OU-HET 288
hep-th/9801023

Euclidean Path Integral, D0-Branes and Schwarzschild Black Holes in Matrix Theory

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Abstract

The partition function in Matrix theory is constructed by Euclidean path integral method. The D0-branes, which move around in the finite region with a typical size of Schwarzschild radius, are chosen as the background. The mass and entropy of the system obtained from the partition function contain the parameters of the background. After averaging the mass and entropy over the parameters, we find that they match the properties of 11D Schwarzschild black holes. The period β of Euclidean time can be identified with the reciprocal of the boosted Hawking temperature. The entropy S is shown to be proportional to the number N of Matrix theory partons, which is a consequence of the D0-brane background.

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1 Introduction

Recently, Banks, Fischler, Klebanov and Susskind found that Matrix theory [1] can describe the properties of Schwarzschild black holes, including the energy-entropy relation, the Hawking temperature and the physical size up to numerical factors of order unity [2, 3, 4]. Their analysis was done in the limit that the entropy S of Schwarzschild black holes is proportional to the number N of Matrix theory partons. In ref. [5], it was pointed out that the $S \sim N$ limit corresponds to the black hole/black string transition point, and the thermodynamics of Schwarzschild black holes is determined by the mean field dynamics of the induced super Yang-Mills zero modes.

In ref. [6], it was argued that black hole entropy should be independent of the specifics of the boosting procedure. The properties of boosted Schwarzschild black holes can be understood in the framework of an interacting gas of Matrix theory partons. Actually, in ref. [4] the Matrix theory partons – D0-branes, were treated as distinguishable (Boltzmannian) particles to fit the relations of black hole thermodynamics.

To examine the idea of Boltzmann gas of D0-branes, the classical statistical mechanics of an ensemble of D0-branes in toroidally compactified string theory was explored in ref. [7], and it was found that the absence of $1/N!$ factor in the classical Boltzmann partition function is essential for obtaining the correct black hole thermodynamic functions. In ref. [8], the concept of infinite statistics was applied to analyse the properties of Schwarzschild black holes in Matrix theory, and the author found that D0-branes satisfy quantum infinite statistics.

Other related works on Schwarzschild black holes in Matrix theory can be found in refs. [9, 10]. Especially, in [10] the authors explained that the entropy of boosted $(11-p)$-dimensional Schwarzschild black holes equal to the entropy of a black p-brane [2, 3] is precisely the boosted version of the black hole/black string transition when the black hole radius R_S becomes greater than the radius of the compact direction R. They argued that this fact is independent of any property of the 11D M-theory or Matrix theory, but rather it is a result in any theory which contains General Relativity in $(d+1)$ dimensions. Furthermore, the evaporation of Schwarzschild black holes in Matrix theory was discussed in ref. [11], where the Hawking radiation was realized by emission of small clusters of D0-

branes, and it was found that the rate of the Hawking radiation in Matrix theory model of Schwarzschild black holes agrees with the semi-classical rate up to a numerical coefficient of order 1.

On the other hand, Matrix theory purports to be the only candidate for nonperturbative string theories (M-theory), which probably provides a complete quantum theory of gravity (even though there is difficulty to find a general description of Kaluza-Klein compactification of Matrix theory). Thus it is interesting to calculate the entropy of Schwarzschild black holes from first principles of Matrix theory. In the context of Einstein gravity, the Bekenstein-Hawking entropy for Schwarzschild black holes can be derived by Euclidean path integral approach. The partition function $Z(\beta)$ is defined by

$$Z(\beta) = \mathrm{Tr} e^{-\beta H}, \tag{1.1}$$

where H is the Hamiltonian operator. The right hand side of eq. (1.1) can be calculated by path integral method, and one has

$$Z(\beta) = \int \mathcal{D}[\text{path}] e^{-L_E}, \tag{1.2}$$

where L_E denotes the "Euclidean action", and the integral is taken over all Euclidean paths which are periodic in Euclidean time with period β.

Usually one evaluates $Z(\beta)$ by expanding around a minimum of L_E and calculating the contribution to $Z(\beta)$ in the one-loop approximation. In the case of Schwarzschild black hole background, Gibbons and Hawking [12] calculated the path integral in the tree-level approximation, and found that the entropy derived from the partition function in this approximation is precisely the Bekenstein-Hawking entropy. The following questions then naturally arise: Is it possible to calculate the Bekenstein-Hawking entropy for Schwarzschild black holes by Euclidean path integral method in Matrix theory? What is the proper background for Schwarzschild black holes in Matrix theory?

