The Bernstein conjecture, its failure, and the 8-brane

G W Gibbons

DAMTP, Cambridge

Duff-fest, London, Apr 2009

It is a great honour to be invited here to celebrate Mike Duff's birthday. I would like to thank the organisers for making this possible.

Mike's many contributions to the theory of extended objects and their rôle in fundamental physics are well known. Thanks to his work, that of his collaborators, and that of many others, we now have a fairly complete global picture. However there remain some obscure points. In this talk I will describe one of them and invite Mike to clear it up.

What I am about to describe is based on joint ongoing, and as yet unpublished work with Ke-ichi Maeda and Umpei Myamoto begun last summer.

The simplest model of a brane we can contemplate is a minimal surfaces in Euclidean space $\mathbb{E}^3$ have been extensively studied since the pioneering work of Thomas Young and of Laplace. In Monge, or non-parametric, gauge * the surface is specified by the height function $z = z(x, y)$ above some plane. The non-parametric minimal surface equation governing the function $z(x, y)$ is

$$\partial_x\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \partial_y\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,. \tag{1}$$

*often called *static* gauge

A famous result of Bernstein asserts that the only single valued solution of (1) defined for all $(x, y) \in \mathbf{R}^2$ is a plane. It may also be shown that the planar solution is a minimizer of the area functional among compactly supported variations of the surface.

In terms of brane theory, this means that the "classical ground state", i.e., the static minimum of the energy functional for a membrane in three dimensional Euclidean space $\mathbf{E}^3$, which may be thought of as a static configuration in 4-dimensional Minkowski spacetime $\mathbf{E}^{3,1}$, is smooth and indeed planar. From the world volume point of view the classical ground state of the membrane preserves (2+1)-dimensional Poincaré invariance and may be thought of as a copy of (2+1)-dimensional Minkowski spacetime $\mathbf{E}^{2,1}$.

It is natural to conjecture that Bernstein's theorem remains valid for a minimal p-dimensional hypersurface in $(p+1)$-dimensional Euclidean space $\mathbf{E}^{p+1}$. In other words the classical ground state of a p-*brane* in $(p+1+1)$-dimensional Minkowski spacetime $\mathbf{E}^{p+1,1}$ should be flat and invariant under the action of the $(p+1)$-dimensional Poincaré group $E(p,1)$. Remarkably, although true for $p \leq 7$ it fails for $p+1 \geq 9$ * . In other words the classical ground state of an 8-brane in 10-dimensional Minkowski spacetime spontaneously breaks 10-dimensional Poincaré invariance. The proof rests on the fact that in $\mathbf{E}^8$ and above, a minimal hypersurface which is a minimizer of the p-volume functional among compactly supported variations need not be smooth. There are rather explicit counterexamples called *minimal cones*. Their existence leads to the conclusion that Bernstein's theorem fails in $\mathbb{E}^9$.

*E. Bombieri, E. de Giorgi and E. Giusti, Minimal Cones and the Bernstein Problem, *Inventiones math.* **7**, 243-268 (1969).

There seems to be no discussion of the significance of this fact by M/String theorists. The breakdown of regularity of minimal hypersurfaces of flat space extends to minimal hypersurfaces of curved Riemannian manifolds and has consequences for proofs of the positive energy theorem which make essential use of minimal surfaces as a technical tool * . We therefore chose to examine the behavior of minimal surfaces in higher dimensions and in curved spaces in some explicit detail. In particular we wanted to see whether the existence of various critical dimensions which has been noted in related contexts is of a universal nature and related to the the breakdown of Bernstein's theorem and the existence of minimal cones.

*R. Schoen and S. T. Yau, "Positivity Of The Total Mass Of A General Space-Time," *Phys.Rev. Lett.* **43**, 1457 (1979)
"On the Proof of the Positivity Mass Conjecture in General Relativity," *Comm. Math. Phys.* **65**, 45 (1979)

To make progress we assume sufficient symmetry that we are reduced to solving an ordinary differential equations in an appropriate quotient space $X = \mathbf{E}^n/G$, a ploy known to mathematicians as *equivariant variational theory*. Typically the brane equations of motion reduce to finding geodesics in X with respect to a suitable metric g on X, induced by the p-volume functional.

The p-brane will be p-volume minimizing if the corresponding geodesic γ is length minimizing. A necessary condition that a geodesic joining points a and b be length minimizing is that γ contains no points between a and b conjugate to either. The existence of such conjugate points is governed by the Jacobi or geodesic deviation equation, solutions of which depend on the curvature of X.

In the case that X is 2-dimensional, the sign of the Gauss curvature K is important. If for example K is negative in the vicinity of γ, then it can contain no conjugate points and hence must be locally length minimizing. In the cases we shall consider the Gauss curvature is actually positive and a more detailed examination is required. One might have thought that positive Gauss curvature would lead to a second variation or Hessian of indefinite sign. However the situation is more subtle since the effective metric governing the variational principle is incomplete and becomes singular near a conical point and compensatory terms can arise which in low dimension which render the Hessian positive definite.

