The Origin of Space-Time as Seen from Matrix Model Simulations

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The AdS/CFT correspondence, or more generally the gauge/gravity duality, is a remarkable conjecture obtained from superstring theory with various D-brane backgrounds. According to this conjecture, a higher-dimensional curved space-time emerges from supersymmetric gauge theory in lower-dimensional flat space-time. In the first part of this article, we review Monte Carlo studies of U(N) supersymmetric gauge theories, which confirmed the gauge/gravity duality for various observables. In particular, Monte Carlo results for thermodynamic quantities enable us to understand the microscopic origin of the black hole entropy associated with the dual geometry. We also discuss results for Wilson loops and correlation functions, which agree nicely with the predictions from the gravity side. In the second part, we review recent developments in a nonperturbative formulation of superstring theory, which may be regarded as a counterpart of the lattice gauge theory in QCD. In particular, we discuss Monte Carlo results for the Lorentzian matrix model, which suggest that (3+1)-dimensional expanding universe emerges dynamically from type IIB superstring theory in (9+1) dimensions if one treats the theory nonperturbatively.

§1. Introduction

Monte Carlo calculations based on lattice gauge theory have been playing a central role in studying various properties of QCD in a fully nonperturbative manner. In this article, we would like to show that similar developments are starting to take place in superstring theory, which is a unified theory for all the matters and the fundamental interactions including gravity. As a theory of quantum gravity, the notion of space-time had to undergo a revolutionary change. It is by now widely appreciated that matrices are the fundamental degrees of freedom of superstring theory at the nonperturbative level, and the space-time only emerges effectively as a derived concept at low energy or at long distances.

One of the developments that manifest this idea is the gauge/gravity duality. (See Ref. 1) for a comprehensive review.) The original conjecture is known as the AdS/CFT correspondence, which was put forward by Maldacena in 1997,[2] but it was soon generalized to non-conformal cases.[3] These dualities are arrived at by considering two different descriptions of D-brane backgrounds in superstring theory. One is the gauge theory, which describes the open strings with both ends attached to the D-branes. The other is a classical solution of supergravity, which describes the closed string degrees of freedom in the bulk sourced by the D-branes. Such a remarkable statement that relates gauge theory and gravity theory is possible precisely because superstring theory naturally contains both of them.

When one considers the gauge theory at finite temperature, the dual supergravity solution can have a geometry with an event horizon, which is characteristic to black holes (or "black branes", more generally). It is well known that black holes have thermodynamic properties although their microscopic origin has long been a mystery.

The gauge/gravity duality provides a very clear and explicit answer to this problem. The thermodynamic properties of a black hole can be understood as those of the dual gauge theory, which is considered to describe the interior structure of the black hole. Monte Carlo calculation of thermodynamic quantities in the gauge theory indeed reproduced the black hole thermodynamics.[4]–[6] In particular, Ref. 5) shows that the duality holds including α' corrections, which represent the effects of closed strings having finite length. These results demonstrate that the gauge theory describes correctly the quantum space-time structure at the center of the black hole.

The correspondence between gauge theory and gravity was extended to the level of operators, which is of particular importance in using the duality to study strongly coupled gauge theories by much simpler calculations in supergravity. In particular, explicit prescriptions for obtaining Wilson loops[7] and correlation functions[8] were proposed. The predictions obtained by such prescriptions have also been confirmed by direct Monte Carlo calculations on the gauge theory side.[9]–[11]

Monte Carlo studies mentioned above deal with the simplest case of D0-branes, which corresponds to one-dimensional $U(N)$ super Yang-Mills theory (SYM) with 16 supercharges. In fact, it is possible to extend these works to higher dimensions by extending the idea of the large-N reduction[12] to a curved space,[13] while circumventing a well-known problem[14] in the original idea. This unconventional regularization scheme, as opposed to the lattice, enables calculations respecting supersymmetry maximally. Some preliminary results are obtained[15]–[17] for the case of D3-branes, which corresponds to four-dimensional $\mathcal{N} = 4$ $U(N)$ SYM.