In the present paper, the partition function $Z(\beta)$ in Matrix theory is constructed by Euclidean path integral method. Since the works in refs. [2]-[9] strongly suggest that Schwarzschild black holes consist of D0-brane gas, as the first simple exploration, we consider D0-branes for our background. To compute the partition function $Z(\beta)$, we expand the Lagrangian of Matrix theory to quadratic order in the fluctuation around the background, and integrate out the off-diagonal matrix elements which correspond to the degrees of freedom of the virtual strings stretched among different D0-branes. From the resulting partition function $Z(\beta)$, we read off the mass and entropy of the system, which contain the parameters of the background. Since the typical size of the background should be smaller than or equal to Schwarzschild radius [4], the average values of the parameters of the background can be obtained. After using these average values of the parameters of the background in the mass and entropy of the system, we find that they match the properties of 11D Schwarzschild black holes. From the consistency of our formalism, we show that the period β of Euclidean time can be identified with the reciprocal of the boosted Hawking temperature.

In refs. [2, 3, 4], the analysis was done in the limit that the entropy S of Schwarzschild black holes is proportional to the number N of Matrix theory partons. In the present case, the entropy of the system is derived from the partition function $Z(\beta)$ without any extra assumption, and the $N \sim S$ limit is found to be a consequence of the fact that D0-branes are exploited as the background. Thus it is consistent with the picture that D0-brane gas can describe the properties of 11D Schwarzschild black hole states in the region $S \sim N$.

The layout of the paper is as follows. In the next section we consider the D0-branes as the background, discuss the restrictions on its parameters, and construct the partition function $Z(\beta)$ in Matrix theory by Euclidean path integral method. In sect. 3 we calculate the mass, entropy, temperature and the typical size of the system from the resulting partition function $Z(\beta)$ and the consistency of our model. In the derivation, we do not refer to any information about Schwarzschild black holes except that we require the typical size of the system to be of order of Schwarzschild radius R_S. As a result, we show that the D0-brane background can be interpreted as 11D Schwarzschild black hole states in Matrix theory. Finally in sect. 4, we present our discussions of some related issues.

2 Construction of partition function $Z(\beta)$ in Matrix theory

In ref. [1], the authors proposed that M-theory in the infinite momentum frame is described as a system of $N \to \infty$ "partons", represented by D0-branes as the carriers of longitudinal momentum. However, as discussed in ref. [13], Matrix theory is best thought of as the Discretized Light-Cone Quantization (DLCQ) of M-theory, *i.e.*, compactification on a light-like circle of radius R. The dynamics is dictated by the quantum mechanical Lagrangian with $U(N)$ gauge symmetry [1]

$$\mathcal{L} = \frac{1}{2g}\mathrm{Tr}\left\{D_0X^iD_0X^i + \frac{1}{2}[X^i,X^j]^2 + i\theta^\dagger D_0\theta + \theta^\dagger\gamma_i[X^i,\theta]\right\}, \tag{2.1}$$

where we have set the string scale $l_s = 1$, and $X^i(i = 1,\cdots,9)$ and θ are bosonic and fermionic hermitian $N \times N$ matrices, respectively. Since the above Lagrangian possesses gauge symmetry, to calculate the partition function $Z(\beta)$ we have to add gauge fixing and corresponding ghost terms [14, 15]. Following ref. [1], let us rewrite the gauge fixed Lagrangian in the unit of 11D Planck length l_P [14, 15]:

$$\begin{aligned}\mathcal{L}_T &= \mathrm{Tr}\left\{\frac{1}{2R}D_0X^iD_0X^i + \frac{R}{4l_P^6}[X^i,X^j]^2 + i\theta^\dagger D_0\theta\right.\\ &\qquad\left. + \frac{R}{l_P^3}\theta^\dagger\gamma_i[X^i,\theta] - \frac{1}{2R}(\bar{D}^\mu A_\mu)^2 + \mathcal{L}_g\right\},\end{aligned} \tag{2.2}$$

where $\bar{D}^\mu A_\mu = \partial^\mu A_\mu - \frac{iR}{l_P^3}[\bar{X}^\mu, A_\mu]$, $A^i = X^i$, $\bar{X}^\mu$ is the expectation value of A^μ, $\mathcal{L}_g$ is the ghost term and R is the compactification radius of the light-like coordinate X^- [13].

