Hydrodynamics and Gravity

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- In this talk I outline a precise map between (perhaps) the two most important and best studied nonlinear partial differential equations in physics; Einstein's equations of general relativity and the equations of hydrodynamics. The setting for this 'duality' is the AdS/CFT correspondence of string theory.
- All D dimensional CFTs admit a classical D dimensional hydrodynamical description at large distances and times compared to an equilibriation scale ($\propto T^{-1}$). However AdS/CFT asserts that particular CFTs always admit a classical $D+1$ dimensional gravitational description.
- It must be that $D+1$ dimensional gravitational equations reduce to D dimensional hydrodynamics at long wavelengths. This talk: direct check without using AdS/CFT.

- Under appropriate circumstances the appropriate gravitational equations for AdS/CFT are Einstein's equations with a negative coslmological constant

$$R_{MN} - \frac{R}{2} g_{MN} = \frac{d(d-1)}{2} g_{MN} : \quad : M, N = 1 \ldots d+1$$

- The black brane at temperature T and velocity u_μ are a d parameter set of exact solutions of these equations

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

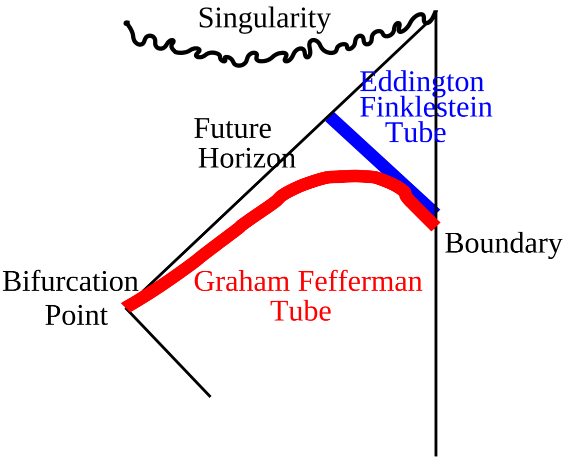
$$f(r) = 1 - \left(\frac{4\pi T}{d\ r}\right)^d ; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

- These solutions have a horizon at $r = \frac{4\pi T}{d}$. They describe a thermalized field theory moving with temperature T and at uniform velocity u^μ.

- Einstein's equations allow us to study deviations from thermal equilibrium. Natural first question: what is the spectrum of linearized fluctuations about thermal equilibrium?.
- If we impose the requirement of regularity of the future horizon, the anwer is given by gravitational 'quasinormal modes'. Discrete infinity of such modes labeled by integers. For the n^{th} mode $\omega = \omega_n(k)$. Frequency complex corresponding to decay.
- It follows from conformal invariance that $\omega_n(0) = \frac{f(n)}{T}$. $f(n) \neq 0$ except for the 4 Goldstone modes corresponding to variations of T and u_μ. Infact Policastro Starinets and Son demonstrated that the dispersion relation for these Goldstone modes at small k takes the form predicted by fluid dynamics -(shear and sound waves) provided $\frac{\eta}{s} = \frac{1}{4\pi}$

- Let us search for solutions to Einstein's equations that 'locally' approximate black branes but with space varying velocities and temperatures. More precisely we search for bulk solution tubewise approximated by black branes.
- Causality, and the requirement of regularity, both suggest that the tubes follow null ingoing geodesics

Penrose diagram



- In particular let us define

$$ds^2 = g^{(0)}_{MN} dx^M dx^N = -2u_\mu(x) dx^\mu dr + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu dx^\nu - r^2 f(r, T(x)) u_\mu(x) u_\nu(x) dx^\mu dx^\nu$$

- Metric generally regular but not solution to Einstein's equations. However solves equations for constant $u^\mu, T, g_{\mu\nu}$. Consequntly appropriate starting point for a perturbative soln of equations in the parameter $\epsilon(x)$.

- That is we set

$$g_{MN} = g^{(0)}_{MN}(\epsilon x) + \epsilon g^{(1)}_{MN}(\epsilon x) + \epsilon^2 g^{(2)}_{MN}(\epsilon x) \ldots$$

  and attempt to solve for $g^{(n)}_{MN}$ order by order in ϵ.
- Perturbation expansion surprisingly simple to implement. Nonlinear partial differential equation $\rightarrow \frac{d(d+1)}{2}$ ordinary differential equations, in the variable r at each order and each boundary point.