An important aspect of the gauge/gravity duality is that higher-dimensional curved space-time emerges from the gauge theory in lower-dimensional flat space-time. The extra spatial dimensions are actually described in gauge theory by the scalar fields in the adjoint representation of $U(N)$ gauge group, which are represented by $N \times N$ matrices. This is an example of "emergent space", which appears in various contexts of string theory.[18] One of the important open questions is whether one can extend the idea to emergent space-time instead of just space. Here we quote a sentence from Seiberg's lecture[18] in 2005: *Understanding how time emerges will undoubtedly shed new light on some of the most important questions in theoretical physics including the origin of the Universe.*

The second development we would like to review in this article concerns a nonperturbative formulation of superstring theory, which is considered as a counterpart of the lattice gauge theory in the case of QCD. In particular, we discuss recent Monte Carlo results for a Lorentzian matrix model,[19] which actually show that *the emergent space-time* seems to be naturally realized. Back in 1996, Ishibashi, Kawai, Kitazawa and Tsuchiya proposed a matrix model, which is called the type IIB matrix model, as a nonperturbative definition of type IIB superstring theory in ten dimensions.[20] The model can be obtained formally from SYM that appears in the aforementioned examples of the gauge/gravity duality by dimensionally reducing them to zero dimension. Thus one obtains ten bosonic matrices and sixteen fermionic matrices, which do not have dependence on space-time coordinates. The entire space-time is expected to emerge dynamically from the ten bosonic matrices.

Until quite recently, however, it was common to study the type IIB matrix model

after making a "Wick rotation". This amounts to replacing the Hermitian matrix A_0 in the temporal direction by $A_0 = iA_{10}$, and treating the Hermitian matrix A_{10} on equal footing as the matrices A_i ($i = 1, \cdots 9$) in the spatial directions. The Euclidean model obtained in this way has manifest SO(10) symmetry, and it is well defined as Monte Carlo studies with small matrices demonstrate.[21] In fact the partition function was proven to be finite for arbitrary matrix size.[22] In Ref. 23), perturbative expansion around the diagonal configurations $A_\mu = \mathrm{diag}(x_{1\mu}, \cdots, x_{N\mu})$ was studied and the low-energy effective theory for the diagonal elements was discussed. In particular, it was speculated that configurations with the N points $\{\vec{x}_i; i = 1, \cdots, N\}$ distributed on a four-dimensional hypersurface in ten-dimensional Euclidean space may be favored due to some nontrivial interactions in the low-energy effective theory. If that really happens, it implies that the SO(10) symmetry is spontaneously broken down to SO(4) and that four-dimensional space-time is generated dynamically.

Monte Carlo studies of the Euclidean model is difficult due to the sign problem since the Pfaffian that appears from integrating out the fermionic matrices is complex in general. Monte Carlo studies of the model omitting the phase of the Pfaffian show that the SO(10) symmetry is not spontaneously broken.[24] In fact the phase of the Pfaffian has an effect of favoring lower-dimensional configurations.[25] It is expected that such an effect can be studied by Monte Carlo simulation in the near future by using a new method to overcome the sign problem[26),27]

As an alternative approach to this issue, the Gaussian expansion method was proposed.[28] Recently, the free energy was calculated by assuming that the SO(d) symmetry ($2 \le d \le 7$) remains unbroken, and it was found that $d = 3$ gives the minimum.[29] Another important observation from the Gaussian expansion method was that the extent of space-time in the extended d directions and that in the shrunken $(10 - d)$ directions turn out to have a finite ratio even in the large-N limit.[29] While these results reveal interesting dynamical properties of the Euclidean model, which can also be understood intuitively from the viewpoint of the low-energy effective theory, the connection to our real space-time is not very clear.

Motivated by these results for the Euclidean model, Kim, J.N. and Tsuchiya[19] studied the type IIB matrix model *without making the Wick rotation*. The action has an SO(9,1) symmetry instead of SO(10). The reason why no one dared to study this *Lorentzian model* before beyond the classical level[30] was that the bosonic action S_b is not positive definite unlike the Euclidean case, and therefore the system seemed to be highly unstable. Moreover, the bosonic action becomes a pure phase in the integrand of the partition function as in the path integral formulation of quantum field theories in Minkowski space. Therefore it seemed just impossible to get anything sensible out of the Lorentzian model without making a Wick rotation.

On the other hand, it is known from many examples that the Wick rotation is subtle in theories including gravity. For instance, the Lorentzian quantum gravity has been pursued within the dynamical triangulation approach[31] motivated from earlier studies of the Euclidean gravity, and the results turned out to be quite different. More recently, the worm hole scenario as a solution to the cosmological constant problem was reconsidered in the Lorentzian quantum gravity,[32] and the results provided a consistent picture, which was not available in the original Euclidean formulation.

