E(Full, nonlinear, GIMG) ≥ 0.

(The 35 min. Title)

(?, Prev arXiv)

A birthday Greeting for Mike @ 60

[and all 3 of the world's pros are here!]
(you know who you are, G.P.S.!)

The problem, and its solution, for non-pros* in 35 m.

[*if any; Hull, Stelle, Townsend ... will recognize some methodology]

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1. What is TMG (and C…)?

Cosmological Topologically Massive Gravity

TMG - dates from 1982 [SD, Jackiw & Templeton]

but is undergoing an inflationary phase (≳30 papers/1 yr) and lots of (technical) controversy far exceeding its importance.

The new work is centered on CTMG (also old, SD 84) and its various "phases", with a possible (hotly disputed) discontinuity. This I will not discuss in the talk!

Next, the action.

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$$I_{CTMG} = (-) I_E + I_{CS} + I_\Lambda$$

$$= \int d^3x \left[\left\{ \text{"}-\text{"} R - \Lambda \right\} \right] + \frac{1}{\mu} \int d^3x \, \varepsilon \left[\Gamma \partial \Gamma + \frac{2}{3} \Gamma\Gamma\Gamma \right]$$

"$-$" : "Wrong" sign, opposite to I_{EH}, in $D=4$.

↳ why, see below.

$$\delta_g I = 0 \Rightarrow \mathcal{G}^{\mu\nu} \equiv (-G^{\mu\nu} + \Lambda g^{\mu\nu}) + \frac{1}{\mu} C^{\mu\nu} = 0$$

$$C^{\mu\nu} \equiv \varepsilon^{\mu\alpha\beta} D_\alpha \left(R_\beta{}^\nu - \frac{1}{4} \delta_\beta^\nu R \right) \equiv \text{"Cotton"}$$

$\equiv$ Weyl in 3D: $C^{\mu\nu} = 0 \Leftrightarrow g_{\mu\nu} = \phi \eta_{\mu\nu}$.

$$\underbrace{C^\mu_\mu \equiv 0}_{\text{manifest}} \equiv \underbrace{C^{\mu\nu}}_{\text{2 lines.}} \equiv \underbrace{D_\mu C^{\mu\nu}}_{\text{Since } I_{CS} \text{ is invariant}}$$

All this is in Eisenhart and Cotton $\approx$ 1900 !

NB! Pure GR $\Leftrightarrow$ flat outside $T_{\mu\nu}$.

$${}^{(3)}G^{\mu\nu} = \frac{1}{4} \varepsilon^{\mu\alpha\beta} R_{\alpha\beta\lambda\sigma} \varepsilon^{\lambda\sigma\nu}$$

No excitations, not (much) excitement!

(Sign of I_E dull too)

So: $\begin{cases} I_E\ (+I_\Lambda) \rightarrow \text{flat (or AdS)} \\ I_{CS} \rightarrow \text{conform flat.} \end{cases}$

Both dull. BUT $I_E \oplus I_{CS}$ IS fun.

Paradoxical features, at linear level already:

E.g.: 1) ∂^3 order ($C^{\mu\nu} \sim \partial^3 g_{\mu\nu}$) but ghost-free.

2) Gauge invariant but **massive** ($m \sim \mu$) excitation(s)

1 local graviton helicity +/− 2.

3) $E(\text{exact}) \geq 0$

Iff use "−" I_E.

4) Finite-range excitations, but long-range constraints!

Wait, there's more: SUGRA

SU GRAS: → Important for us: key to E70 ⑤

$(I_E + I_\Lambda)$, $(I_E + I_{CS})$, $(I_E + I_{CS} + I_\Lambda)$

$\underbrace{(I_E + I_\Lambda), (I_E + I_{CS})}_{\text{SD+Kay '83}}$ $\underbrace{(I_E + I_{CS} + I_\Lambda)}_{\text{SD '84}}$

The fermions are as expected:

$I_{3/2} = \int \bar{\psi}_\mu f^\mu(\psi) \qquad f^\mu = \frac{1}{2}\varepsilon^{\mu\alpha\beta}\partial_\alpha \psi_\beta$.

So $\delta I_{3/2} = 0 \Rightarrow f^{\mu\nu} = 0 \Rightarrow$ pure gauge ~~no particle~~ (like I_G)

$I_{FCS} \sim \frac{1}{m}\int f^\mu \gamma_\nu \gamma_\mu f^\nu$: also dull alone.

