Conformal Supergravity *Tree Amplitudes from Open Twistor String Theory*

For Michael Duff's 60th Birthday

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CONFORMAL ANOMALIES AND THE RENORMALIZABILITY PROBLEM IN QUANTUM GRAVITY

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It is argued that conformal invariance cannot in general solve the problem of renormalization in quantum gravity. This is due to the presence of conformal anomalies.

It has recently been suggested by several authors [1, 2] that the long-outstanding problem of renormalizing quantum gravity might be solved by appealing to conformal invariance. The purpose of this note is to point out that this is unlikely to be the case since conformal invariance is not a good symmetry at the quantum level [3], i.e., the presence of conformal anomalies in these models will spoil renormalizability.

The author of ref. [1] considered an Einstein-Yang-Mills Lagrangian which was invariant under the Weyl conformal transformation of the metric tensor

$$\delta g_{\mu\nu}(x) = \sigma(x) g_{\mu\nu}(x), \tag{1}$$

with σ arbitrary, in addition to the usual general coor-

may occur. Indeed, in ref. [3], the present authors showed, using dimensional regularization, that Lagrangian models initially invariant under the transformation (1) do in fact exhibit anomalous behaviour and that conformal invariance is violated. Having lost conformal invariance in this way, one cannot then invoke it to restrict the possible types of counterterm. This is analogous to the situation in the unified gauge models of weak and electromagnetic interactions where the γ_5 anomalies may spoil renormalizability [4].

There are two further points concerning ref. [3] that should be emphasized. Firstly, at the one-loop level the counterterms will be correctly invariant

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Dear Louise,

Thanks for your letter and for sending me the info. on the Arizona Conference ...

An interesting evening on the radio last night: First Graham Greene was on (a rare visit to England said the interviewer – I was lucky) and said his favourite novel was; yes, the 'Honorary Consul'! Then later, who else was on but Steven Weinberg talking about the 'unification of the forces of nature' (well, 3 of them, anyway) plus ~~[illegible]~~ Llewellyn (?) Smith for good measure on the ψ's.

My friend Deser is coming to King's for 3 months in January & we'll be sharing an office! (If you can't beat 'em, join 'em) Maybe I'll see him at Harvard. Got a preprint request from Kallosh in Moscow, in whose paper Copper & I ... wrote back in friendly terms, in order to maintain Detente. (Apparently she's a woman!) ...

Twistor String is a Finite Theory:

$D = 4$, $\mathcal{N} = 4$ Conformal Supergravity coupled to $\mathcal{N} = 4$ SYM

Dipole Ghosts

Tree Amplitudes

Comparison with Einstein Gravity

Weak-Weak Duality, Toward a QCD String

Open Twistor String

The world sheet action with Euclidean signature is

$$S = S_{YZ} + S_{\text{ghost}} + S_G \qquad \text{where } S_G \text{ has } c = 28$$

$$S_{YZ} = \int d^2z \, \left(Y^{Iz} D_z \bar{Z}^I + Y^{I\bar{z}} D_{\bar{z}} Z^I \right)$$

with $D_\mu = \partial_\mu - iA_\mu$ and $1 \le I \le 8$.

The equations of motion for S_{YZ} are

$$D_{\bar{z}} Z = D_z \bar{Z} = 0, \qquad D'_z Y^z = D'_{\bar{z}} Y^{\bar{z}} = 0$$

together with the constraints $Y^{\bar{z}} Z = Y^z \bar{Z} = 0$.

The end condition on the open string

$$n_z Y^z \delta \bar{Z} = -n_{\bar{z}} Y^{\bar{z}} \delta Z$$

is satisfied by the boundary conditions

$$\bar{Z} = UZ, \qquad Y^z n_z = -U^{-1} Y^{\bar{z}} n_{\bar{z}}$$

where $U = e^{2i\alpha}$, $|U| = 1$.

World-sheet Field Content

The fields $Z^J, 1 \leq J \leq 8$, comprise four boson fields, $\lambda^a, \mu^a, 1 \leq a \leq 2$, and four fermion fields $\psi^M, 1 \leq M \leq 4$.