The crucial trick to make the Lorentzian matrix model accessible by Monte Carlo simulation is to integrate out the scale factor of the bosonic matrices first,[*] which essentially converts the phase factor e^{iS_b} into a constraint $S_b \approx 0$. This is possible since the action of the type IIB matrix model is homogeneous[**] with respect to the matrices. The model one obtains in this way does not have the sign problem since the Pfaffian that appears from integrating out fermionic matrices is real in the Lorentzian case.

First of all, Monte Carlo studies confirmed that the Lorentzian matrix model is not well defined as it is. It was found that the extents in the temporal and spatial directions, which are represented by $\frac{1}{N}\mathrm{tr}\,(A_0)^2$ and $\frac{1}{N}\mathrm{tr}\,(A_i)^2$, respectively, both tend to diverge. One therefore has to put cutoffs on these quantities. However, it turned out that the two cutoffs can be removed in the large-N limit in such a way that the results scale in N. The theory thus obtained turned out to have no parameters other than the scale parameter, which can be naturally identified as the string scale. This is a highly nontrivial property of the Lorentzian matrix model, which supports its validity as a nonperturbative formulation of superstring theory.

Another important observation in the Monte Carlo studies is that the eigenvalue distribution of the matrix A_0 representing the time direction extends as one takes the large-N limit explained above. Supersymmetry plays a crucial role here. (For the bosonic model, the eigenvalue distribution of the matrix A_0 has finite extent even in the large-N limit.)

Moreover, by making an SU(N) transformation in such a way that the temporal matrix A_0 is diagonalized, the nine-dimensional space represented by A_i ($i = 1, \cdots, 9$) is found to exhibit a sensible "time evolution". In fact, the space remains small and SO(9) symmetric from the infinite past until some "critical time", at which only three directions start to expand rapidly. This implies that the rotational SO(9) symmetry in the spatial directions is spontaneously broken down to SO(3) at the critical time, which may be identified as "the birth of our Universe". The so-called initial condition problem is not an issue in the present framework since even the time evolution is an emergent concept. The mechanism of the spontaneous symmetry breaking (SSB) relies crucially on the noncommutativity of space, and it seems totally different from the Euclidean case, in which the SSB is considered to be caused by the phase of the Pfaffian. In the Lorentzian model, the Pfaffian is real as we mentioned above.

The rest of this article is organized as follows. In section 2 we review the developments related to the gauge/gravity duality. In particular, we discuss its direct tests based on Monte Carlo studies of supersymmetric gauge theories. In section 3 we review the developments related to a nonperturbative formulation of superstring theory. In particular, we discuss how (3+1)-dimensional expanding universe emerges from the Lorentzian matrix model. In section 4 we conclude with a summary and future prospects.

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[*] The same procedure was also used in Ref. 21) for simulating the Euclidean model.

[**] Note that the gauge theories before dimensional reduction do not have this property due to the existence of the derivative terms.

§2. Monte Carlo studies of the gauge/gravity duality

The gauge/gravity duality[1] is conjectured by considering D-brane backgrounds in superstring theory. D-branes are known to be consistent backgrounds in superstring theory, and they may be considered as a counterpart of solitons in field theory. Note that solitons do not appear in perturbative expansion around the trivial vacuum, and they are considered as nonperturbative objects. From that point of view, D-branes are expected to capture some nonperturbative aspects of string theory. Indeed D-branes played a crucial role in finding the duality web of superstring/M theory and in constructing a nonperturbative formulation of superstring/M theory.

D-brane can extend in p dimensions, and it is characterized as a hypersurface on which strings can end on. ("D" stands for the Dirichlet boundary condition imposed at the boundary of the worldsheet of an open string.) Let us consider an open string attached to the D-brane propagating along it. In Fig. 1 on the left, we describe such a process diagrammatically. If one slices the diagram in the orthogonal direction, one notices that the same process can be viewed as emission of a closed string. This is an example of the so-called open-string/closed-string duality. Note here that an open string and a closed string include a gauge particle and a graviton, respectively, as massless modes.
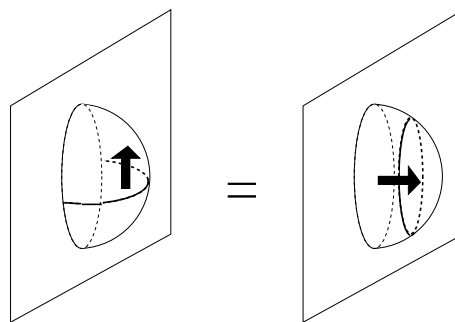

Fig. 1. On the left, an open string attached to the D-brane is propagating along it. On the right, the same process is viewed as emission of a closed string from the D-brane.