$I_{3/2} + I_{FCS} \Rightarrow I \sim \int \bar{\eta}(\not{\partial} + \mu)\eta$,

helicity $\pm 3/2$ $\quad m \neq 0$

$I_\Lambda \to (\sqrt{-\Lambda})^{-1}\int \bar{\psi}_\mu \gamma_\alpha \psi_\nu \varepsilon^{\mu\alpha\nu}$ as usual in 4D (PKT '77)

[I_{FCS} also conform invariant: $\delta\psi_\mu = \gamma_\mu \alpha(x)$ like I_{CS} essential for E70 proof]

END of (C)TMG Summer School.

2. SUSY $\Longleftrightarrow$ $E>0$.

There is no question that linearized (C)TMG has localized complete set of $E>0$ excitations, $E \sim \int \omega \, |a(\vec{k})|^2$

$$\omega = \sqrt{k^2+\mu^2}.$$

But what about the full theory?

We outline the $E>0$, stability, arguments at 2 levels: a) Formal SUSY + more telling $\rightarrow$ b) concrete derivation.

This is a very old story, in 4D: SUGRA has $E>0$ $\Leftrightarrow$ GR ditto
(SD + Teitelboim 1977) (Witten '81 ..)

How does this "descend" to D=3 TMG?

Review Global SUSY:

$$E = \frac{1}{2}\,\mathrm{tr}\,Q^2 \geq 0$$ ← iff positive Hilbert space metric.

, Q = Supercharge (see below)

$E = 0 \Rightarrow Q = 0 \Leftrightarrow$ vacuum

[As always, if we flip the sign of the action, all becomes negative E!]

SUGRA: more complicated, local.

Just as E is both the definition and value of total energy, so is Q.

Roughly: In any gauge theory: $\nabla^2 \varphi = \rho$ $\quad E \equiv \int d^{(n-1)}x\, \nabla^2 \varphi \equiv \oint dS_i \nabla_i \varphi$.

but value is $\int d^{(n-1)}x\, \rho = E$

But we know all this for GR, and ditto for SUGRA:

(S.D., L.Teitelboim)

$$G^0{}_\mu = T^0_\mu \Rightarrow G^0_\mu(\mathrm{lin}) = -G^0_\mu(\mathrm{nonlin}) + T^0_\mu$$

$$R^0(\psi) = J^0_{\mathrm{ext}} \Rightarrow R^0(\mathrm{lin}) = -J(\mathrm{nonlin})$$

$$R^\mu_{3/2} \equiv \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho \psi_\sigma$$

$$J^0 \sim \varepsilon^{0ijk} \gamma_5 \gamma_i \partial_j \psi_k$$

$$Q \equiv \int J^0 d^3x \sim \oint dS_j \, \underbrace{\varepsilon^{ijk} \gamma_5 \gamma_i}_{\gamma_0 \sigma^{ij}} \psi_k$$

Now

$$\delta_\alpha Q(\alpha) = \{\alpha Q, \alpha Q\} = \bar{\alpha}_0 \gamma^\mu \alpha_0 P_\mu$$

$$\gamma^\mu P_\mu = \{Q, \bar{Q}\} \gamma^0$$

$\mathrm{tr}\,\gamma^0 >$ $\quad P^0 = \frac{1}{2} \mathrm{tr}\, Q Q \geq 0$ + Hilbert caveat

regain "global susy."

So here in $D=4$,

$$\gamma \cdot P \; \alpha \, \delta Q(\alpha) = \oint dS_i \, \bar{\alpha} \, \sigma^{ij} \partial_j \alpha$$

$$\Rightarrow \int d^3r \, D_i [\bar{\alpha} \sigma^{ij} D_j \alpha]$$

$$= \underbrace{\int (\bar{\alpha}_i \sigma^{ij} \alpha_j)}_{} + \int \bar{\alpha} \, \underbrace{\sigma^{ij} [D_i, D_j] \alpha}_{G^0{}_\mu \gamma^\mu}$$

(Weitzenbock ≡)

$(\gamma^i \psi_i = 0) \Leftrightarrow$ Witten gauge: $\not{D} \alpha = 0$

$$E = \int d^3x \, |\nabla \alpha|^2 \geq 0 \quad \Big\} \geq 0 \text{ dominant energy cond.}$$

and $= 0 \Rightarrow \alpha = \alpha_0 \Rightarrow$ flat.