- It turns out that all equations can be solved analytically (and rather simply). Upon solving the equations we find that the perturbative procedure spelt out above can be implimented at n^{th} order *only* when an integrability condition of the form $\partial_\mu T^{\mu\nu}_{n-1}(u^\mu(x), T(x))$ where $T^{\mu\nu}_{n-1}(u^\mu(x), T(x))$ is a specific function of temperature and velocities and their spacetime derivatives to $(n-1)^{th}$ order. This function is determined directly from Einstein's equations by the perturbative procedure.
- For every $u^\mu(x)$ and $T(x)$ that satisfies this Fluid Dynamical equation we have a solution to Einstein's equations. The map from fluid dynamics to gravity is locally invertable assuming regularity of the future event horizon.

Second order boundary stress tensor

The dual stress tensor corresponding to this metric is given by ($4\pi T = b^{-1} d$)

$$\begin{aligned} T_{\mu\nu} &= p\left(g_{\mu\nu} + d u_\mu u_\nu\right) \\ &\quad - 2\eta\left[\sigma_{\mu\nu} - \tau_\pi u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} - \tau_\omega\left(\sigma_\mu{}^\lambda \omega_{\lambda\nu} - \omega_\mu{}^\lambda \sigma_{\lambda\nu}\right)\right] \\ &\quad + \xi_\sigma\left[\sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu}\right] + \xi_C C_{\mu\alpha\nu\beta} u^\alpha u^\beta \end{aligned}$$

$$p = \frac{1}{16\pi G_{d+1} b^d} \quad ; \quad \eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{d+1} b^{d-1}}$$

$$\tau_\pi = (1 - H_1(1)) b \quad ; \quad \tau_\omega = H_1(1) b \quad ; \quad \xi_\sigma = \xi_C = 2\eta b$$

- Note that gravity reduces to fluid dynamics with particular (holographically determined) values for dissipative parameters. As we have seen the schematic form of the fluid stress tensor is

$$T_{\mu\nu} = aT^d(g_{\mu\nu} + du_\mu u_\nu) + bT^{d-1}\sigma_{\mu\nu} + T^{d-2}\sum_{i=1}^{5} c_i S^i_{\mu\nu}$$

- a is a thermodynamic parameter. b is related to the viscosity: we find $\eta/s = 1/(4\pi)$. c_i coefficients of the five traceless symmetric Weyl covariant two derivative tensors are second order transport coefficients. Value disagree with the predictions of the Israel Stewart formalism.
- Recall that results universal. Should yield correct order of magnitude estimate of transport coefficients in any strongly coupled CFT.

- Our solutions are singular at $r = 0$. Quite remarkably it is possible (under certain conditions) to demonstrate that these solutions have event horizons and to explicitly determine the event horizon manifold order by order in the derivative expansion. This horizon shields the $r = 0$ singularity from the boundary.
- Our control over the event horizon, together with the classic area increase theorem of general relativity, can be used to derive an 'entropy current' for our fluid flows that is local and has positive divergence.

Entropy Current at second order

Explicitly this entropy current is given to second order by

$$4\,G_{d+1}\,b^{d\,-\,1}\,J_S^{\mu} = [\,1\,+\,b^2\,(\,A_1\,\sigma^{\alpha\beta}\,\sigma_{\alpha\beta}\,+\,A_2\,\omega^{\alpha\beta}\,\omega_{\alpha\beta}\,+\,A_3\,\mathcal{R}\,)\,]\,u^{\mu} \\ +\,b^2\,[\,B_1\,\mathcal{D}_{\lambda}\,\sigma^{\mu\lambda}\,+\,B_2\,\mathcal{D}_{\lambda}\,\omega^{\mu\lambda}\,]$$

where

$$A_1 = \frac{2}{d^2}(d+2) - \frac{K_1(1)d + K_2(1)}{d}\,, \qquad A_2 = -\frac{1}{2d}, \qquad B_2 = \frac{1}{d-2}$$

$$B_1 = -2A_3 = \frac{2}{d\,(d-2)}$$

Charged Fluid Dynamics

- The story described above was developed over four years ago. It has since been generalized in many directions. The two most interesting generalizations both require the addition of a Maxwell field (in addition to the graviton) to the bulk.
- The addition of a bulk Maxwell field endows the boundary theory with a conserved global charge. Equilibrium states in such a system are labeled by a charge density together with the energy density and velocity; in the bulk these equilibrium configurations are given by charged AdS-Reisnner Nordstorm black branes.