This observation lies at the heart of the gauge/gravity duality.

In order to formulate the gauge/gravity duality, we need to consider N D-branes lying on top of each other and take a particular low-energy limit so that the open strings attached to the D-branes and the closed strings in the bulk are decoupled. Then one has two independent descriptions of D-branes. One is the $(p + 1)$-dimensional U(N) SYM, which describes the open strings attached to the D-branes. The other is a solution to supergravity, which describes the closed string degrees of freedom in the bulk sourced by the D-branes. Note that the bulk ten-dimensional space-time is curved since D-branes emit gravitons. In order for the supergravity description to be valid, one needs to take certain limits on the gauge theory side.

- The so-called 't Hooft large-N limit with fixed $\lambda \equiv g_{\mathrm{YM}}^2 N$. (The string loop corrections are suppressed by $1/N$.)
- The large-λ limit. (The α' corrections, which are due to strings having finite extent, are suppressed by some powers of $1/\lambda$.)

A stronger version of the conjecture claims that the gauge/gravity duality holds including the string loop corrections and the α' corrections. If that is the case, one

can say that the gauge theory, which is well defined for all N and λ, actually defines superstring theory on the particular curved background nonperturbatively.

In addition to this perspective, there are various reasons why the gauge/gravity duality are considered very interesting. First of all, it is a realization of an old idea by 't Hooft,[33] which states that the large-N gauge theory is equivalent to some classical string theory, although in those days people may not have anticipated that the string theory actually lives in a curved space-time. It is interesting that the curved space-time emerges from a gauge theory in a flat space. This aspect of the duality is often referred to as the *emergent space*.[18] In the gauge/gravity duality, one typically obtains the anti-de Sitter space. If one considers the gauge theory at finite temperature, one can have a black-hole-like geometry.[3),34] Therefore, one may explain the microscopic origin of the black hole thermodynamics in terms of gauge theory. One can also use the duality in the opposite direction, and study strongly coupled gauge theories, which are relevant to hadron and condensed matter physics, from a curved space-time.

Since the gauge/gravity duality is a strong-weak duality, it is important to study gauge theories in the strongly coupled regime. Monte Carlo simulation can be a powerful tool for such purposes. However, the problem is that the gauge theories we are interested in have supersymmetry, which is broken by the lattice. This can be seen immediately if one recalls the supersymmetry algebra $\{Q, \bar{Q}\} \propto P_\mu$, where the generators for translation appear on the right hand side. Since the translational symmetry is broken by the lattice regularization, one necessarily breaks supersymmetry. The best one can do is to restore supersymmetry in the continuum limit by fine-tuning some parameters in the action, which requires a lot of efforts, however.

Recently there are considerable developments in "lattice supersymmetry", which can be categorized into two classes. One is the construction of lattice actions with various symmetries. For instance, one can preserve one supercharge by using the so-called topological twist. The other one, which we discuss in what follows, is to use a regularization different from the lattice.

In the case of D0-branes, which corresponds to the supersymmetric gauge theory in 1 dimension with 16 supercharges, one can regularize the theory using momentum cutoff after fixing the gauge appropriately.[35] Black hole dynamics have been reproduced including the α' corrections,[4),5] and the gauge/gravity duality for the Wilson loops[9] and the correlation functions[10),11] has been confirmed.

Notably, one can extend this approach to 3d and 4d gauge theories by using the idea of large-N reduction.[13] In the 4d case, the gauge theory becomes superconformal and the number of supersymmetries enhances from 16 to 32. This superconformal theory is interesting on its own right, but it is also studied intensively in the context of the AdS/CFT correspondence, which is a typical case of the gauge/gravity duality.[2] The non-lattice simulation of the 4d superconformal theory requires no fine-tuning, unlike the previous proposals based on the lattice regularization.[36] As we will see, preliminary results for the Wilson loops and the correlation functions are promising.[15)–17]