As usual, we choose $\bar{X}^0 = 0$ and $\bar{X}^i$ to satisfy the equations of motion. We then expand (2.2) to quadratic order in the fluctuations around the background fields $X^i = \bar{X}^i + \phi^i$, $A^0 = \phi^0$ [15]:

$$\begin{aligned}\mathcal{L}_T &= \mathrm{Tr}\left\{\frac{1}{2R}(\dot{\bar{X}}^i)^2 + \frac{R}{4l_P^6}[\bar{X}^i,\bar{X}^j]^2 + \frac{1}{2R}\left[(\partial_0\phi^i)^2 - (\partial_0\phi^0)^2\right] - \frac{2i}{l_P^3}\dot{\bar{X}}^i[\phi^0,\phi^i]\right.\\ &\quad + \frac{R}{2l_P^6}\left([\bar{X}^i,\phi^j]^2 + [\bar{X}^i,\phi^j][\phi^i,\bar{X}^j] + [\bar{X}^i,\phi^i]^2 - [\phi^0,\bar{X}^i]^2 + [\bar{X}^i,\bar{X}^j][\phi^i,\phi^j]\right)\\ &\quad\left. + \frac{1}{2R}\partial_0C^*\partial_0C + \frac{R}{2l_P^6}[C^*,\bar{X}^i][\bar{X}^i,C] + i\theta^\dagger\partial_0\theta + \frac{R}{l_P^3}\theta^\dagger\gamma_i[X^i,\theta]\right\},\end{aligned} \tag{2.3}$$

where the first two terms are the classical parts corresponding to the tree-level results, and the rests are quadratic in fluctuations contributing at the one-loop level.

Now we consider the proper background $\bar{X}^i$, which can capture the essential physics of Schwarzschild black holes. From previous discussions in [2]-[9], we know that Schwarzschild black holes consist of D0-brane gas interacting via the long range static forces [14]. To describe the dilute D0-brane gas, the D0-branes should be far apart from each other. For the first simple exploration, we choose N D0-branes as the background. For simplicity, we assume that they are moving around in one direction:[1]

$$\begin{aligned}\bar{X}^1 &= \text{diag.}(v_1 t, v_2 t, \cdots, v_k t, \cdots, v_l t, \cdots, v_N t),\\ \bar{X}^2 &= \text{diag.}(b_1, b_2, \cdots, b_k, \cdots, b_l, \cdots, b_N),\\ \bar{X}^i &= 0, \quad 3 \le i \le 9.\end{aligned} \tag{2.4}$$

Since we work in Euclidean time with period β, the separations between k-th and l-th D0-branes in $\bar{X}^1, \bar{X}^2$ directions are given by $v_{kl}\beta$ and b_{kl} with

$$v_{kl} = v_k - v_l; \quad b_{kl} = b_k - b_l. \tag{2.5}$$

What is the restriction on the parameters of the background (2.4)? In order to describe the properties of Schwarzschild black holes in Matrix theory, the background should satisfy a number of properties [4]. For example, the typical size of the background should be of the magnitude of the order of Schwarzschild radius R_S at least in the sense of the average, which imposes the following restriction on the parameters of the background for $|b_{kl}| >> 1$ and $|v_{kl}| << 1$:

$$\langle |b_{kl}| \rangle \sim \beta \langle |v_{kl}| \rangle \sim \beta \langle |v_k| \rangle \sim R_S, \tag{2.6}$$

where $\langle\ \rangle$ denotes average over the parameters. Eqs. (2.6) indicates that D0-branes move around in the finite region with a typical size of Schwarzschild radius R_S. Furthermore, the partons saturate the uncertainty bound [4, 5]:

$$R_S \frac{\langle |v_k| \rangle}{R} \sim R_S \frac{\langle |v_{kl}| \rangle}{R} \sim 1, \tag{2.7}$$

[1] Physical results are essentially the same if the D0-branes are moving in various directions.

i.e.

$$\langle |v_k| \rangle \sim \langle |v_{kl}| \rangle \sim \frac{R}{R_S}. \tag{2.8}$$

Combining (2.6) and (2.8), we find

$$\beta \sim \frac{R_S^2}{R}. \tag{2.9}$$

Here we should note that eq. (2.9) is obtained from the consistency of the model, and that we have not used any datum of Schwarzschild black holes, such as Hawking temperature. As we will see below, the period β of Euclidean time can be identified with the reciprocal of the boosted Hawking temperature, which gives another evidence that the present model indeed describes the boosted 11D Schwarzschild black holes.