The basic example of this setup is on $\mathbf{E}^{2k+2}$ where $p = 2k+1$, and $G = SO(k+1) \times SO(k+1)$ with the standard action on $\mathbf{E}^{2k+2} = \mathbf{E}^{k+1} \times \mathbf{E}^{k+1}$ with flat metric

$$h = dx^2 + x^2 d\Omega_k^2 + dy^2 + y^2 d\Omega_k^2 \,, \tag{2}$$

where $d\Omega_k^2$ is the standard round metric on S^k. The induced metric g is

$$g = d\ell^2 = (xy)^{2k}(dx^2 + dy^2)\,. \tag{3}$$

The orbit of the straight line $x = y$ under the action of $SO(k+1) \times SO(k+1)$ is a $(2k+1)$-dimensional minimal cone with a singularity at the origin, $x = y = 0$.

A smooth minimal surface would depart from this straight line.

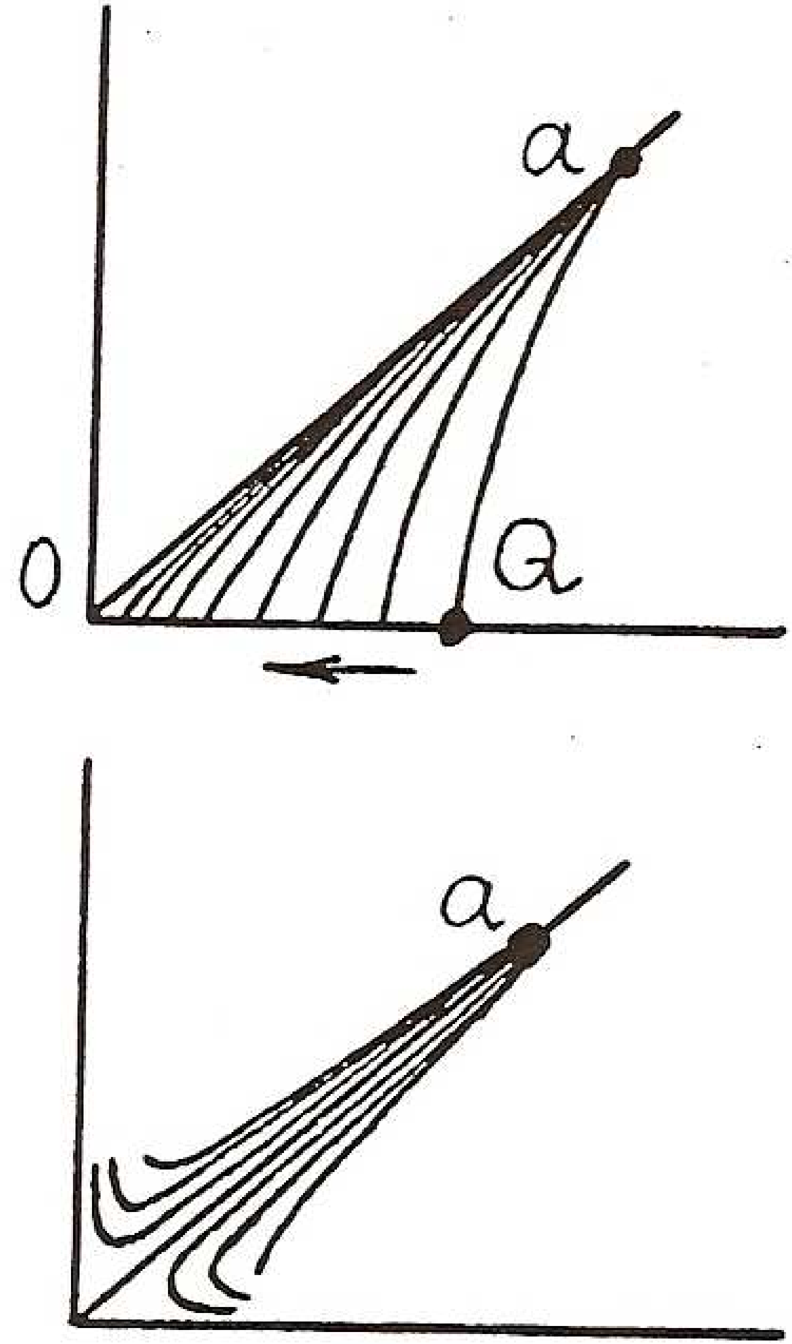

Figure 20

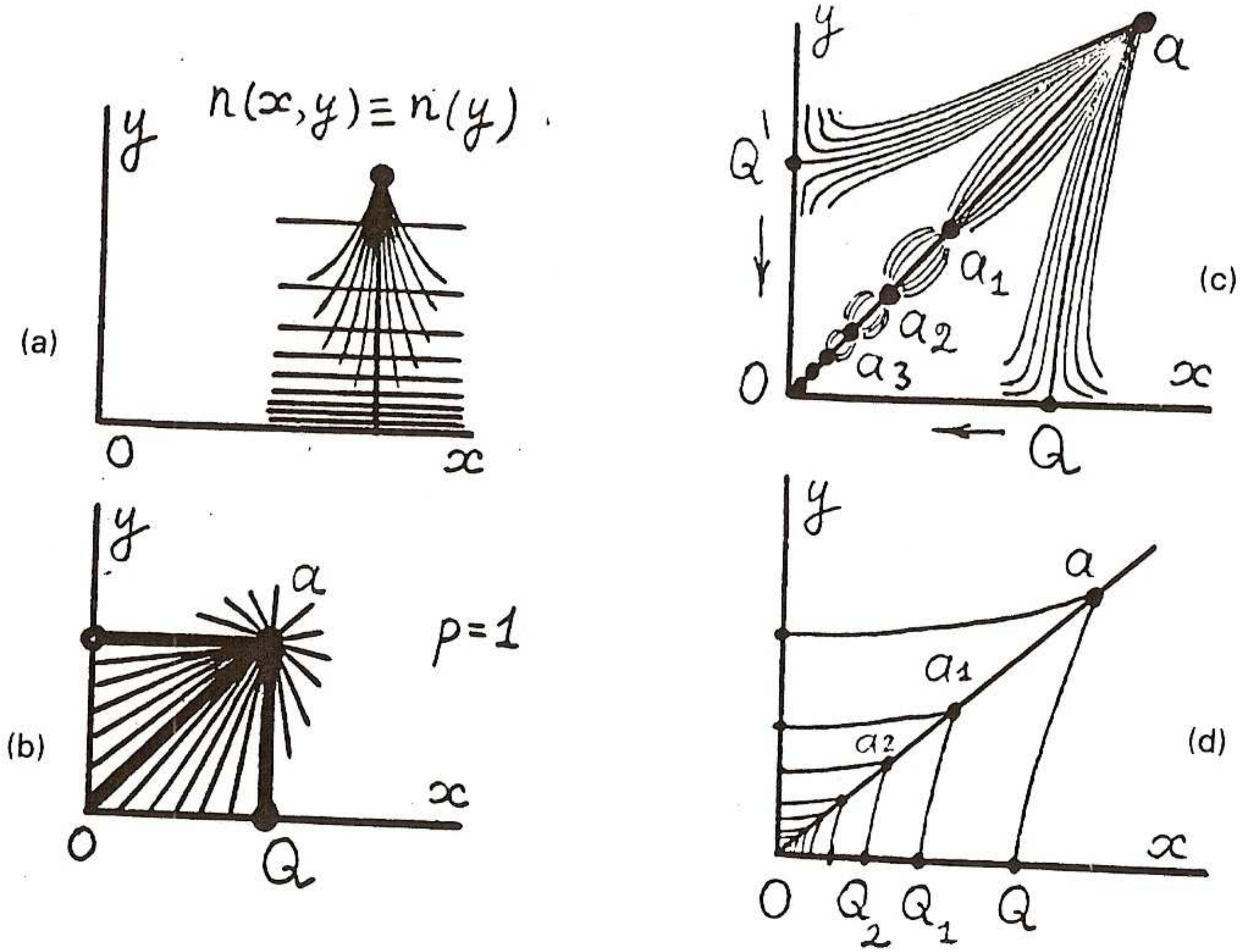

Figure 18a–d

A study of the second variation shows that this singular cone is $(2k+1)$-volume minimizing as long as $k \geq 3$.