2.1. *Non-lattice simulation of 1d SYM with 16 supercharges*

Let us start with the D0-brane case, which corresponds to the 1d U(N) SYM with 16 supercharges. The action is given by $S = S_{\mathrm{b}} + S_{\mathrm{f}}$, where

$$S_{\mathrm{b}} = \frac{1}{g^2} \int_0^\beta dt \ \mathrm{tr} \left\{ \frac{1}{2} \Big(D X_i(t) \Big)^2 - \frac{1}{4} [X_i(t), X_j(t)]^2 \right\} \ , \tag{2·1}$$

$$S_{\mathrm{f}} = \frac{1}{g^2} \int_0^\beta dt \ \mathrm{tr} \left\{ \frac{1}{2} \Psi_\alpha D \Psi_\alpha - \frac{1}{2} \Psi_\alpha (\gamma_i)_{\alpha\beta} [X_i, \Psi_\beta] \right\} \ . \tag{2·2}$$

The covariant derivative is denoted as $D = \partial_t - i\,[A(t), \ \cdot \ ]$. $X_j(t)$ ($j = 1, \cdots, 9$) and $\Psi_\alpha(t)$ ($\alpha = 1, \cdots, 16$) are $N \times N$ Hermitian matrices, and the theory has SO(9) symmetry. When we are interested in finite temperature, we impose periodic boundary conditions on $X_j(t)$ and anti-periodic boundary conditions on $\Psi_\alpha(t)$. Then the temperature is given by $T \equiv \beta^{-1}$, where β is the extent in the Euclidean time (t) direction. The 't Hooft coupling constant is defined by $\lambda \equiv g^2 N$, which has the dimension of mass cubed. The physics of the system is determined only by the dimensionless coupling constant $\lambda_{\mathrm{eff}} \equiv \frac{\lambda}{T^3}$. Therefore one can take $\lambda = 1$ without loss of generality. With this convention, the low T regime corresponds to the strongly coupled regime, which is expected to have the dual gravity description,[3] whereas the high T regime is essentially weakly coupled, and the high temperature expansion (HTE) is applicable.[37]

In non-lattice simulation,[35] we introduce an upper bound on the Fourier mode as $X_i(t) = \sum_{n=-\Lambda}^{\Lambda} \tilde{X}_{i,n} e^{i\omega n t}$, where $\omega = \frac{2\pi}{\beta}$, and similarly for the fermions. This idea does not work usually because it breaks gauge invariance. (Recall that the Fourier mode is not a gauge invariant concept.) However, in 1d, one can fix the gauge nonperturbatively in the following way. We first take the static diagonal gauge $A(t) = \frac{1}{\beta} \mathrm{diag}(\alpha_1, \cdots, \alpha_N)$, in which the gauge field is constant in time and diagonal. By following the usual Faddeev-Popov procedure, one obtains

$$S_{\mathrm{FP}} = -\sum_{a<b} 2 \ln \left| \sin \frac{\alpha_a - \alpha_b}{2} \right| \tag{2·3}$$

as a term to be added to the action. The above gauge choice does not fix the gauge symmetry completely, and there is a residual symmetry given by

$$\alpha_a \mapsto \alpha_a + 2\pi\nu_a \ , \qquad \tilde{X}_{i,n}^{ab} \mapsto \tilde{X}_{i,n-\nu_a+\nu_b}^{ab} \ , \qquad \tilde{\Psi}_{\alpha,n}^{ab} \mapsto \tilde{\Psi}_{\alpha,n-\nu_a+\nu_b}^{ab} \ , \tag{2·4}$$

which represents a topologically nontrivial gauge transformation corresponding to the gauge function $g(t) = \mathrm{diag}(e^{i\omega\nu_1 t}, \cdots, e^{i\omega\nu_N t})$. This residual gauge symmetry can be fixed by imposing $-\pi < \alpha_a \le \pi$. One can then introduce the Fourier mode cutoff Λ. Since there is no UV divergence in this 1d model, one can take the $\Lambda \to \infty$ limit naively, and one retrieves the original gauge theory with 16 supercharges.

The system with finite Λ can be simulated efficiently *) by using the standard RHMC algorithm.[38] In particular, the Fourier acceleration[39] can be implemented

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*) Strictly speaking, the Pfaffian one obtains from integrating out the fermions is complex in general. However, there is numerical evidence[11] that the phase can be omitted without altering the results. Complete justification is left for future investigations.

without extra cost since we are dealing with the Fourier modes directly as the fundamental degrees of freedom. This is crucial in reducing the critical slowing down at large Λ. (The same theory is also studied using the standard lattice approach.[6]) However, from the results obtained so far, the non-lattice simulations seem to be far more efficient in obtaining the continuum limit.)

