Holographic Renormalization

for non-relativistic systems

Jan de Boer, Amsterdam

Royal Society, April 23, 2012

Mostly based on:

Baggio, JdB, Holsheimer arxiv:1107.5562
Baggio, JdB, Holsheimer arxiv:1112.6416

Lifshitz backgrounds are interesting candidate gravitational duals of strongly coupled non-relativistic conformal field theories.

Koroteev, Libanov
Kachru, Liu, Mulligan

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2$$

Dual to field theories with anisotropic scale invariance

$$t \to \lambda^z t, \qquad x \to \lambda x$$

These geometries have various issues:

- For z<1, they violate NEC (Hoyos, Koroteev)
- There is no regular global version of Lifshitz
- There is a null singularity as $r \to -\infty$ in the deep IR. Not clear whether and how these are resolved in string theory (Horowitz, Way) (Harrison, Kachru, Wang)
- What are the right boundary conditions to impose?
- Do the usual recipes of AdS/CFT such as holographic renormalization carry over?

Schrödinger backgrounds are also interesting gravitational duals of non-relativistic systems.

Son
Balasubramanian, McGreevy

$$ds^2 = -\beta \frac{dt^2}{r^{2z}} + \frac{2dt\,du + d\vec{x}^2 + dr^2}{r^2}$$

Dual to field theories with anisotropic scale invariance

$$t \to \lambda^z t, \qquad x \to \lambda x, \qquad u \to \lambda^{2-z} u$$

Clearly, z=2 is a distinguished value

These geometries have various issues:

- For various z and choices of sign of β problems with geodesic completeness, singularities and tidal forces. (Blau, Hartong, Rollier)

- Problematic issues with the null coordinate, can be partially cured by turning on a finite temperature.

- What are the right boundary conditions to impose?

- Do the usual recipes of AdS/CFT such as holographic renormalization carry over?

Virtually all known Schrödinger solutions are obtainable as deformations of AdS solutions.

If one deforms a CFT by a null vector operators and null spin-two operators one obtains non-relativistic theories

$$S_{\rm CFT} + g_1 \int \mathcal{O}_- + g_2 \int \mathcal{O}_{--}$$

$$\Delta = d + z - 1 \qquad\qquad \Delta = d + 2(z-1)$$

Kraus, Perlmutter

No higher order terms are generated for CFTs with a gravitational dual.

Holographic computations in the Schrödinger case:

The main point is that one should work in momentum space in the u-direction.

Henkel, Unterberger
Fuertes, Moroz
Volovich, Wen
Leigh, Hoang
Barnes, Vaman, Wu
Guica, Skenderis, Taylor, van Rees

Non-relativistic correlators can for example be obtained from AdS/CFT correlators by Fourier fransforming with respect to a light-cone coordinate.

$$G(x^+, x^i) = \int dx^- \frac{1}{|x^+x^- + x^i x^i|^\Delta} e^{-ipx^-}$$

There is similar to DLCQ. There are some issues with compact vs noncompact x^-.

For the rest many things work as in AdS/CFT.

For the remainder, will look at Lifshitz solutions instead.

Correlation functions for scalar operators can again be computed in a fairly straightforward way.

$$\langle \mathcal{O}(t,x)\mathcal{O}(t',x')\rangle = \frac{1}{|x-x'|^{2\Delta}} f\left(\frac{|t-t'|}{|x-x'|^{z}}\right)$$

Kachru, Liu, Mulligan
Balasubramanian, McGreevy
Keränen, Thorlacius

An important issue in understanding Lifshitz spacetimes has to do with the allowed boundary conditions on fields and in particular the metric.

Best studied in a concrete example.

Setup (for simplicity; other Lagrangians could also be considered):

Taylor

$$S = S_{\rm grav} + S_A + S_\phi$$

$$S_{\rm grav} = \int d^{d+1}x\sqrt{-g}\,(R - 2\Lambda) + \int d^d\xi\sqrt{-\gamma}\;2K,$$

$$S_A = \int d^{d+1}x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right),$$

$$S_\phi = \int d^{d+1}x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right)$$

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2$$

$$A = \sqrt{-\alpha_0}e^{zr}dt, \quad \Lambda = -\tfrac{1}{2}(z^2 + z + 4), \quad m^2 = 2z, \quad \alpha_0 = -2\tfrac{z-1}{z}$$

We are going to cut off the space at a finite value of r and evaluate the on-shell value S_{cl} of the effective action subject to Dirichlet boundary conditions.