- Following the general procedure described in this talk, it has been established that the AdS Einstein -Maxwell equations reduce, in an appropriate long wavelength limit, to the equations of charged relativistic hydrodynamics.
- The procedure yields expressions for the stress tensor and charge currents as a function of local temperatures, velocities and chemical potentials.
- We find a surprise here even at first order in the derivative expansion. In addition to the usual diffusive currents, in $d = 4$ we find a term in the charge current proportional to $\epsilon_{\mu\nu\rho\sigma}\omega^{\nu\rho}u^{\sigma}$. This is important because this term was ignored by Landau and Lifshitz and perhaps all authors subsequently.

Superfluidity

- Another generalization is to the study of superfluid hydrodynamics. Superfluidity arises in systems with a conserved global charge when the equilibrium state includes a condensate of an operator of nonzero global charge.
- The bulk dual description of the equilibrium state of a superfluid is a charged black brane emmersed in charged scalar ‘hair’. Such hairy black brane solutions exist in certain AdS models, and have been intensively studied. Once again repeating the procedure above on these solutions results in the equations of Landau-Tiza superfluidity, with one new term that was missed by previous studies, and a new structure for parity odd fluids.

- As we have described above, computations within gravity have revealed that the 'standard' equations of hydrodynamics are incomplete in certain situations (some allowed terms have been missed).
- It is clear from this observation that a complete and systematic 'theory of hydrodynamics' is at present lacking.
- We will list two general principles that constrain the equations of fluid dynamics and whose analysis may eventually lead to a complete 'theory' of fluid dynamics.

The Entropy positivity constraint

- Fluid constitutive relations must be consistent with existence of an entropy current whose divergence is positive in every fluid flow in every consitent background, including in curved space.
- This principle turns out to be surprisingly constraining. At first order in ordinary relativistic charged fluid dynamics it sets two of the five symmetry allowed constitutive parameters to zero. At second order in uncharged fluid dynamics it kills 5 out of the 15 parameters . At first order in parity invariant superfluid dynamics it kills 26 of the 47 parameters.

Consistency with Equilibrium

- All well behaved systems are expected to equilibriate under appropriate circumstances. In particular, a relativistic system formulated on a spacetime with a timelike killng vector field must equilibriate. Moreover the equilibrium must be generated by a partition function.
- The existence of a stationary solution to the equations of hydrodynamics is not automatic in an arbitrary stationary background, but instead imposes constraints on constitutive relations.
- It has very recently been demonstrated in four nontrivial examples that these constraints match with those of entropy positivity. Always true? Implications for the 2nd law?

Consequences for gravity

- The second and third constraints listed above are kinematical in AdS/CFT. However the first constraint is more interesting, and has only been proved for 2 derivative Einstein gravity in the bulk.
- It is possible that the kinematical constraints completely imply the first set of constraints. This would suggest that the area increase theorem of gravity can be generalized to an entropy increase theorem for arbitrary higher derivative gravity, on purely kinematical grounds.
- Another possibility, however, is that the constraints from entropy positivity are cannot be derived from kinematical considerations. Their validity could conceivably constrain higher derivative corrections to Einstein's equations.

- Within the fluid gravity correspondence, it is possible to show that there exist nontrivial gravitational configurations in which the entropy production vanishes in Einstein gravity
- It is possible that requirement that entropy remain either constant, or increase in such configurations will yield nontrivial constraints on possible higher derivative corrections to Einstein's equation.
- This suggests the exciting - though perhaps unlikely - possibility of an infinite number of completely new purely low energy constraints on structure of higher derivative corrections to Einstein gravity whose origin lies in the thermodyanical nature of gravity in configurations with a horizon.

Conclusions

- Asymptotically AdS_{d+1} gravity reduces, in the long wavelength limit, to the equations $d+1$ dimensional Navier Stokes equations with gravitationally determined dissipative parameters.
- The gravitationaly determined fluid dynamical system has no free parameters, and so enjoys a degree of universality. Values of these parameters are interesting and sometimes surprising.
- This map allows us to reword open problems in fluid dynamicsas problems in gravity. Potential for new insights?
- It would be very interesting to extend this connection past the large N (or classical) limit.