The gauge invariance insures that the Z^J are effectively projective coordinates in the target space $\mathbb{CP}^{3|4}$.

In a gauge where A_z and $A_{\bar{z}}$ are zero, the mode expansion is

$$Z(z) = \sum Z_n z^{-n}, \quad Y(z) = \sum Y_n z^{-n-1}, \quad J^A(z) = \sum J^A_n z^{-n-1}.$$

$$[\![Z^I_m, Y_{Jn}]\!] = \delta^I_J \delta_{m,-n}, \quad [J^A_m, J^B_n] = i f^{AB}{}_C J^C_{m+n} + km\delta_{m,-n}\delta^{AB}.$$

$[\![,]\!]$ denote anticommutators when $I, J \geq 5$, otherwise commutators The vacuum satisfies $\Phi_n|0\rangle = 0$ for $n > -\mathcal{J}$. So $Z^I_n|0\rangle = 0$ for $n \geq 1$.

Gauge Transformations with Instanton Number $d = 0, 1, 2, \ldots$

The current associated with the abelian gauge transformation is

$$J(z) = -\sum_{J=1}^{8} : Y^J(z) Z_J(z) : = -\sum_{\substack{J=1 \\ m}}^{8} a_m^J z^{-m-1} = -\sum_m a_m z^{-m-1},$$

$$X^J(z) = q_0^J + a_0^J \log z - \sum_{n \neq 0} \frac{1}{n} a_n^I z^{-n}, \quad Z^J(z) =: e^{-X^J(z)} :, \quad e^{\pm q_0} = \prod_{J=1}^{8} e^{\pm q_0^J}$$

Gauge transformation with winding number d:

$$g(z) = z^d e^{-\sum_n f_n z^{-n}}, \quad U_g = e^{dq_0} e^{\sum_n f_{-n} a_n}, \quad U_g Z(z) U_g^{-1} = g(z) Z(z)$$

$$\langle 0 | U_g V_1(z_1) V_2(z_2) \ldots V_n(z_n) | 0 \rangle = \langle 0 | e^{dq_0} V_1(z_1) V_2(z_2) \ldots V_n(z_n) | 0 \rangle$$

Tree

Scalar Products

fermions:

$$\langle 0|Z_0|0\rangle = 1 = \int dZ_0 Z_0, \quad Z_0|0\rangle = e^{-q_0}|0\rangle, \quad \langle 0|e^{dq_0} Z_{-d}\dots Z_0|0\rangle = 1$$

(Tree amplitude will vanish unless number of negative helicity modes is $d+1$).

bosons:

$$\langle 0|f(Z_0)|0\rangle = \int dZ_0 f(Z_0), \qquad \text{or, equivalently,} \qquad \langle 0|e^{ikZ_0}|0\rangle = \delta(k)$$

$$\langle 0|e^{dq_0} \exp\left\{ i\sum_{j=0}^{d} k_j Z_{-j} \right\}|0\rangle = \prod_{j=0}^{d} \delta(k_j)$$

Vertex Opertors

Physical state $|\Psi\rangle = \lim_{z\to 0} V(\Psi, z)|0\rangle$.

Gluon vertex operator $V^A(\Psi, z) = f(Z(z))J^A(z)$ describes the dependence on the mean position of the string in twistor superspace $\mathbb{CP}^{3|4}$, $\mathcal{Z}' = (\pi^a, \omega^a, \theta^M)$:

$$W(z) = \int \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a)\delta(k\mu^a(z) - \omega^a) \prod_{M=1}^{4} (k\psi^M(z) - \theta^M)\frac{dk}{k}$$

Multiply by polarizations: $A(\theta) = A_+ + \theta^1\theta^2\theta^3\theta^4 A_-$
Fourier transform on ω^a, integrate over θ^M, then

$$V_-^A(z) = \int dk k^3 \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^a(z)\bar{\pi}_a} \psi^1(z)\psi^2(z)\psi^3(z)\psi^4(z) J^A(z)$$

$$V_+^A(z) = \int \frac{dk}{k} \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^a(z)\bar{\pi}_a} J^A(z)$$

