Relativistic-Newtonian correspondence of the zero-pressure but weakly nonlinear cosmology

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Perturbations

- Perturbation types:

  - Scalar: density fluctuations
  - Vector: rotation
  - Tensor: gravitational waves

  Decouple to the linear order (in Friedmann universe)
  However, they couple in the second-order!

- Evolution:

  - Scalar type: conserved in the super-sound horizon scale.
  - Vector type: angular momentum conservation.
  - Tensor type: conserved in the super horizon scale.

Why we assume the linearity?

- Because
  - Very small CMB temperature anisotropy: $\frac{\delta T}{T} \sim 10^{-5}$
  - The large scale clustering of galaxies is nearly linear as the scale becomes large.
  - Valid in the early universe and the large scale in the present.
  - However, as the scale becomes smaller, the distribution of galaxies apparently shows quasi-linear to fully non-linear structures.

Perturbed Friedmann Universe

- Metric (Bardeen 1988):

$$ds^2 = -a^2(1+2\alpha)\,d\eta^2 - 2a^2(\beta_{,\alpha} + B^{(v)}_\alpha)d\eta dx^\alpha + a^2\left[g^{(3)}_{\alpha\beta}(1+2\varphi) + 2\gamma_{,\alpha|\beta} + 2C^{(v)}_{(\alpha|\beta)} + 2C^{(t)}_{\alpha\beta}\right]dx^\alpha dx^\beta.$$

- Energy momentum tensor:

$$\tilde{T}^0_0 \equiv -\mu - \delta\mu, \quad \tilde{T}^0_\alpha \equiv (\mu + p)(-v_{,\alpha} + v^{(v)}_\alpha), \quad \tilde{T}^\alpha_\beta \equiv (p + \delta p)\,\delta^\alpha_\beta + \Pi^\alpha_\beta.$$

- Pressure-less, irrotational fluid (comoving gauge):

$$\tilde{T}^0_0 = -\tilde{\mu}, \quad \tilde{T}^0_\alpha = 0 = \tilde{T}^\alpha_\beta.$$

- Background: (homogeneous and isotropic)

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\mu - \frac{\text{const.}}{a^2} + \frac{\Lambda}{3}, \qquad \mu \propto a^{-3}.$$

  - Relativistic (Friedmann 1922)
  - Newtonian (Milne-McCrea 1933)
  - In the pressure-less medium, they coincide!

- Linear perturbations:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0.$$

  - Relativistic (Lifshitz 1946)
  - Newtonian (Bonner 1957)
  - In the pressure-less medium, they coincide!

Weakly Nonlinear Perturbations

- Newtonian: (Peebles 1980)

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta \mathbf{u}) \, ,$$

$$\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u} + \frac{1}{a}\nabla\delta\Phi = -\frac{1}{a}\mathbf{u} \cdot \nabla\mathbf{u},$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G\delta\varrho,$$

give

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = -\frac{1}{a^2}\left[a\nabla \cdot (\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) \, .$$

valid to fully order.

We prove the correspondence

Using the Covariant equations (fully nonlinear)

$$\dot{\tilde{\mu}} + \tilde{\mu}\tilde{\theta} = 0,$$
$$\dot{\tilde{\theta}} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0,$$

← Energy conservation

← Raychaudhury equation

where $\dot{\tilde{\mu}} \equiv \tilde{\mu}_{,a}\tilde{u}^a$, $\tilde{\theta} \equiv \tilde{u}^a{}_{;a}$, etc.

gives

$$\left(\frac{\dot{\tilde{\mu}}}{\tilde{\mu}}\right)^{\tilde{\cdot}} - \frac{1}{3}\left(\frac{\dot{\tilde{\mu}}}{\tilde{\mu}}\right)^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - 4\pi G\tilde{\mu} + \Lambda = 0.$$

To the second order

- Identifying

$$\delta\mu_v \equiv \delta\varrho, \quad \delta\theta_v \equiv \frac{1}{a}\nabla\cdot\mathbf{u}.$$

Comoving gauge

- We have

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}),$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u}\right) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a^2}u_{\alpha,\beta} + \dot{C}^{(t)}_{\alpha\beta}\right).$$

- Second-order perturbation:
  - Newtonian (Peebles 1980)
  - Relativistic (Noh-Hwang 2004; K=0, Λ≠0, irrotational):

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}\nabla^\alpha u^\beta + \dot{C}^{(t)\alpha\beta}\right).$$

  - Except for the gravitational wave contribution, the relativistic zero-pressure fluid perturbed to the second-order in a flat Friedmann background coincides exactly with the Newtonian one.

Third-order perturbations

(Physical Rev. D. 72, 044012 (2005))

- To the third-order,

$$\delta\mu_v \equiv \delta\rho, \quad \delta\theta_v \equiv \frac{1}{a}\nabla\cdot\mathbf{u}.$$

- Pure scalar type perturbation gives

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$$
$$+\frac{1}{a^2}\left\{a\left[2\varphi_v\mathbf{u} - \nabla\left(\Delta^{-1}X\right)\right]\cdot\nabla\delta\right\}^{\cdot} - \frac{4}{a^2}\nabla\cdot\left[\varphi_v\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]$$
$$+\frac{2}{3a^2}\varphi_v\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) + \frac{\Delta}{a^2}\left[\mathbf{u}\cdot\nabla\left(\Delta^{-1}X\right)\right] - \frac{1}{a^2}\mathbf{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\mathbf{u},$$
$$X \equiv 2\varphi_v\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi_v + \frac{3}{2}\Delta^{-1}\nabla\cdot\left[\mathbf{u}\cdot\nabla(\nabla\varphi_v) + \mathbf{u}\Delta\varphi_v\right].$$

- Non-vanishing pure relativistic corrections are φ_v order higher than the Newtonian ones.

- For general Λ

$$\dot{\varphi}_v = 0.$$

- The CMB temperature anisotropy shows $(\Lambda = 0)$

$$\frac{\delta T}{T} \sim \frac{1}{3}\delta\Phi \sim \frac{1}{5}\varphi_v \sim 10^{-5}.$$

SUMMARY

- Equations for the relativistic pressure-less fluid (except for the gravitational wave contribution) in a flat Friedmann universe coincide exactly with the Newtonian ones even to the second-order perturbations.

- Up to the second order, the relativistic density and the velocity perturbation variables are identified. Though, we do not identify the relativistic variable corresponding to the Newtonian gravitational potential.

- The cosmological constant is included.

- We show that the Newtonian dynamics is applicable even to the horizon scale in a pressure-less case.

- The third-order correction terms show the pure general relativistic effects, and are independent of the horizon scale. It depends only on the linear order gravitational potential perturbation.

$$\frac{\delta T}{T} \sim \frac{1}{3}\delta\Phi \sim \frac{1}{5}\varphi_v \sim 10^{-5}.$$

- Now one can use the large scale numerical simulations more reliably even as the simulation scale approaches near the horizon.

So far we assumed

- Flat Friedmann background model
- Pressure-less
- Irrotational
- Single component

In progress

- Relaxing the above assumptions!
  - Background Curvature
  - Pressure: relativistic even to the linear order
  - Rotation
  - Multi-component fluids (irrotation, pressure-less)

lead to pure general relativistic effects!