1 n_2\, V_{p+1}\, \tan\pi\nu_1\, \tan\pi\nu_2}{(8\pi^2\alpha')^{\frac{1+p}{2}}} \int_0^\infty \frac{dt}{t}\, e^{-\frac{Y^2}{2\pi\alpha' t}}\, t^{-\frac{7-p}{2}} \\
&\quad \times \frac{1}{\eta^6(it)} \frac{\theta_1^2\left(\frac{\nu_1-\nu_2}{2}\middle| it\right)\, \theta_1^2\left(\frac{\nu_1+\nu_2}{2}\middle| it\right)}{\theta_1(\nu_1|it)\theta_1(\nu_2|it)}, \\
&= \frac{2^4\, n_1 n_2\, V_{p+1}\, C_F \sin^2\frac{\pi(\nu_1-\nu_2)}{2}\, \sin^2\frac{\pi(\nu_1+\nu_2)}{2}}{(8\pi^2\alpha')^{\frac{1+p}{2}}} \\
&\quad \times \int_0^\infty \frac{dt}{t}\, e^{-\frac{Y^2}{2\pi\alpha' t}}\, t^{-\frac{7-p}{2}} \prod_{n=1}^\infty \frac{1}{(1-|z|^{2n})^4} \\
&\quad \times \prod_{j=1}^2 \frac{(1-e^{\pi i(\nu_1+(-)^j\nu_2)}|z|^{2n})^2(1-e^{-\pi i(\nu_1+(-)^j\nu_2)}|z|^{2n})^2}{(1-e^{2\pi i\nu_j}|z|^{2n})(1-e^{-2\pi i\nu_j}|z|^{2n})},
\end{aligned}
\tag{4.1}
$$

Class II

In obtaining the above, we have used the following Jacobi identity

$$2\,\theta_1^2\left(\frac{\nu_1-\nu_2}{2}\Big|\tau\right)\theta_1^2\left(\frac{\nu_1+\nu_2}{2}\Big|\tau\right)=\theta_3(\nu_1|\tau)\theta_3(\nu_2|\tau)\theta^2(0|\tau)$$
$$-\theta_4(\nu_1|\tau)\theta_4(\nu_2|\tau)\theta_4^2(0|\tau)-\theta_2(\nu_1|\tau)\theta_2(\nu_2|\tau)\theta_2(0|\tau), \qquad (4.2)$$

we have also defined

$$\lambda_j = e^{2i\pi\nu_j} \qquad (4.3)$$

with $j = 1, 2$ and we have

$$\tan\pi\nu_j = \begin{cases} i|f_j| & \text{for electric flux,} \\ |f_j| & \text{for magnetic flux,} \end{cases} \qquad (4.4)$$

So for an electric flux, $\nu_j = i\nu_{j0}$ with $0 < \nu_{j0} < \infty$ since $0 < |f_j| < 1$ while for a magnetic flux $\nu_j = \nu_{j0}$ with $0 < \nu_{j0} < 1/2$ since $0 < |f_j| < \infty$.

Class II

A few properties of the amplitude:

- The amplitude (4.7) is reduced to the previous one (3.7) when we set ν_1 or $\nu_2 \to 0$, therefore it is more general.
- The nature of force remains the same as before and can also be discussed similarly.
- For the present case, the force can vanish only for the fluxes being both magnetic, i.e., Case II and this happens when $n_1 n_2 > 0$ and $|f_1| = |f_2|$. The underlying system preserves 1/4 SUSY.
- There are a few new things occurring at small Y (or small t) for Case II, i.e., the case with one electric flux and one magnetic flux, which is our focus next.

Class II

For Case I, the small t physics of the amplitude can be best described in terms of annulus amplitude which can be obtained as before using (3.12). We have now

$$\begin{aligned}\Gamma &= \frac{4\,n_1 n_2\,V_{p+1}\ \tanh\pi\nu_{10}\ \tan\pi\nu_{20}}{(8\pi^2\alpha')^{\frac{1+p}{2}}}\int_0^\infty \frac{dt'}{t'}\,e^{-\frac{Y^2 t'}{2\pi\alpha'}}\,t'^{\frac{3-p}{2}}\\ &\times\frac{(\cos\pi\nu_{10}t' - \cosh\pi\nu_{20}t')^2}{\sin(\pi\nu_{10}t')\sinh(\pi\nu_{20}t')}\\ &\times\prod_{n=1}^{\infty}\frac{1}{(1-|z|^{2n})^4(1-e^{2\pi\nu_{20}t'}|z|^{2n})(1-e^{-2\pi\nu_{20}t'}|z|^{2n})}\\ &\times\frac{\prod_{j=1}^{2}(1-e^{\pi(i\nu_{10}+(-)^j\nu_{20})t'}|z|^{2n})^2(1-e^{-\pi(i\nu_{10}+(-)^j\nu_{20})t'}|z|^{2n})^2}{1-2|z|^{2n}\cos 2\pi\nu_{10}t' + |z|^{4n}}\end{aligned} \tag{4.5}$$