Conformal Graviton Vertex Operators

Vertex Operator	**Helicities**
$V_F(z) = f^{\dot{a}}(Z(z))Y_{\dot{a}}(z)$	$(2,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 0)$
$V_G(z) = g_a(Z(z))\partial\lambda^a(z)$	$(0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ -2)$
$V_{F'}(z) = f^a(Z(z))Y_a(Z) + \hat{f}^{\dot{a}}(Z(z))Y_{\dot{a}}(z)$	$(2,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 0)$
$V_{G'}(z) = g_{\dot{a}}(Z(z))\partial\mu^{\dot{a}}(z) + \hat{g}_a(Z(z))\partial\lambda^a(z)$	$(0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ -2)$
$V_f(z) = f^m(Z(z))Y_m(z) + \tilde{f}^{\dot{a}}(Z(z))Y_{\dot{a}}(z)$	$(\frac{3}{2},\ 1,\ \frac{1}{2},\ 0,\ -\frac{1}{2})$
$V_g(z) = g_m(Z(z))\partial\psi^m(z) + \tilde{g}_a(Z(z))\partial\lambda^a(z)$	$(\frac{1}{2},\ 0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2})$
$V_\Phi^A(z) = V_\phi(Z(z))J^A(z)$	$(\pm 1,\ 4(\pm\frac{1}{2}),\ 6(0))$

$$\frac{\partial}{\partial Z^J} f^J = 0, \qquad Z^J g_J = 0.$$

N=4 Conformal Gravity Particle Content

Helicity	$SU(4)_R$ **Representation**
$0, 0$	1
$-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$	$\bar{4}$
0	$\overline{10}$
$-1, -1, 0$	6
$-\frac{1}{2}$	20
$-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}$	4
$1, -1$	15
$2, 2, 1, -1, -2, -2$	1
$\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{3}{2}$	$\bar{4}$
$\frac{1}{2}$	$\overline{20}$
$1, 1, 0$	6
0	10
$\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	4
$0, 0$	1

Dipole Pairs of Helicities

For the dipole conformal supergravity states, vertex operators are

$$V_F(z) = f^{\dot a}(Z(z))\, Y_{\dot a}(z), \qquad V_{F'} = f^a(Z(z)) Y_a(Z) + \hat f^{\dot a}(Z(z)) Y_{\dot a}(z)$$

$$f^{\dot a}(Z(z)) = i \int \frac{dk}{k^2} \bar\pi^{\dot a} \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\bar\pi_{\dot b}\mu^{\dot b}(z)}$$

$$\times \left[e_2 + k\psi^b \eta_{\frac{3}{2}b} + \frac{k^2}{2}\psi^b\psi^c T_{1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \Lambda_{\frac{1}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 \bar C_0 \right],$$

$$f^a(Z(z)) = \bar s^a \int \frac{dk}{k^2} \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\bar\pi_{\dot b}\mu^{\dot b}(z)}$$

$$\times \left[e_2' + k\psi^b \eta'_{\frac{3}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \Lambda'_{\frac{1}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 \bar C_0' \right]$$

$$\hat f^{\dot a}(Z(z)) = \ldots \qquad \bar s_a \pi^a = 1$$

Tree Amplitudes

$$\langle A^{A_1}_{-1}(z_1)A^{A_2}_{-1}(z_2)C(z_3)\rangle_{\rm tree} = \int \langle 0|e^{q_0}A^{A_1}_{-1}(z_1)A^{A_2}_{-1}(z_2)C(z_3)|0\rangle \prod_{r=1}^{3} dz_r/d\gamma_M d\gamma_S$$

$$= -\delta^4(\Sigma\pi_r\bar\pi_r)\langle 12\rangle^2\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C_{0(3)}$$

$$= \delta^4(\Sigma\pi_r\bar\pi_r)\epsilon_1^-\cdot p_2\,\epsilon_2^-\cdot p_1\,\delta^{A_1A_2}C_{0(3)}.$$

$$\langle A^{A_1}_{-1}(z_1)A^{A_2}_{-1}(z_2)C'(z_3)\rangle_{\rm tree} = \langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\frac{C'_{0(3)}}{2p_3^0}\frac{\partial}{\partial P^0}\delta^4(\Sigma\pi_r\bar\pi_r)$$

where $P^0 = \sum_{r=1}^3 p_r^0$.