To perform one-loop calculation of the partition function $Z(\beta)$, we take the following form for the fluctuation fields ϕ^μ, θ and C^α:

$$\phi^\mu = \begin{pmatrix} 0 & a^\mu_{12} & a^\mu_{13} & & & \cdots & & & a^\mu_{1,N-1} & a^\mu_{1,N} \\ a^{\mu\dagger}_{12} & 0 & a^\mu_{23} & & & \cdots & & & a^\mu_{2,N-1} & a^\mu_{2,N} \\ & & & & & \cdots & & & & \\ a^{\mu\dagger}_{1m} & a^{\mu\dagger}_{2m} & \cdots & a^{\mu\dagger}_{m-1,m} & & 0 & a^\mu_{m,m+1} & \cdots & a^\mu_{m,N-1} & a^\mu_{m,N} \\ & & & & & \cdots & & & & \\ a^{\mu\dagger}_{1N} & a^{\mu\dagger}_{2N} & & & & \cdots & & & a^{\mu\dagger}_{N-1,N} & 0 \end{pmatrix}, \tag{2.10}$$

where $\mu = t, 1, \cdots, 9$ and $\phi^t = i\phi^0$, and

$$\theta = \begin{pmatrix} 0 & \psi_{12} & \psi_{13} & & & \cdots & & & \psi_{1,N-1} & \psi_{1,N} \\ \psi^{\dagger}_{12} & 0 & \psi_{23} & & & \cdots & & & \psi_{2,N-1} & \psi_{2,N} \\ & & & & & \cdots & & & & \\ \psi^{\dagger}_{1m} & \psi^{\dagger}_{2m} & \cdots & \psi^{\dagger}_{m-1,m} & & 0 & \psi_{m,m+1} & \cdots & \psi_{m,N-1} & \psi_{m,N} \\ & & & & & \cdots & & & & \\ \psi^{\dagger}_{1N} & \psi^{\dagger}_{2N} & & & & \cdots & & & \psi^{\dagger}_{N-1,N} & 0 \end{pmatrix},$$

$$C^\alpha = \begin{pmatrix} 0 & C^\alpha_{12} & C^\alpha_{13} & & & \cdots & & & C^\alpha_{1,N-1} & C^\alpha_{1,N} \\ C^{\alpha\dagger}_{12} & 0 & C^\alpha_{23} & & & \cdots & & & C^\alpha_{2,N-1} & C^\alpha_{2,N} \\ & & & & & \cdots & & & & \\ C^{\alpha\dagger}_{1m} & C^{\alpha\dagger}_{2m} & \cdots & C^{\alpha\dagger}_{m-1,m} & 0 & C^\alpha_{m,m+1} & \cdots & & C^\alpha_{m,N-1} & C^\alpha_{m,N} \\ & & & & & \cdots & & & & \\ C^{\alpha\dagger}_{1N} & C^{\alpha\dagger}_{2N} & & & & \cdots & & & C^{\alpha\dagger}_{N-1,N} & 0 \end{pmatrix}, \tag{2.11}$$

where $\alpha = 1, 2$. The diagonal elements of the fluctuation fields are set to zero because their contribution to the partition function can be ignored in the one-loop approximation on the background (2.4).

Inserting (2.4), (2.10) and (2.11) into (2.3), and changing the time $t \to it$ and velocity $v \to -iv$, we have