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- This amplitude has an infinite number of simple poles occurring on the positive real t'-axis at $t'_k = k/\nu_0$ with $k = 1, 2, \cdots$.
- Therefore this amplitude has an imaginary part which is given as sum of the residues of these simple poles. It gives the rate of pair production of open strings per unit worldvolume as

$$\begin{aligned}
\mathcal{W} \;&=\; \frac{8 n_1 n_2 \tanh \pi\nu_{10} \tan \pi\nu_{20}}{\nu_{10}} \sum_{k=1}^{\infty} (-)^{k+1} \left(\frac{\nu_{10}}{8k\pi^2\alpha'}\right)^{\frac{1+p}{2}} \\
&\times e^{-\frac{kY^2}{2\pi\nu_{10}\alpha'}} \frac{\left[\cosh \frac{k\pi\nu_{20}}{\nu_{10}} - (-)^k\right]^2}{\frac{\nu_{10}}{k} \sinh \frac{k\pi\nu_{20}}{\nu_{10}}} \\
&\times \prod_{n=1}^{\infty} \frac{\left[1 - 2(-)^k e^{-\frac{2nk\pi}{\nu_{10}}} \cosh \frac{k\pi\nu_{20}}{\nu_{10}} + e^{-\frac{4nk\pi}{\nu_{10}}}\right]^4}{\left[1 - e^{-\frac{2nk\pi}{\nu_{10}}}\right]^6 \left[1 - e^{-\frac{2k\pi}{\nu_{10}}(n-\nu_{20})}\right] \left[1 - e^{-\frac{2k\pi}{\nu_{10}}(n+\nu_{20})}\right]}
\end{aligned}$$

(4.6

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- The above rate reduces to the one (3.15) given in Class I when we set $\nu_{20} \to 0$ (due to the magnetic flux) and $\nu_{10} = \nu_0$.
- The rate is highly suppressed by the separation and the integer k and for each given k the corresponding term appears likely enhanced by both ν_{10} and ν_{20}.
- The latter is particularly evident for large magnetic flux for which $\nu_{20} \to 1/2$ and the front factor $\tan \pi\nu_{20} \to \infty$.
- The odd k gives positive contribution while the even k gives negative contribution to the above rate. $k = 1$ term gives the leading positive contribution to the rate.
- The presence of magnetic flux appears to enhance the rate

Class II

For small electric flux, i.e., small ν_{10} and fixed ν_{20}, the rate can be approximated by the leading (k = 1) term as

$$\mathcal{W} \approx \frac{4 n_1 n_2 \pi}{\nu_{10}} \left(\frac{\nu_{10}}{8\pi^2 \alpha'}\right)^{\frac{1+p}{2}} e^{-\frac{Y^2}{2\pi\nu_{10}\alpha'}} e^{\frac{\pi\nu_{20}}{\nu_{10}}} \tan \pi\nu_{20}, \tag{4.7}$$

which is greatly enhanced by a factor of $e^{\frac{\pi\nu_{20}}{\nu_{10}}} \tan \pi\nu_{20}/(8\nu_{10})$ in comparison with the rate given in Class I.

Let us make some numerical estimations. Take $\nu_{20} = 2/5$, $\nu_{10} = 1/50$ and the enhance factor given above is then

$$e^{\frac{\pi\nu_{20}}{\nu_{10}}} \frac{\tan \pi\nu_{20}}{8\,\nu_{10}} = e^{20\pi} \frac{25 \tan 0.4\pi}{4} \sim 3.6 \times 10^{28}. \tag{4.8}$$

Class II

A few sample calculations (note $p \geq 3$) for the rate as

$$
\begin{aligned}
(2\pi\alpha')^{\frac{p+1}{2}}\mathcal{W} &\approx n_1 n_2 \left(\frac{\nu_{10}}{4\pi}\right)^{\frac{p-1}{2}} e^{-\frac{Y^2 - 2\pi^2\nu_{20}\alpha'}{2\pi\nu_{10}\alpha'}} \tan\pi\nu_{20} \\
&\approx n_1 n_2 \left(\frac{\nu_{10}}{4\pi}\right)^{\frac{p-1}{2}} \tan\pi\nu_{20}, \\
&\approx \begin{cases} 0.489 & \text{for}\, p = 3, \\ 0.028 & \text{for}\, p = 4, \end{cases}
\end{aligned} \tag{4.9}
$$

where we have taken $Y = \pi\sqrt{2\nu_{20}\alpha'} \approx 2.81\sqrt{\alpha'}$, i.e., on the order of string scale, $n_1 = 10$ and $n_2 = 10$.

Class II

Another new singularity at large t' when $Y - \pi\sqrt{2\nu_{20}\alpha'} \to 0^-$ as

$$\lim_{t'\to\infty} \frac{e^{-\frac{Y^2 t'}{2\pi\alpha'}}(\cos\pi\nu_{10}t' - \cosh\pi\nu_{20}t')^2}{\sinh(\pi\nu_{20}t')} \sim \lim_{t'\to\infty} e^{-\frac{t'}{2\pi\alpha'}(Y^2 - 2\pi^2\nu_{20}\alpha')}, \tag{4.10}$$

which signals also the onset of tachyonic instability as in the pure magnetic case. Note that this is associated with the real part of the amplitude. When this happens, for a weak electric flux with a large magnetic flux, the rate of pair production also diverges. So when $y > \pi\sqrt{2\nu_{20}\alpha'}$, the pair production of open strings is the only process to lower the system energy but as $Y \to \pi\sqrt{2\nu_{20}\alpha'}$ both the tachyonic instability and the instability of pair production start to occur (in addition to the strong electric flux divergence which is independent of the separation).

Mike

Happy Birthday!