In AdS/CFT, we can write

$$S_{\rm cl} = S_{\rm loc} + \Gamma$$

Where $S_{\rm loc}$ contains all power-law divergences and is local. Moreover, it admits a derivative expansion.

Key idea 1: assume that Lifshitz solutions are consistent, and can also be holographically renormalized using local counterterms

Key idea 2: Solve the Hamilton-Jacobi equations for S_{cl}. Insist that Lifshitz space-time is a solution. This will fix the counterterms, as well as the boundary conditions on the fields.

JdB, Verlinde, Verlinde

One can analyze all counterterms order by order in a derivative expansion. They are all local, generally covariant expressions of the metric γ on a fixed r-slice that do not explicitly on r.

$$\{S_{\rm loc} + \Gamma, S_{\rm loc} + \Gamma\} = \mathcal{L}$$

Notice that everything is completely covariant. At the end of the day we are going to evaluate these covariant quantities on a metric γ of the form

$$ds^2 = -e^{2zr} dt^2 + e^{2r} d\vec{x}^2$$

and as $r \to \infty$ this will generate non-covariant terms

Counterterms

The lowest order, non-derivative counterterm is of the form

$$S_{\rm loc} = \int_{\Sigma_r} d^d x \sqrt{-\gamma}\, U(\alpha, \phi)$$

where $\alpha = A_a A^a$ is the unique scalar that can be made out of the gauge field.

Using this, the leading contribution near the boundary to the various radial derivatives reads (d=3 from now on)

$$\begin{aligned}
\partial_r \gamma_{ab} &= \left(\frac{1}{2}U - \alpha\,\frac{\partial U}{\partial \alpha}\right)\gamma_{ab} + 2A_a A_b\,\frac{\partial U}{\partial \alpha}, \\
\partial_r A_a &= -2A_a\,\frac{\partial U}{\partial \alpha}, \\
\partial_r \phi &= -\frac{\partial U}{\partial \phi}.
\end{aligned}$$

Since Lifshitz itself has to be a solution, this fixes the constant and linear terms in U:

$$U(\alpha,\phi) = 2(z+1) - \tfrac{z}{2}(\alpha-\alpha_0) + \mathcal{O}(\phi^2, (\alpha-\alpha_0)^2)$$

Next one can use the HJ equations to find the quadratic terms in U. These in turn determine the leading behavior of $(\alpha-\alpha_0)$ and ϕ :

This yields

$$\partial_r\phi = \lambda_\phi\phi, \qquad \partial_r(\alpha-\alpha_0) = \lambda_\alpha(\alpha-\alpha_0)$$

with

$$\lambda_\phi = -\tfrac{1}{2}((z+2) \pm \sqrt{(z+2)^2 + 4\mu^2})$$

$$\lambda_\alpha = -\tfrac{1}{2}((z+2) \pm \sqrt{(z+2)^2 + 8(z-1)(z-2)})$$

The scalar field result can also be obtained using the linearized field equations, the result for the composite field is not quite as easily obtainable.

Boundary conditions: to have a well-defined series of counterterms, we need that $\sqrt{-\gamma} \sim e^{(z+2)r}$ which does e.g. not allow arbitrary $\gamma_{ti} \sim e^{2zr}$ which according to a linearized analysis could be there. Similarly,

$$\gamma_{ab} A^a A^b = \alpha_0 + \alpha_1 e^{\lambda_\alpha r} + \dots$$

This can not be formulated as in AdS as boundary conditions on individual components – boundary conditions are nonlinear.

One can also formulate this in terms of frame fields. Ross

Ambiguities

When solving the HJ equations, one sometimes runs into ambiguities. These appear when there are special relations between scaling dimensions, like e.g.

$$\lambda_\alpha + 2\lambda_\phi = -(z+2)$$

which shows that the dual theory has a marginal operator.

Consistency check: We explicitly computed some solutions of the field equations perturbatively and verified that the counterterms indeed remove divergences (even for z=2)

Cheng, Hartnoll, Keeler

Higher derivative counterterms: can also be done but very tedious.

What is the interpretation of the bulk metric and gauge field in the dual non-relativistic field theory?