$$\begin{aligned} \mathcal{L}_E \;=\; & \sum_{l=1}^{N} \frac{v_l^2}{2R} + \frac{1}{R}\Bigg\{ \sum_{k<l}^{N} (a^{t\dagger}_{kl}, a^{1\dagger}_{kl}) \begin{pmatrix} -\partial_t^2 + \frac{R^2(b_{kl}^2 + v_{kl}^2 t^2)}{l_P^6} & -\frac{2iRv_{kl}}{l_P^3} \\ \frac{2iRv_{kl}}{l_P^3} & -\partial_t^2 + \frac{R^2(b_{kl}^2 + v_{kl}^2 t^2)}{l_P^6} \end{pmatrix} \begin{pmatrix} a^t_{kl} \\ a^1_{kl} \end{pmatrix} \\ & + \sum_{i=2}^{9} a^{i\dagger}_{kl} \left[-\partial_t^2 + \frac{R^2(b_{kl}^2 + v_{kl}^2 t^2)}{l_P^6} \right] a^i_{kl} + \sum_{\alpha=1}^{2} C^{\alpha\dagger}_{kl} \left[-\partial_t^2 + \frac{R^2(b_{kl}^2 + v_{kl}^2 t^2)}{l_P^6} \right] C^\alpha_{kl} \\ & + \psi^\dagger_{kl} \left[\partial_t - \frac{R(b_{kl}\gamma_2 + v_{kl} t \gamma_1)}{l_P^3} \right] \psi_{kl} \Bigg\}, \end{aligned} \tag{2.12}$$

where t is the Euclidean time. Note that eq. (2.12) shows that in the one-loop approximation the elements a^i_{kl}, C^α_{kl} and ψ_{kl} decouple from other components $a^i_{k'l'}, C^\alpha_{k'l'}$ and $\psi_{k'l'}$ ($k \neq k'$ and $l \neq l'$).

According to eq. (1.2), we define the partition function $Z(\beta)$ as

$$Z(\beta) = \int \prod_{k<l}^{N} \prod_{i=1}^{9} \prod_{\alpha=1}^{2} \mathcal{D}a^i_{kl} \mathcal{D}a^{i\dagger}_{kl} \mathcal{D}a^t_{kl} \mathcal{D}a^{t\dagger}_{kl} \mathcal{D}C^\alpha_{kl} \mathcal{D}C^{\alpha\dagger}_{kl} \mathcal{D}\psi_{kl} \mathcal{D}\psi^\dagger_{kl} \exp\left[-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \mathcal{L}_E dt \right], \tag{2.13}$$

with the (anti-)periodic boundary conditions[2]

$$\Phi_{kl}\left(-\frac{\beta}{2}\right) = \pm \Phi_{kl}\left(\frac{\beta}{2}\right), \Phi^\dagger_{kl}\left(-\frac{\beta}{2}\right) = \pm \Phi^\dagger_{kl}\left(\frac{\beta}{2}\right), \tag{2.14}$$

where Φ denotes a^i, a^t, C^α and ψ and the sign is $+(-)$ for bosons and ghosts (fermions).

[2]It is customary to take the range of Euclidean time t from 0 to β. We have chosen it from $-\frac{\beta}{2}$ to $\frac{\beta}{2}$ for our convenience. Not only can this be justified but also does not affect our later estimate of the order of magnitudes of various physical quantities.

After integrating over Φ_{kl} and $\Phi^\dagger_{kl}$, we have [14]-[21]

$$Z(\beta) = e^{-(\Gamma_0+\Gamma_1)}, \tag{2.15}$$

where Γ_0 is the contribution from the tree-level and Γ_1 is that from the one-loop. They can be expressed as

$$\Gamma_0 = \beta \sum_{l=1}^{N} \frac{v_l^2}{2R}, \tag{2.16}$$

and

$$\begin{aligned}
\Gamma_1 \;=\; & -\sum_{k<l}^{N} \ln \left\{ \det{}^{-6}\left[-\partial_t^2 + \frac{R^2(b_{kl}^2+v_{kl}^2t^2)}{l_P^6}\right] \det{}^{-1}\left[-\partial_t^2 + \frac{R^2(b_{kl}^2+v_{kl}^2t^2)}{l_P^6} + \frac{2Rv_{kl}}{l_P^3}\right] \right. \\
& \times \det{}^{-1}\left[-\partial_t^2 + \frac{R^2(b_{kl}^2+v_{kl}^2t^2)}{l_P^6} - \frac{2Rv_{kl}}{l_P^3}\right] \det{}^{4}\left[-\partial_t^2 + \frac{R^2(b_{kl}^2+v_{kl}^2t^2)}{l_P^6} + \frac{Rv_{kl}}{l_P^3}\right] \\
& \left. \times \det{}^{4}\left[-\partial_t^2 + \frac{R^2(b_{kl}^2+v_{kl}^2t^2)}{l_P^6} - \frac{Rv_{kl}}{l_P^3}\right] \right\}.
\end{aligned} \tag{2.17}$$

The interval of t is $-\frac{\beta}{2} \le t \le \frac{\beta}{2}$.