Let us first discuss the phase structure that appears when one changes the temperature. As is well known, the Polyakov line serves as an order parameter for the spontaneous breaking of the center symmetry. Fig. 2 (Left) shows the results.[4] At high temperature the data agree nicely with the HTE[37] including the next-leading order. As the temperature decreases below $T \sim 3$, the data start to deviate, and at low temperature below $T \sim 0.9$, the data can be fitted to the characteristic behavior of the "deconfined phase"

$$\langle |P| \rangle = \exp\left(-\frac{a}{T} + b\right) \ . \tag{2.5}$$

In the temperature regime investigated, we find no phase transition. This is in sharp contrast to the bosonic model,[40]–[42] which undergoes a phase transition to the "confined phase" at $T \sim 0.9$. The absence of the phase transition is consistent with analyses on the gravity side.[34],[43]
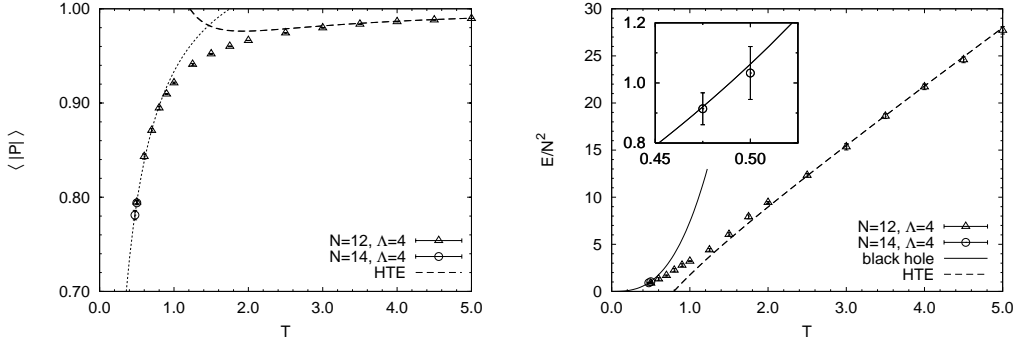

Fig. 2. (Left) The Polyakov line is plotted against T. The dashed line represents the result of HTE up to the next leading order for $N = 12$.[37] The dotted line represents a fit to eq. (2.5) with $a = 0.15$ and $b = 0.072$. (Right) The energy (normalized by N^2) is plotted against T. The dashed line represents the result obtained by HTE up to the next leading order for $N = 12$.[37] The solid line represents the asymptotic power-law behavior at small T predicted by the gauge/gravity duality.

2.2. Black hole thermodynamics from 1d SYM

Let us turn to a quantitative prediction from the gauge/gravity duality. Given the dual geometry, one can use Hawking's theory of black hole thermodynamics to obtain various thermodynamic relations such as[44]

$$\frac{1}{N^2}\left(\frac{E}{\lambda^{1/3}}\right) = c\left(\frac{T}{\lambda^{1/3}}\right)^{14/5} \ , \qquad c = \frac{9}{14}\left\{4^{13}15^2\left(\frac{\pi}{7}\right)^{14}\right\}^{1/5} = 7.41\cdots \ . \tag{2.6}$$
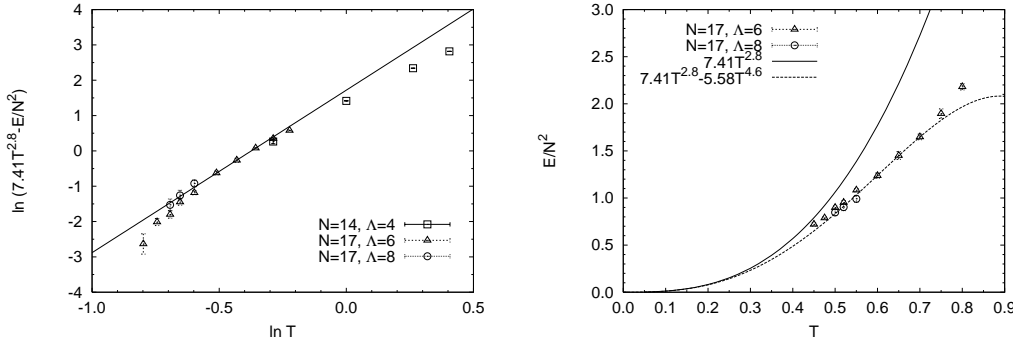
Fig. 3. (Left) The deviation of the internal energy $\frac{1}{N^2}E$ from the leading term $7.41\,T^{\frac{14}{5}}$ is plotted against the temperature in the log-log scale for $\lambda = 1$. The solid line represents a fit to a straight line with the slope 4.6 predicted from the α' corrections on the gravity side. (Right) The internal energy $\frac{1}{N^2}E$ is plotted against T for $\lambda = 1$. The solid line represents the leading asymptotic behavior at small T predicted by the gauge/gravity duality. The dashed line represents a fit to the behavior (2·7) including the subleading term with $C = 5.58$.