Penrose spinors :

$< rs >= \pi_r^a\pi_{ra}$, $\quad [rs] = \bar\pi_r^{\dot a}\bar\pi_{r\dot a}$, $\quad p_{ra\dot a} = \pi_{ra}\bar\pi_{r\dot a}$.

Polarizations: $\quad \epsilon_r^+ = A_r^+\bar s_{ra}\bar\pi_{r\dot a}$, $\quad \epsilon_r^- = A_r^-\pi_{ra}s_{r\dot a}$

Conformal gravity has higher derivative equations of motion.

$(\partial_\mu \partial^\mu)^2 C(x) = 0, \qquad \text{then} \quad C(x) = e^{ip\cdot x} + A \cdot x \; e^{ip\cdot x}.$

The momentum operator acts on the dipole states as $-i\frac{\partial}{\partial x^{a\dot{a}}}$.

$-i\frac{\partial}{\partial x^{a\dot{a}}} e^{ip\cdot x} = p_{a\dot{a}} e^{ip\cdot x}$

$-i\frac{\partial}{\partial x^{a\dot{a}}} A \cdot x \; e^{ip\cdot x} = p_{a\dot{a}} A \cdot x \; e^{ip\cdot x} - i A_{a\dot{a}} e^{ip\cdot x}$

The dipole pair is comprised of a plane wave state $e^{ip\cdot x}$ that diagonlizes the momentum operator, and a state $A \cdot x \; e^{ip\cdot x}$ that is not an eigenstate of momentum. Choose $iA_0 = \frac{C_0'}{2C_0 p^0}$

Translational Invariance of Dipole Trees

the momentum operator acts on the coupling $\langle A^{A_1}_{-1}(z_1)A^{A_2}_{-1}(z_2)C'(z_3)\rangle_{\rm tree}$ as

$$P^0\langle A^{A_1}_{-1}(z_1)A^{A_2}_{-1}(z_2)C'(z_3)\rangle_{\rm tree}-\frac{C'_{0(3)}}{C_{0(3)}2p^0_3}\langle A^{A_1}_{-1}(z_1)A^{A_2}_{-1}(z_2)C(z_3)\rangle_{\rm tree}$$

$$=\langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\frac{C'_{0(3)}}{2p^0_3}\;P^0\frac{\partial}{\partial P^0}\delta(P^0)\delta^3(P^i)$$

$$+\frac{C'_{0(3)}}{2p^0_3}\langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\delta(P^0)\delta^3(P^i)$$

$$=0$$

Comparison with Einstein Gravity

$$\langle e_{-2}(z_1)e_{-2}(z_2)e_2(z_3)e_2(z_4)\rangle_{CG} = \langle 12\rangle^4 \prod_{j=3,4}\sum_{k\neq j}\frac{[jk]\langle k\xi\rangle^2}{\langle jk\rangle\langle j\xi\rangle^2}$$

$$= -\frac{\langle 12\rangle^4[32]\langle 21\rangle\left(\langle 43\rangle\langle 21\rangle-\langle 23\rangle\langle 41\rangle\right)[42]\langle 21\rangle\left(\langle 34\rangle\langle 21\rangle-\langle 24\rangle\langle 31\rangle\right)}{\langle 31\rangle^2\langle 41\rangle^2\langle 34\rangle^2\langle 23\rangle\langle 42\rangle}$$

$$= \frac{\langle 12\rangle^4[34]^4}{(s_{12})^2} = \frac{s_{23}s_{24}}{s_{12}}\ \langle e_{-2}(1)e_{-2}(2)e_2(3)e_2(4)\rangle_{Einstein},$$