The calculation of the above determinants is closely related to that in ref. [21] where the finite time amplitudes in Matrix theory was discussed. In order to derive the leading order result, it is sufficient to make the adiabatic approximation; namely, we retain only the contribution from the ground state. Thus we make the approximation

$$\det\left[-\partial_t^2 + \omega^2(t)\right] \simeq \exp\left[\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \omega(t)dt\right]. \tag{2.18}$$

We consider large Schwarzschild black holes with $b_{kl} >> 1$ and $v_{kl} << 1$. The leading term for Γ_1 can then be written as [21]

$$\Gamma_1 = -\frac{15 l_P^9}{16R^3} \sum_{k<l}^{N} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt \frac{v_{kl}^4}{(b_{kl}^2+v_{kl}^2t^2)^{7/2}}. \tag{2.19}$$

Substituting (2.16) and (2.19) into (2.15), the partition function becomes

$$Z(\beta) = \exp\left\{-\left[\beta\sum_{l=1}^{N}\frac{v_l^2}{2R} - \frac{15G_{11}}{16R^3}\sum_{k<l}^{N}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt \frac{v_{kl}^4}{(b_{kl}^2+v_{kl}^2t^2)^{7/2}}\right]\right\}, \tag{2.20}$$

where $G_{11} = l_P^9$.

Up to now, we have arrived at the first goal, that is, we have successfully constructed the partition function $Z(\beta)$ in the one-loop approximation in Matrix theory. In the next section, we will use it to calculate the energy and entropy of the system, and compare them with the properties of 11D Schwarzschild black holes.

3 The energy, entropy and temperature of the system

Let us first consider the energy of the system. From eq. (2.20), we can express it as

$$\begin{aligned}\tilde{E}_e &= -\frac{\partial \ln Z(\beta)}{\partial \beta} \\ &= \sum_{l=1}^{N} \frac{v_l^2}{2R} - \frac{15G_{11}}{16R^3} \sum_{k<l}^{N} \frac{v_{kl}^4}{(b_{kl}^2 + v_{kl}^2\beta^2/4)^{7/2}},\end{aligned} \tag{3.1}$$

where the tilde means that the energy has not been averaged over the parameters. Here we point out that in deriving (3.1), we have not made any approximation in (2.20). So we use the subscript "e" to denote the exact calculation.[3]

Now let us average the energy $\tilde{E}_e$ over the parameters by using eqs. (2.6)-(2.9):

$$\begin{aligned}E_e &= \langle \tilde{E}_e \rangle \\ &= \sum_{l=1}^{N} \frac{\langle v_l^2 \rangle}{2R} - \frac{15G_{11}}{16R^3} \sum_{k<l}^{N} \frac{\langle v_{kl}^4 \rangle}{\langle (b_{kl}^2 + v_{kl}^2\beta^2/4)^{7/2} \rangle} \\ &\sim \frac{NR}{2R_S^2} - \frac{1}{2}\left(\frac{4}{5}\right)^{\frac{7}{2}} \frac{15G_{11}RN^2}{16R_S^{11}}.\end{aligned} \tag{3.2}$$

On the other hand, since $v_{kl}^2/b_{kl}^2 << 1$, we expand (2.20) in the power of v_{kl}^2/b_{kl}^2 to get

$$-\ln Z(\beta) \simeq \beta \sum_{l=1}^{N} \frac{v_l^2}{2R} - \frac{15G_{11}\beta}{16R^3} \sum_{k<l}^{N} \frac{v_{kl}^4}{b_{kl}^7} + \frac{15G_{11}\beta^3}{16R^3} \sum_{k<l}^{N} \frac{7v_{kl}^6}{24b_{kl}^9}. \tag{3.3}$$