The gauge/gravity duality predicts that this should be reproduced by 1d SYM in the large-N limit at low T.[3] The importance of this prediction is that, if it is true, it explains the microscopic origin of the black hole thermodynamics, meaning that the 1d SYM provides the quantum description of the states inside the black hole.

In Fig. 2 (Right) we plot the internal energy,[4] which is defined by $E = \frac{\partial}{\partial\beta}(\beta\mathcal{F})$ in terms of the free energy $\mathcal{F}$. At $T \gtrsim 3$ the data agree with the HTE.[37] As one goes to lower temperature, the data points approach the solid line, which represents the result (2·6) obtained from the 10d black hole. (See Refs. 45) for earlier studies based on the Gaussian approximation.)

From Fig. 2 (Right) alone, it is not clear whether the gauge theory results continue to follow the line predicted from gravity at lower T. In fact, simulations at lower T are difficult, since one has to increase Λ proportionally to $1/T$, and at the same time one has to increase N to avoid the run-away behavior due to finite N.[4] Instead of lowering T, we determined the power of the subleading term as[5]

$$\frac{1}{N^2}\left(\frac{E}{\lambda^{1/3}}\right) = c\left(\frac{T}{\lambda^{1/3}}\right)^{14/5} - C\left(\frac{T}{\lambda^{1/3}}\right)^{23/5}, \qquad (2\cdot7)$$

from gravity. This was derived by considering higher derivative corrections in the supergravity action due to the effects of strings having finite extent (α' corrections). The coefficient C of the subleading term is calculable in principle, but it requires the full information of the higher derivative corrections, which are yet to be determined. By using (2·7), however, we can already make a nontrivial test of the gauge/gravity duality.[5] In Fig. 3 (Left) we plot the discrepancy $7.41T^{14/5} - E/N^2$ against T in the log-log scale, which reveals that the power of the subleading term is indeed consistent with the predicted value $23/5 = 4.6$. In Fig. 3 (Right) we find that the data at $T \lesssim 0.7$ can be nicely fitted to the form (2·7) with $C = 5.58$. Note also that the $\Lambda = 6$ data seem to suffer from some finite Λ effects at low T. From this point of

view, we consider that the $\Lambda = 4$ data points at low T in Fig. 2 (Right), which seem to be on the curve of the leading order result from gravity, also suffer from finite Λ effects. Now we know that actually the subleading term in (2·7) should be taken into account for precise agreement.

The fact that the internal energy goes to zero as a power of T towards $T = 0$ is closely related to the existence of the threshold bound state in the 1d gauge theory.[46] The leading power-law behavior $T^{14/5}$ can actually be understood by considering that excitations around the threshold bound state have energy of the order of $N^{-5/9}$ as suggested from the effective Hamiltonian. In the case of 1d SYM with four and eight supercharges, it is considered that the threshold bound state does not exist. Indeed, Monte Carlo studies show that the internal energy decreases faster as $\sim e^{-cT}$ as T goes to zero.[47]

2.3. *Schwarzschild radius from Wilson loop in 1d SYM*

As another prediction from the gauge/gravity duality, let us consider the Wilson loop, which winds around the temporal direction once. Unlike the usual Polyakov line, we consider the one involving the adjoint scalar as

$$W \equiv \frac{1}{N} \operatorname{tr} \mathcal{P} \exp \left[i \int_0^\beta dt \{ A(t) + i\, n_i\, X_i(t) \} \right] \;, \qquad (2·8)$$

where n_i is a unit vector in 9d, which can be chosen arbitrarily due to the SO(9) invariance. This object can be calculated on the gravity side by considering the minimal surface spanning the loop in the dual geometry.[7] For the present model, the result is given by[9]

$$\ln W = \frac{\beta R_{\mathrm{Sch}}}{2\pi \alpha'} = \kappa \left(\frac{T}{\lambda^{1/3}} \right)^{-3/5} \;, \qquad (2·9)$$

where R_{Sch} is the Schwarzschild radius of the dual black hole geometry and

$$\kappa = \frac{1}{2\pi} \left\{ \frac{16\sqrt{15}\pi^{7/2}}{7} \right\}^{2/5} = 1.89 \cdots \;. \qquad (2·10)$$