Quantum Lifshitz model

$$S = \int dt d^2x \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_i \partial_i \phi)^2 \right)$$

This theory has conserved currents associated to translational invariance. These do not form a symmetric energy-momentum tensor.

It is also possible to couple the theory to metric $g_{\mu\nu}$ and timelike unit vector n_μ with $n_\mu n^\mu = -1$

Define

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

$$\Delta\phi = \partial_\mu(h^{\mu\nu}\partial_\nu\phi) + \frac{1}{2}h^{\mu\nu}h^{\rho\sigma}\partial_\mu\phi\partial_\beta h_{\rho\sigma}$$

then the action

$$S = \int dt d^2x\sqrt{-g}\left(\frac{1}{2}(n^\mu\partial_\mu\phi)^2 - \frac{1}{2}(\Delta\phi)^2\right)$$

couples the Lifshitz model to a metric and gauge field in the same way as one would expect from the bulk description.

The variation of this action with respect to the metric yields a symmetric tensor which is not conserved.

Combinations of variations with respect to the metric and with respect to the timelike unit vector can be used to obtain the components of the conserved currents.

Conformal anomalies

Conformal anomalies are a very useful characterization of conformal field theories and control many properties (entanglement entropy, finite-size effects, etc)

What is the right generalization to non-relativistic theories?

Inspired by the Lifshitz geometry, consider the metric

$$ds^2 = -N(t,x)^z dt^2 + h_{ij}(t,x) dx^i dx^j$$

then the theory is somehow expected to be invariant under

$$N \to \Omega(t,x) N, \qquad h_{ij} \to \Omega(t,x) h_{ij}$$

but this symmetry can be broken by a conformal anomaly.

The conformal anomaly can be expressed as:

$$\mathcal{A} = z\langle T_t{}^t\rangle + \langle T_i{}^i\rangle$$

The conformal anomaly must transform in the right way under global scale transformations, ie have weight –(z+2), must be covariant under diffeomorphisms in time and under diffeomorphisms in space.

We restrict to z=2 from now on. There are then 15 possible terms. Three time-dependent ones

$$h^{ij}\frac{1}{N}\partial_t(\frac{1}{N}\partial_t h_{ij}), \qquad \frac{1}{N^2}(h^{ij}\dot{h}_{ij})^2, \qquad h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li}.$$

and there are 12 with only spatial derivatives

$G_1 = R^2$ $\quad$ $G_2 = \Delta R$ $\quad$ $G_3 = (\frac{1}{N}\Delta N)R$

$G_4 = (\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N)R$ $\quad$ $G_5 = ((\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N))^2$ $\quad$ $G_6 = (\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N)(\frac{1}{N}\Delta N)$

$G_7 = (\frac{1}{N}\Delta N)^2$ $\quad$ $G_8 = (\frac{1}{N}\partial^i N)(\frac{1}{N}\partial^j N)(\frac{1}{N}\nabla_i\partial_j N)$ $\quad$ $G_9 = (\frac{1}{N}\partial^i N)\frac{1}{N}\partial_i\Delta N$

$G_{10} = \frac{1}{N}\nabla_i\partial_j N\frac{1}{N}\nabla^i\partial^j N$ $\quad$ $G_{11} = \frac{1}{N}\Delta^2 N$ $\quad$ $G_{12} = \frac{1}{N}\partial^i N\partial_i R$

Not all terms can appear in the conformal anomaly: WZ consistency conditions imply that the anomaly is itself invariant under local scale transformations up to total derivatives.

Moreover, some anomalies can be canceled by adding suitable counterterms to the action.

Result:

$$\mathcal{A} = C_1 \frac{1}{N^2} \left((h^{ij}\dot{h}_{ij})^2 - 2h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li} \right) +$$

$$C_2 \left(R + \frac{1}{N}\Delta N - \frac{1}{N^2}\partial_i N \partial^i N \right)^2$$

The Lifshitz model

$$S = \int dt d^2 x N \sqrt{h} \left(\frac{1}{2} N^{-2} (\partial_t \phi)^2 - \frac{1}{2} (\frac{1}{\sqrt{h}} \partial_i \sqrt{h} h^{ij} \partial_j \phi)^2 \right)$$

Can easily be coupled to the relevant background fields that scale as before.

Of interest for a variety of systems like e.g. the quantum dimer model.