Note: SUGRA gave pure GR answer!

Finally, Back to $D=3$!

a) pure GR: Same form exactly:

$$Q = \int J^0 d^2x = \int d^2x\, \varepsilon^{0ij} \partial_i \psi_j = \oint dS_i \varepsilon^{0ij} \psi_j,$$

$\varepsilon^{0ij} = \gamma^0 \sigma^{ij}$ also. ($\gamma^0 = \sigma^3$ etc)

so $E = \int d^2x\, |\nabla \alpha|^2 \geq 0$? NOT QUITE!

But now we know $G_{\mu\nu} = 0$ on shell,

so $\not{D}\alpha = 0 \Rightarrow \nabla_0^2 \alpha = 0 \Rightarrow \alpha = \alpha_0$!

So $E = 0$, as should (same $T_{\mu\nu}$)

b) TMG (at last!).

(Skip $\Lambda \neq 0$ for a minute!)

$Q = \int J^0$ Has 2 parts: $\int J^0_{3/2} + \int J^0_{FCS}$

$$J^0_F = \varepsilon^{0ij} [\psi_j + \tfrac{1}{\mu} D_i (\gamma_\nu \partial_j f^\nu)]$$

$$Q_F = \oint dS_i\, \varepsilon^{0ij} [\psi_j + \tfrac{1}{\mu} \gamma_\nu \partial_j f^\nu] dx^2$$

(keep vol form)

($\Lambda = 0$ cont'd) (10)

Now Write

$$\alpha Q = \oint E^{0ij} \alpha \psi_j \, dS_i \underbrace{+ \frac{1}{M} \int d^3x \, D_i \alpha \gamma_\mu \gamma_j f^\nu \mathcal{E}^{0ij}}$$

[keep in volume form, not as long-range flux part:] convenient.

Now $\alpha \delta Q(\alpha) \to \oint dS_i \underset{\gamma^0 \sigma^{ij}}{E^{0ij}} \alpha D_j \alpha +$

$$\int d^2x \; C^0_\mu \bar{\alpha} \gamma^\mu \alpha$$

by Weitzenbock $=$ (+ Γ-ology), $\delta f \sim \Gamma \bar{R} \alpha$!

$$\to E = \underbrace{\int d^2x (D_i \bar{\alpha} \, \gamma^{ij} D_j \alpha)}_{\text{as in 4D}} + \int d^2x \, \mathcal{G}^0_\mu \bar{\alpha} \gamma^\mu \alpha$$

Einstein part of:

$$\mathcal{G}^0_\mu \equiv G^0_\mu + \underset{= T^0_\mu \text{ matter}}{C^0_\mu}$$

That's it: $\not{D} \alpha = 0 \Rightarrow E = \int d^2x \, |\nabla \alpha|^2 \geq 0$

and not flat, so $\gamma^i \nabla_i \alpha = 0 \Rightarrow (\nabla^2 + R)\alpha = 0$

and $\alpha \neq \alpha_0$

Finally, $\Lambda \neq 0$: very simple: essentially all you do is write $D_\mu \alpha = \delta \psi_\mu$ as $\Longrightarrow (D_\mu + m\gamma_\mu)\epsilon \equiv \bar{D}_\mu \epsilon$
$m^{-1} \equiv \sqrt{-\Lambda}$

$\Longrightarrow$ {PK Townsend ('77), S. D. Zumino, '77}

and $[\bar{D}, \bar{D}] \sim (R + \Lambda)$

and note that the conformal invariance of I_{FCS} allows $D \to \bar{D}$ and some minor, well-known, use of AdS Killing vectors as.

So: $\underline{\underline{C}}$ TMG + TMG $\Leftrightarrow E \geq 0$
For the full nonlinear theory

QED

[This is true for all values of (μ, Λ) including $\mu\sqrt{-\Lambda} = 1$.]

#: $E = 0 \Longrightarrow$ Vacuum, AdS.

Summery

All's well in $D = 3, 4, \ldots$

re-Happy $3 \times 4 \times 5$, Mike!