which has fewer poles than Einstein gravity, since the Berends Giele Kuijf expression for Einstein gravity tree amplitudes as a product of Yang-Mills trees is

$$\langle e_{-2}(1)e_{-2}(2)e_2(3)e_2(4)\rangle_{Einstein} = s_{12}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\ \frac{\langle 12\rangle^3}{\langle 24\rangle\langle 43\rangle\langle 31\rangle}$$

$$= (s_{12}s_{23}s_{24})^{-1}\ \langle 12\rangle^4[34]^4.$$

Conformal three-graviton coupling for plane wave states vanishes:

$$\langle e_{-2}(z_1)e_{-2}(z_2)e_2(z_3)\rangle_{CG} = \delta^4(\Sigma\pi_r\bar{\pi}_r)\,\langle 12\rangle^6\,\frac{[23]}{\langle 23\rangle\langle 31\rangle^2}\,e_{-2(1)}e_{-2(2)}e_{2(3)} = 0$$

$$\langle e_{-2}(z_1)e_{-2}(z_2)e_2(z_3)\rangle_{Einstein} = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,e_{-2(1)}e_{-2(2)}e_{2(3)},$$

But the dipole partners do couple,

$$\langle e_{-2}(z_1)e_{-2}(z_2)e'_2(z_3)\rangle_{CG} = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,e_{-2(1)}e_{-2(2)}e'_{2(3)},$$

$$\langle e_{-2}(z_1)e'_{-2}(z_2)e_2(z_3)\rangle_{CG} = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,e_{-2(1)}e'_{-2(2)}e_{2(3)}.$$

J. Broedel and B. Wurm, 0902.0550 [hep-th],
C. Hull, L. Mason, and M. Abou-Zeid, hep-th/0606272.

Conformal graviton-2 dilaton coupling for plane wave states vanishes:

$$\langle C(z_1)\bar{C}(z_2)e_{-2}(z_3)\rangle_{CG} = \delta^4(\Sigma\pi_r\bar{\pi}_r)\,\langle 13\rangle^2\,\frac{[23]\langle 23\rangle^3}{\langle 12\rangle^2}\,C_{0(1)}\,\bar{C}_{0(2)}\,e_{-2(3)} = 0$$

$$\langle C(z_1)\bar{C}(z_2)e_{-2}(z_3)\rangle_{Einstein} = \frac{\langle 13\rangle^2\langle 23\rangle^2}{\langle 12\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,e_{-2(1)}\,e_{-2(2)}\,e_{2(3)},$$

The dipole partners couple similarly,

$$\langle C(z_1)\bar{C}(z_2)e'_{-2}(z_3)\rangle_{CG} = \frac{\langle 13\rangle^2\langle 23\rangle^2}{\langle 12\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,C_{0(1)}\,\bar{C}_{0(2)}\,e'_{-2(3)},$$

$$\langle C(z_1)\bar{C}'(z_2)e_{-2}(z_3)\rangle_{CG} = \frac{\langle 13\rangle^2\langle 23\rangle^2}{\langle 12\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,C_{0(1)}\,\bar{C}'_{0(2)}\,e_{-2(3)},$$

$$\langle C'(z_1)\bar{C}(z_2)e_{-2}(z_3)\rangle_{CG} = \frac{\langle 13\rangle^2\langle 23\rangle^2}{\langle 12\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,C'_{0(1)}\,\bar{C}_{0(2)}\,e_{-2(3)}.$$

Weak-Weak Duality, Toward a QCD String

The conformal supergraviton states have zero norm in our basis, but non-trivial inner products. A different basis would have both positive and negative norm states.

To avoid the lack of unitarity, could require a GSO-like projection to a positive definite Hilbert space.

A further projection might avoid the gravity altogether in the twistor string, leaving a string description of only Yang-Mills.

Happy Birthday Michael
and Many Happy Returns

2009 - 1973 = 36 2009 + 36 = 2045...