We would like to compute the conformal anomaly of this model. Rewrite the action as

$$S = \int dt d^2 x N \sqrt{h} \, (\phi D \phi)$$

The partition function is obtained from the determinant of D. This can be computed using zeta-function regulation in terms of

$$\zeta(s, f, D) = \mathrm{Tr}_{L^2}(f D^{-s})$$

The regularized effective action is then

$$W = -\frac{1}{2}\zeta'(0, 1, D) - \frac{1}{2}\log(\mu^2)\zeta(0, 1, D)$$

One can also define the heat kernel

$$K(t, f, D) = \mathrm{Tr}_{L^2}(f e^{-tD})$$

Which is related via

$$\zeta(s, f, D) = \Gamma(s)^{-1} \int_0^\infty dt t^{s-1} K(t, f, D)$$

The heat kernel expansion is of the form

$$K(\epsilon, f, D) = \sum_{k=0}^{\infty} \epsilon^{\frac{k}{2}-1} \tilde{a}_k(f, D)$$

and the conformal anomaly is then related to the finite term in the heat kernel expansion via

$$\delta W = -2\tilde{a}_2(\delta\rho, D)$$

To compute the heat kernel, we write it as

$$K = \int \frac{d\omega d^2 k}{(2\pi)^3} \int dt d^2 x e^{-i\omega t - ikx} f e^{-\epsilon D} e^{i\omega t + ikx}$$

Ceresole, Pizzochero, van Nieuwenhuizen

A tedious computation shows that

$$C_1 = -\frac{1}{256\pi}, \qquad C_2 = 0$$

Recall that

$$\mathcal{A} = C_1 \tfrac{1}{N^2} \left((h^{ij}\dot{h}_{ij})^2 - 2h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li} \right) + C_2 \left(R + \tfrac{1}{N}\Delta N - \tfrac{1}{N^2}\partial_i N \partial^i N \right)^2$$

Conformal anomaly from gravity

Recall that from the HJ equation we obtained a relation between local counterterms and a finite Remainder

$$\{S_{\rm loc}, S_{\rm loc}\} - \mathcal{L} = \mathcal{H}_{\rm rem}$$

and that

$$2\{S_{\rm loc}, \Gamma\} \simeq \mathcal{H}_{\rm rem}$$

The latter equation can be expanded as

$$\begin{aligned}
\partial_r \Gamma &= \int d^dx \left(\dot{\gamma}_{ab} \frac{\delta \Gamma}{\delta \gamma_{ab}} + \dot{A}_a \frac{\delta \Gamma}{\delta A_a} + \dot{\phi} \frac{\delta \Gamma}{\delta \phi} \right) \\
&= \int d^dx \left(2z\hat{\gamma}_{tt} \frac{\delta \Gamma}{\delta \hat{\gamma}_{tt}} + 2\hat{\gamma}_{ij} \frac{\delta \Gamma}{\delta \hat{\gamma}_{ij}} + z\hat{A}_t \frac{\delta \Gamma}{\delta \hat{A}_t} + \lambda_\phi^- \hat{\phi} \frac{\delta \Gamma}{\delta \hat{\phi}} \right) + \ldots \\
&= \int d^dx \sqrt{-\hat{\gamma}} \left(z\langle T_t{}^t \rangle + \langle T_i{}^i \rangle + z\hat{A}_t \langle \mathcal{O}_A \rangle + \lambda_\phi^- \hat{\phi} \langle \mathcal{O}_\phi \rangle \right) + \ldots,
\end{aligned}$$

To compute the conformal anomaly, we need to find the local counterterms, which is a purely algebraic problem, but unfortunately very tedious, and the evaluate these on the metric $ds^2 = -e^{2zr}dt^2 + e^{2r}d\vec{x}^2$ in the limit $r \to \infty$.

Result:

$$C_1 = -\frac{1}{128\pi G}, \qquad C_2 = 0$$

See also Griffin, Horava, Melby-Thompson

Conclusions

- Extend to other values of d,z and other theories.

- Explain C2=0? Renormalization of z? Detailed balance?

  Adams, Maloney, Sinha, Vazquez

- How restrictive is bulk covariance for the class of non-relativistic theories with a graviational dual?

- Any connection to ground state wave function of Lifshitz theory?

- Anomaly of DLCQ theories?

- Explore the physical consequences of these anomalies