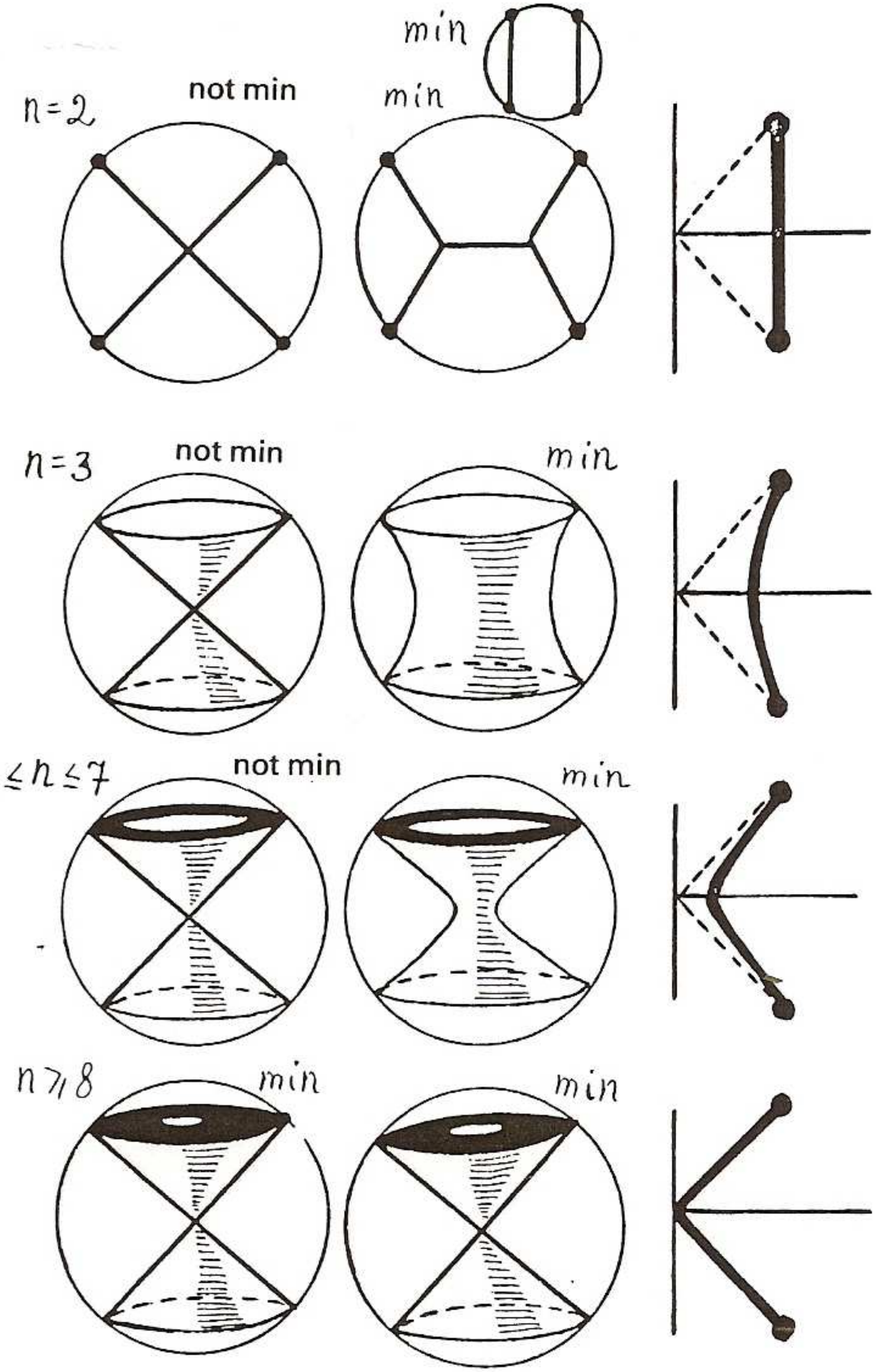

Figure 16

The Jacobi or equation of geodesic deviation is

$$\frac{d^2\eta}{d\ell^2} + K\eta = 0\,. \tag{4}$$

where K is Gaussian curvature. For a general metric $g = v(x,y)^{2/\lambda}(dx^2 + dy^2)$, K is

$$K = -\frac{2}{\lambda v^{2/\lambda}}(\partial_x^2 + \partial_y^2)\ln v. \tag{5}$$

For the present case, $v(x,y) = x^m y^n$ and $\lambda = 1$, we have

$$K = \frac{1}{x^{2m}y^{2n}}\left(\frac{m}{x^2} + \frac{n}{y^2}\right), \tag{6}$$

which is positive definite!

However

along the cone the Gaussian curvature and proper distance are given by

$$K = \frac{2m^{n+1}}{n^n x^{2(m+n+1)}}, \qquad \ell = \frac{n^{n/2}(m+n)^{1/2}}{m^{(n+1)/2}(m+n+1)^{1/2}} x^{m+n+1}. \tag{7}$$

Thus

$$\frac{d^2\eta}{d\ell^2} + \frac{c}{\ell^2}\eta = 0, \qquad c = \frac{2(m+n)}{(m+n+1)^2} \tag{8}$$

which has solutions

$$\eta = \ell^{\beta_\pm}, \quad \beta_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-4c}\right). \tag{9}$$

Thus, these solutions oscillate for $2 \leq m+n \leq 5$, while does not for $m+n \geq 6$.

Bombieri et all deduced from this that in space dimension ≥ 8 a minimal hypersurface can be singular, i.e. non-smooth. More over they showed that minimal comes can be absolute minimizers of the area functionl among all competing surfaces with the same boundary.

In fact it appears that these minimal cones are related to Calibrations, Bogomoln'yi bounds and and Susy Branes. Precisely how remains to be elucidated. By considering groups which can act on spheres, one finds a classification of co-dimension one minimal cones with symmetry when they are minimizers.