From eq. (3.3), the energy can be read off as

$$\begin{aligned}\tilde{E}_p &= -\frac{\partial \ln Z(\beta)}{\partial \beta} \\ &= \sum_{l=1}^{N} \frac{v_l^2}{2R} - \frac{15G_{11}}{16R^3} \sum_{k<l}^{N} \frac{v_{kl}^4}{b_{kl}^7} + \frac{15G_{11}\beta^2}{16R^3} \sum_{k<l}^{N} \frac{7v_{kl}^6}{8b_{kl}^9},\end{aligned} \tag{3.4}$$

where the subscript "p" denotes that the energy is obtained from the perturbative $Z(\beta)$, in contrast to that in eq. (3.1). We then have

$$\begin{aligned}E_p &= \langle \tilde{E}_p \rangle \\ &\sim \frac{NR}{2R_S^2} - \frac{1}{16} \frac{15G_{11}RN^2}{16R_S^{11}}.\end{aligned} \tag{3.5}$$

[3]By "exact", we mean that no approximation is made to eq. (2.20), which was obtained in the approximation (2.18) and (2.19). This is to be contrasted with the perturbative result (3.4) below.

Comparing (3.5) with (3.2), we find that $\beta^2\langle v_{kl}^2/b_{kl}^2\rangle \sim 1$ and higher order terms have the same order of magnitudes as the lower terms, but they just affect the coefficient of the second term in (3.5). Thus in order to estimate the order of magnitudes and also dependence on various physical quantities (up to numerical coefficients of order unity), we can use eq. (3.3) to calculate the energy and entropy of the system. From eq. (3.3), the Helmholtz free energy is given by

$$\begin{aligned}\tilde{F} &= -\frac{1}{\beta}\ln Z(\beta) \\ &= \sum_{l=1}^{N}\frac{v_l^2}{2R} - \frac{15G_{11}}{16R^3}\sum_{k<l}^{N}\frac{v_{kl}^4}{b_{kl}^7} + \frac{15G_{11}\beta^2}{16R^3}\sum_{k<l}^{N}\frac{7v_{kl}^6}{24b_{kl}^9},\end{aligned} \tag{3.6}$$

Then the entropy of the system is

$$\begin{aligned}\tilde{S} &= \beta^2\frac{\partial\tilde{F}}{\partial\beta} \\ &= \frac{15G_{11}\beta^3}{8R^3}\sum_{k<l}^{N}\frac{7v_{kl}^6}{24b_{kl}^9}.\end{aligned} \tag{3.7}$$

After averaging $\tilde{S}$ over the parameters, one has

$$S = \langle\tilde{S}\rangle \sim \frac{G_{11}N^2}{R_S^9}. \tag{3.8}$$

Here we emphasize that the entropy of the system is derived directly from the partition function $Z(\beta)$ defined in sect. 2.

To compare our results with previous estimates in refs. [2]-[9], we apply the virial theorem to (3.2) to find

$$R_S \sim (G_{11}N)^{1/9}. \tag{3.9}$$

Inserting (3.9) into eqs. (2.9), (3.5) and (3.8) yields

$$E \sim \frac{(G_{11}^{-1/9}N^{8/9})^2 R}{N}, \tag{3.10}$$

$$S \sim N, \tag{3.11}$$

$$\beta \sim \frac{(G_{11}N)^{2/9}}{R}. \tag{3.12}$$

Eq. (3.11) is the condition $N \sim S$ assumed in refs. [2, 3, 4], but here we have derived it without any extra assumption. It is just a consequence from the fact that D0-branes are chosen as the background. Since the energy E in Matrix theory is the light-cone energy related to the mass M of a boosted object by [2]-[9]

$$E = \frac{M^2 R}{2N}, \tag{3.13}$$

the mass of the object is

$$M \sim G_{11}^{-1/9} N^{8/9}, \tag{3.14}$$

and the β can be interpreted as the reciprocal of the boosted temperature of the system. This implies that the temperature T in the rest frame can be related with β by

$$\frac{1}{\beta} = \frac{MR}{N} T \sim \frac{R}{(G_{11}N)^{2/9}}. \tag{3.15}$$

Substituting (3.14) into (3.15), we have

$$T \sim \frac{1}{(G_{11}N)^{1/9}} \sim \frac{1}{R_S}. \tag{3.16}$$

Since the entropy, which have an interpretation in terms of total number of states, should not change under the boost, the mass, entropy, temperature and typical size of the system can be collectively written as

$$\begin{aligned} M &\sim G_{11}^{-1/9} N^{8/9}, \\ S &\sim N, \\ T &\sim \frac{1}{R_S}, \\ R_S &\sim (G_{11}N)^{1/9}. \end{aligned} \tag{3.17}$$