Table 1

No	G	dim ψ	H	α	$v^2 = (\mathrm{vol})^2$
1	$SO(r) \times SO(s)$	$r + s$	$SO(r-1) \times SO(s-1)$	$\frac{\pi}{2}$	$x^{2r-2}\, y^{2s-2}$
2	$SO(2) \times SO(k)$	$2k$	$Z_2 \times SO(k-2)$	$\frac{\pi}{4}$	$(xy)^{2k-4}\, (x^2 - y^2)^2$
3	$SU(2) \times SU(k)$	$4k$	$T^1 \times SU(k-2)$	$\frac{\pi}{4}$	$(xy)^{4k-6}(x^2 - y^2)^4$
4	$Sp(2) \times Sp(k)$	$8k$	$(Sp(1))^2 \times Sp(k-2)$	$\frac{\pi}{4}$	$(xy)^{8k-10}(x^2 - y^2)^8$
5	$U(5)$	20	$SU(2) \times SU(2) \times T^1$	$\frac{\pi}{4}$	$(xy)^2\, \{\mathrm{Im}(x + iy)^4\}^8$
6	$SO(3)$	5	$Z_2 \times Z_2$	$\frac{\pi}{3}$	$\{\mathrm{Im}(x + iy)^3\}^2$
7	$SU(3)$	8	T^2	$\frac{\pi}{3}$	$\{\mathrm{Im}(x + iy)^3\}^4$
8	$Sp(3)$	14	$(Sp(1))^3$	$\frac{\pi}{3}$	$\{\mathrm{Im}(x + iy)^3\}^8$
9	$Sp(2)$	10	T^2	$\frac{\pi}{4}$	$\{\mathrm{Im}(x + iy)^4\}^4$
10	G_2	14	T^2	$\frac{\pi}{6}$	$\{\mathrm{Im}(x + iy)^6\}^4$
11	F_4	26	$Spin(8)$	$\frac{\pi}{3}$	$\{\mathrm{Im}(x + iy)^3\}^8$
12	$Spin(10) \times U(1)$	32	$SU(4) \times T^1$	$\frac{\pi}{3}$	$\{\mathrm{Im}(x + iy)^3\}^{20}$

THEOREM 2 (Classification theorem for minimal cones of codimension one).

The unique globally minimal surfaces with boundaries A, where A are the orbits represented in Table 1, are the cones over the following manifolds $A = G/H$:

(a) $S^{r-1} \times S^{s-1} = SO(r) \times SO(s)/SO(r-1) \times SO(s-1)$ in R^{r+s} for $r + s \geq 8$, $s, r \neq 2$,
(b) $SO(2) \times SO(k)/Z_2 \times SO(k-2)$ in R^{2k} for $k \geq 8$,
(c) $SU(2) \times SU(k)/T^1 \times SU(k-2)$ in R^{4k} for $k \geq 4$,
(d) $Sp(2) \times Sp(k)/(Sp(1))^2 \times Sp(k-2)$ in R^{8k} for $k \geq 2$,
(e) $U(5)/SU(2) \times SU(2) \times T^1$ in R^{20}, (f) $Sp(3)/(Sp(1))^3$ in R^{14},
(g) $F_4/Spin(8)$ in R^{26}, (h) $Spin(10) \times U(1)/SU(4) \times T^1$ in R^{32}.

For all the other manifolds G/H indicated in Table 1, their corresponding cones are not minimal, which means that there exists a variation decreasing the volumes of these cones. The globally minimal cones listed above are G-invariant with respect to the corresponding groups indicated in Table 1.

These methods have been used to investigate co-dimension 2 minimal cones. The critical dimensions appear to be less interesting.

The cones CA of codimension two in $\mathbf{R}^N$ over the following locally minimal submanifolds $A^{N-3} = G/H$ in S^{N-1} are the only globally minimal cones with boundary A in $\mathbf{R}^N$:

(1) $S^{r-1} \times S^{r-1} \times S^{r-1}$ in $\mathbf{R}^{3r}$, $r \geq 7$;

(2) $SO(r) \times SO(3)/SO(r-1) \times \mathbf{Z}_2 \times \mathbf{Z}_2$ in $\mathbf{R}^r \times \mathbf{R}^5$, $r \geq 53$;

(3) $SO(r) \times SU(3)/SO(r-1) \times T^2$ in $\mathbf{R}^r \times \mathbf{R}^8$, $r \geq 39$;

(4) $SO(r) \times Sp(3)/SO(r-1) \times (Sp(1))^3$ in $\mathbf{R}^r \times \mathbf{R}^{14}$, $r \geq 74$;

(5) $SO(r) \times Sp(2)/SO(r-1) \times T^2$ in $\mathbf{R}^r \times \mathbf{R}^{10}$, $r \geq 51$;

(6) $SO(r) \times G_2/SO(r-1) \times T^2$ in $\mathbf{R}^r \times \mathbf{R}^{14}$, $r \geq 75$;

(7) $SO(r) \times F_4/SO(r-1) \times \mathrm{Spin}(8)$ in $\mathbf{R}^r \times \mathbf{R}^{26}$, $r \geq 74$;

(8) $SO(r) \times \mathrm{Spin}(10) \times U(1)/SO(r-1) \times SU(4) \times T^1$ in $\mathbf{R}^r \times \mathbf{R}^{32}$, $r \geq 136$.

Starting from their discovery of minimal cones (i.e. 7-branes) in 8 space dimensions, Bombieri et al. went on to argue that the Bernstein conjecture is false in for a miminal 8-brane in 9 space dimensions.

For this, among other things, Bombieri received the Fields medal in 1974.

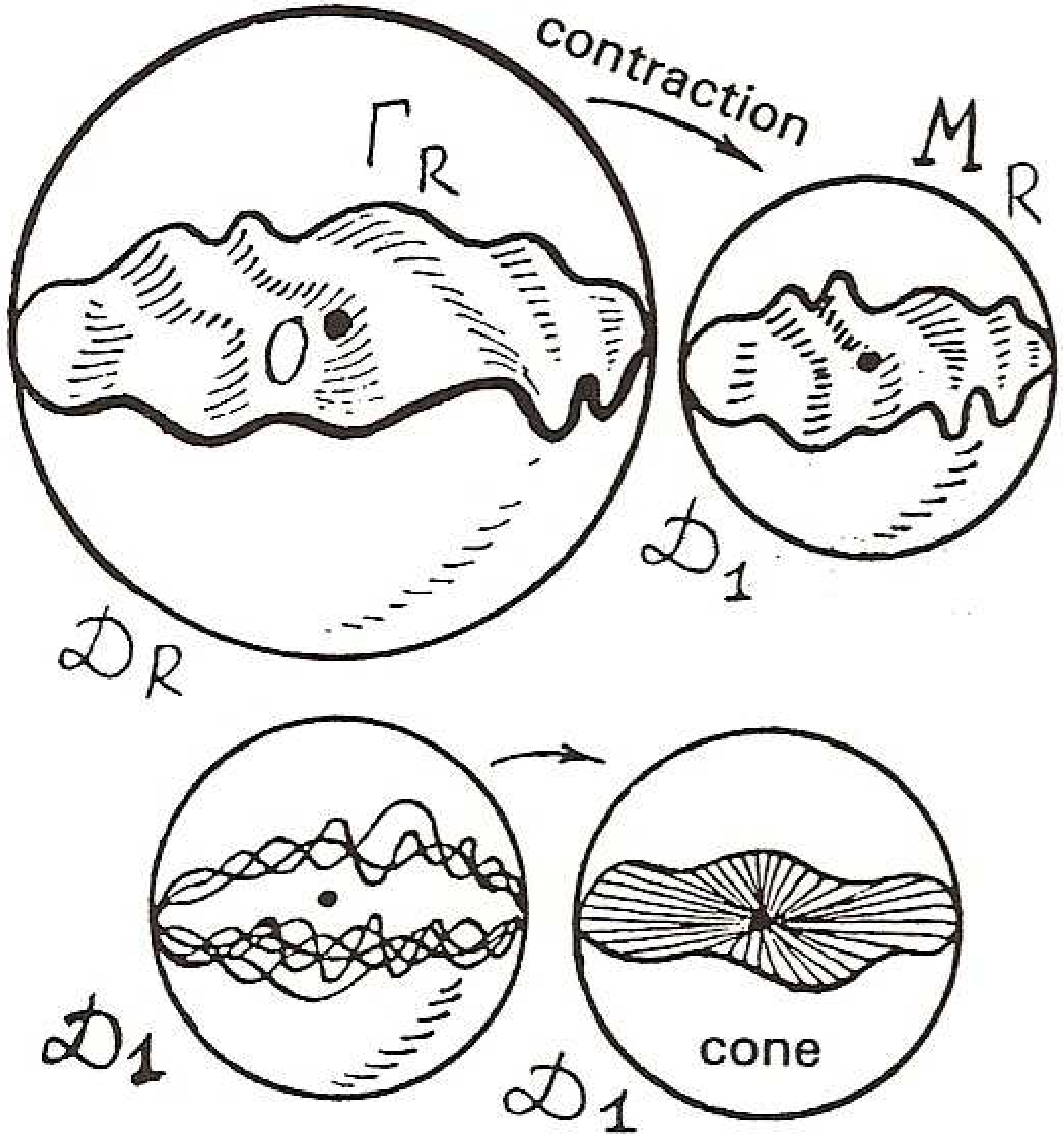
contraction
Γ_R
M_R
$\mathcal{D}_R$
$\mathcal{D}_1$
$\mathcal{D}_1$
$\mathcal{D}_1$
cone
O

Figure 21

In curved spacetime, Frolov * considered a static p-brane in an N-spacetime dimensional Tangherlini black hole†. He shows that this gives a geodesic of the metric

$$g = (r \sin\theta)^{2p-2}\left(dr^2 + r^2 f(r) d\theta^2\right), \quad f(r) = 1 - \left(\frac{r_0}{r}\right)^{N-3}, \quad N = p+2. \tag{10}$$

Here r is a Schwarzschild radial coordinate and θ a co-latitude coordinate.

*V. P. Frolov, "Merger transitions in brane-black-hole systems: Criticality, scaling, and self-similarity," *Phys. Rev.* **D 74**, 044006 (2006) [arXiv:gr-qc/0604114]

†in fact he considered a general case with $N \geq p+2$. Since the number of co-dimensions, $N-p$, does not affect our argument, we only consider the hypersurface case, $N = p+2$. His argument is independent of the specific form of the background solution as long as it has a spherically symmetric non-degenerate horizon.