In deriving the result (3.17), we have not referred to any information about Schwarzschild black holes except that we assume that the typical size of the system is of the order of Schwarzschild radius R_S. Also our model does not contain General Relativity apparently; it is just Matrix theory itself. What we have shown is that we can derive (3.17) solely from the consistency of the model, including Euclidean path integral, backgrounds and restriction on its parameters. From eq. (3.17), it is easy to see that

$$\begin{aligned} R_S &\sim (G_{11}M)^{1/8}, \\ S &\sim G_{11}^{1/8} M^{9/8} \sim N, \\ T &\sim \frac{1}{R_S}. \end{aligned} \tag{3.18}$$

We find that eqs. (3.18) are nothing but the thermodynamic functions of 11D Schwarzschild black holes, which indicates that our model indeed describes 11D Schwarzschild black holes. Since the model does not involve General Relativity manifestly, we conclude that Matrix theory itself contains 11D Schwarzschild black hole states or their disguise. The background we choose is D0-branes, which is closely related to D0-brane gas picture in refs. [2]-[9]; actually it describes the black hole states in the region $S \sim N$.

4 Discussions

So far we have developed Euclidean path integral formalism in Matrix theory to construct the partition function $Z(\beta)$, from which the energy and entropy of the system can be read off. In order to calculate the partition funcution $Z(\beta)$ in the one-loop approximation, the background has been assumed to be D0-branes moving around in the finite region with a typical size of Schwarzschild radius R_S, which can be visualized as a bound state of a large number of D0-branes. In fact, such a bound state can be interpreted as an 11D Schwarzschild black hole state in the language of Matrix theory. As we have shown, with only one assumption that the typical size of the system is of the order of Schwarzschild radius, we can derive the mass, entropy and temperature of the system up to numerical factors of order unity by the consistency of the formalism. Also we have found that they match the properties of 11D Schwarzschild black holes. The assumption $S \sim N$ made in [2, 3, 4] has been clarified in our model, which is a direct consequence of D0-brane background. In other words, the D0-brane gas only describes the properties of 11D Schwarzschild black hole states in the region $S \sim N$.

It is interesting to compare the present Euclidean path integral approach with what Gibbons and Hawking proposed in ref. [12], where they used the metric of Schwarzschild black holes as the background. The Bekenstein-Hawking entropy was read off in the tree-level approximation from the partition function in their calculation. However, in Matrix theory, we have exploited D0-branes as the background, and found that one-loop corrections are enough to find the correct entropy (the contribution from tree-level vanishes). It is quite reminiscent of Enclidean path integral formalism in the calculation of the entropy of black holes from matter fields [22, 23], where the entropy only gets contribution from one-loop corrections, but the background was still chosen as the metric of black holes. Probably, treating D0-branes as the background is the main new feature of the present Euclidean path integral method in Matrix theory.

One of the interesting extensions of our formalism is to check whether it works in various dimensions to describe Schwarzshild black hole states in the corresponding dimension, which is related to finding a general descripition of Kaluza-Klein compactification of Matrix theory [24, 25, 26].

In the above, we have learned that 11D Schwarzschild black hole states in the region $S \sim N$ can be correctly described in Matrix theory by choosing D0-branes as the background. One may ask what happens in 11D Schwarzschild black hole states in Matrix theory in the limit $N >> S$. In ref. [5], it was speculated on that if the transverse size remains constant under boosts, the partons become denser as N increases and strongly interacting clusters will form. The interaction within a cluster should be more "membrane-like" than "graviton-like", since the commutator term in the Matrix-theory Hamitonian is the membrane area element. Now it seems that in our formalism we can examine if the above idea is realized, that is, we can describe 11D Schwarzschild black hole states in Matrix theory in the limit $N >> S$ with certain background. If we choose a number of little nuggets of membranes as the background, one might expect the resulting entropy S will be much less than N, but the mass, entropy, temperature and the typical size of the system still match the properties of the 11D Schwarzschild black holes. Work along this line is in progress and we hope to discuss these issues elsewhere.

Acknowledgement

J.-G.Z. would like to thank T. Nakatsu and K. Oda for discussions. This work was supported in part by Grand-in-aid from the Ministry of Education, Science, Sports and Culture No. 96208.

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