Now Frolov finds a qualitatively different behavior sets in when the spactime dimenson $N \geq 8$ and p is greater than 6. On the face of it this looks different from the result of Bombieri et al.

However a static $p-brane$ in an N-dimensional static Lorentzian manifold,(with periodic imaginary time) may be thought of as a $p+1$-brane in an N-dimensional Riemannian manifold.

Thus from the Riemannian point of view this is when the minimal submanifold has dimension 7 or larger. This agrees with what the analysis of minimal cones in flat space indicates.

To check this we write the Schwarzschild metric as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\Omega^2_{N-3}). \tag{11}$$

Then near the south pole of the horizon,

$$r = r_0 + \xi, \quad \theta = \pi - \eta, \tag{12}$$

with small ξ/r_0 and η. At the leading order

$$ds^2 = -(N-3)\frac{\xi}{r_0}dt^2 + \frac{r_0}{(N-3)\xi}d\xi^2 + r_0^2(d\eta^2 + \eta^2 d\Omega^2_{N-3}). \tag{13}$$

Furthermore

$$x = \sqrt{\frac{4r_0\xi}{N-3}}, \quad y = r_0\eta. \tag{14}$$

Thus, the near horizon metric at the south pole is given by

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + dy^2 + y^2 d\Omega^2_{N-3}, \qquad \kappa = (N-3)/2r_0 \tag{15}$$

Thus, the near-horizon effective 2-dimensional metric in which the geodesic is to be found is

$$g = x^2 y^{2(N-3)}(dx^2 + dy^2). \tag{16}$$

The problem now reduces to one similar to that studied above. Note that the factor x^2 in g comes from the time component of metric (15). Thus, the cone $y = \sqrt{N-3}\ x$ is a geodesic near the horizon, and from the analysis of geodesic deviation, this geodesic corresponds to a minimizer if $N = p + 2 \geq 8$.

This cone solution separates two phases of the brane: one has a Minkowski topology and another a black hole topology. The above cone separates these two phases and the change of stability nature of the brane at $p = 6$ results in the mass scaling law of the black hole on the brane found by Frolov. A holographic application is found in *.

*D. Mateos, R. C. Myers and R. M. Thomson, "Holographic phase transitions with fundamental matter," *Phys. Rev. Lett.* **97**, 091601 (2006) [arXiv:hep-th/0605046]

A minimal surface is a mathematical idealization of something with finite thickness. A model which incorporates this is a non-linear Laplace equation of the form

$$\Delta\phi = V'(\phi)\,, \quad \Delta = \sum_{i=1}^{p+1} \frac{\partial^2}{\partial x_i^2}. \tag{17}$$

If $V(\phi)$ has two critical points at $\phi = \pm 1$, say, at which

$$V'(\pm 1) = V(\pm 1) = 0\,, \tag{18}$$

then a static domain wall is a solution on $\mathbf{E}^{p+1}$, with $\phi \to +1$ as $x_{p+1} \to +\infty$ and $\phi \to -1$ as $x_{p+1} \to -\infty$. If these limits are attained *uniformly* in $\{x_1, x_2, \ldots, x_p\}$, then for all p, all solutions of (17) depend only on x_{p+1}.*

*This is known, for reasons that are only partially clear to G.W.G. as the Gibbons Conjecture and has been proved by a number of people, but not by G.W.G.

If we merely require that $\partial\phi/\partial x_{p+1} > 0$, ϕ is bounded, and that

$$V'(\phi) = -\phi(1-\phi^2) \tag{19}$$

then it is known that for $p < 8$, all solutions of this so-called Allen-Cahn equation are planar.* However if $p \geq 8$, then there are non-planar examples † .

In other words, the behavior of domain walls of finite thickness mirrors that of minimal surfaces.

*This is known for good reasons as de Giorgi's conjecture. He, for good reason, added the *caveat* "at least for $p < 8$".

†M. del Pino, M. Kowalczyk and J. Wei, "On de Giorgi conjecture in dimension $N \geq 9$", arXiv:0806.3141 [math.AP].

Physically one expects that a stable minimal surface, such as a catenoid in $\mathbb{E}^3$, could be mimicked by solution of the Allen-Cahn equation. In fact a numerical simulation by Paul Sutcliffe * showed that starting with a configuration for which $\phi = +1$ in the deep interior of a catenoid and $\phi = -1$ outside it and allowing it to relax to an energy minimizer does lead to a thick catenoidal domain wall. In fact Gòždž and Holyst † have constructed periodic minimal surfaces from Landua-Ginzburg models.

*private communication

†W. Gozdz and R. Holyst, "From the Plateau problem to periodic minimal surfaces in lipids, surfactants and diblock copolymers", cond-mat/9604003; "High Genus Periodic Gyroid Surfaces of Nonpositive Gaussian Curvature" *Phys. Rev. Lett.* **76**, 2726 (1996). [cond-mat/9604013]

A powerful general argument that interfaces in media of a type introduced by Koretweg * should be either planar, spherical or cylindrical has been given by Serrin †. It is of interest to see how it breaks down in our case.

*D. J. Korteweg, Sur la forme que prennat les équations du mouvement des fluides si l'on tient compte des forces capilaires causées par des variations de density *Archives Nèerlandaise des Science Exactes et Naturelles* **6** (1901) 1-24

†J. Serrin, The form of interfacial surfaces in Korteweg's theory of phase equilibria *Quart. Appl. Math.* **41** (1983/84) no. 3, 357-364

Serrin's version of Korteweg's theory starts with the equilbrium condition for the spatial stress tensor

$$\partial_i T_{ij} = 0\,, \tag{20}$$

where T_{ij} is given in terms of a density function $\rho(x)$ by

$$T_{ij} = -P(\rho)\delta_{ij} + v_{ij}\,, \tag{21}$$

where

$$v_{ij} = (\alpha(\rho)\nabla^2\rho + \beta(\rho)|\nabla\rho|^2)\delta_{ij} + (\gamma(\rho)\partial_i\partial_j\rho + \delta(\rho)\partial_i\rho\partial_j\rho) \tag{22}$$

Starting from these equations Serrin derived the over-determined system of equations

$$\nabla^2 \rho = h(\rho)\,, \qquad |\nabla \rho| = g(\rho)\,. \tag{23}$$

It was then shown by Pucci * that solutions $u(x)$ of (23) must have level sets given either by concentric spheres, cylinders or parallel planes.

*Pucci, Patrizia An overdetermined system. *Quart. Appl. Math.* **41** (1983/84), no. 3, 365–367

In detail Serrin defines

$$a = \alpha + \gamma\,, \qquad b = \beta + \delta\,, \qquad c = \gamma' - \delta\,, \tag{24}$$

where $'$ denotes differentiation with respect to ρ.

Then (20) implies that

$$\partial_i(-P + a\nabla^2\rho + (b + \frac{1}{2}c)|\nabla\rho|^2) = (c\nabla^2\rho + \frac{1}{2}c'|\nabla\rho|^2)\partial_i\rho \tag{25}$$

Now if

$$A := bc + \frac{1}{2}(c^2 - ac') \neq 0\,, \tag{26}$$

then he claims to be able to establish that (23) holds for an appropriate choice of $h(\rho)$ and $g(\rho)$.

To this end he defines

$$F = -P + a\nabla^2\rho + (b + \frac{1}{2}c)|\nabla\rho|^2\,, \qquad G = c\nabla^2\rho + \frac{1}{2}c'|\nabla\rho|^2\,. \tag{27}$$

So that

$$\partial_i F = G\partial_i\rho\,. \tag{28}$$

Thus there is a real valued function $f(u)$ such that

$$F = f(\rho)\,, \qquad G = f'(\rho)\,, \tag{29}$$

and hence

$$\nabla^2\rho \;=\; \frac{1}{A}((b + \frac{1}{2}c)f' - c'(f + P)) := h(\rho) \tag{30}$$

$$|\nabla\rho|^2 \;=\; \frac{1}{A}((c(f + P) - af') := g^2(\rho)\,. \tag{31}$$

To see how Serrin's argument can fail consider a single scalar field:

$$T_{ij} = \partial_i \phi \partial_j \phi - \frac{1}{2}\delta_{ij}((\partial_k \phi)^2 + 2V(\phi)) \tag{32}$$

$$\partial_i T_{ij} = \partial_j (\nabla^2 \phi - V'(\phi)) \tag{33}$$

so we just get one equation

$$\nabla^2 \phi - V'(\phi) = 0\,. \tag{34}$$

In Serrin's notation, taking $\rho = 1$, we have $P = -V$,

$$(\alpha, \beta, \gamma, \delta) = (0, -\frac{1}{2}, 0, 1)\,, \tag{35}$$

whence

$$(a, b, c,) = (0, \frac{1}{2}, -1) \tag{36}$$

Thus

$$F = V\,, \qquad\qquad G = \nabla^2\phi, \tag{37}$$

and

$$A = 0\,, \tag{38}$$

which is the case excluded by Serrin, whose method therefore breaks down.

For mininal surfaces, the non-parametric form of the minimal surface equation can be derived by extremising the energy functional

$$E[\phi] = \int F(\phi, \partial_i \phi) d^3 x = \int (\sqrt{1 + |\nabla \phi|^2} - 1) \, d^3 x \, . \tag{39}$$

The stress tensor is

$$T_{ij} = \frac{\partial_i \phi \partial_j \phi}{\sqrt{1 + |\nabla \phi|^2}} - \delta_{ij} (\sqrt{1 + |\nabla \phi|^2} - 1) \, . \tag{40}$$

This is not of the form introduced by Korteweg. Moreover, one has

$$\partial_i T_{ij} = \partial_j \phi (\partial_i (\frac{\partial_i \phi}{\sqrt{1 + |\nabla \phi|^2}})) \tag{41}$$

which vanishes identically, by virtue of the equation of motion. This will always be true of a system obtained by varing an energy functional $F = F(\phi, \partial_i \phi)$.