In Fig. 4 we plot the log of the Wilson loop[9] against $T^{-3/5}$ anticipating (2·9). Indeed, at low temperature (to the right

Fig. 4. The plot of $\langle \log |W| \rangle$ for $\lambda = 1$ against $T^{-3/5}$. The cutoff Λ is chosen as follows: $\Lambda = 12$ for $N = 4$; $\Lambda = 0.6/T$ for $N = 6, 8$; $\Lambda = 4$ for $N = 14$; $\Lambda = 6$ for $N = 17$. The dashed line represents the results of the HTE up to the next-leading order for $N = 14$, which are obtained by applying the method in Ref. 37).

on the figure), we find that the data points can be fitted nicely to a straight line with a slope 1.89 in precise agreement with (2·10). The solid line corresponds to $\langle \log |W| \rangle = 1.89\, T^{-3/5} - 4.58$, where the existence of the constant term can be understood as α' corrections. This result demonstrates that one can extract the information of the dual geometry such as the Schwarzschild radius from the gauge invariant observable (2·8).
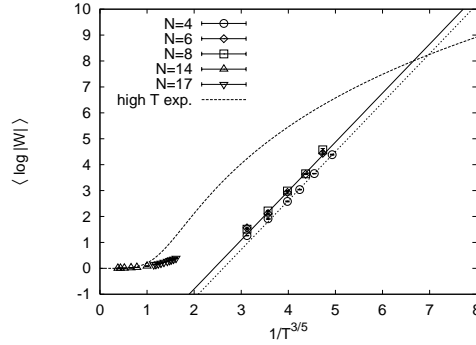
2.4. *Correlation functions in 1d SYM*

One can also predict correlation functions from the gravity side. This was done more than ten years ago by Sekino and Yoneya[48] applying the Gubser-Klebanov-Polykov-Witten prescription[8] to the present 1d SYM case.

For instance, let us consider an operator

$$\mathcal{O}_\ell = \frac{1}{N} \text{Str}\left(F_{ij} X_{i_1} \cdots X_{i_\ell} \right) , \quad (2\cdot11)$$

where $F_{ij} \equiv -i\left[X_i, X_j\right]$ and Str represents the symmetrized trace treating F_{ij} as a single unit. The two-point correlation function of this operator is predicted as

$$\langle \mathcal{O}_\ell(t)\mathcal{O}_\ell(0)\rangle \sim \frac{1}{|t|^p} , \quad p = \frac{4\ell}{5} + 1$$

$$(2\cdot12)$$

at $\lambda^{-1/3} \ll |t| \ll \lambda^{-1/3} N^{10/21}$. In Fig. 5 we plot the two-point correlation function[11] for $\ell = 1, 2, 3, 4$, which agrees precisely with the predicted power-law behavior.[*] What is rather surprising is that the agreement is observed even for

Fig. 5. The log-log plot of the correlator $\left\langle J_\ell^+(t)\, J_\ell^+(0) \right\rangle$ with $\ell = 1, 2, 3, 4$ for $N = 3$. The cutoff parameters are chosen as $\beta = 4$ and $\Lambda = 16$. The straight lines represent the power-law behavior predicted by the gauge/gravity correspondence.

such small N as $N = 3$. In particular, the power-law behavior seems to extend to the far infrared regime, in which the supergravity calculations become invalid. This fact may have important implications on M theory interpretation of the same model. See Ref. 11) for more details as well as results for other operators.

In general, the correlation functions of operators which correspond to supergravity modes on the gravity side show power-law behavior as predicted by the gauge/gravity duality. Some operators are predicted to show unusual infrared diverging behavior such as $|p|^{-6/5}$ in the momentum space. The gauge/gravity duality is confirmed by Monte Carlo calculations even in such cases. The operators corresponding to stringy excited modes are also studied on the gravity side,[50] and they are predicted to have correlation functions with an exotic behavior $e^{-ct^{3/5}}$. Monte Carlo results for these operators are indeed consistent with such a behavior.

2.5. *Extension to higher dimensions based on the large-N reduction*

In this section we discuss how one can extend the works in the previous sections to higher dimensions. Respecting supersymmetry becomes more nontrivial in higher dimensions, and we use the idea of the large-N reduction, which relates gauge theories in higher dimensions to those in lower dimensions in the 't Hooft large-N limit. This,
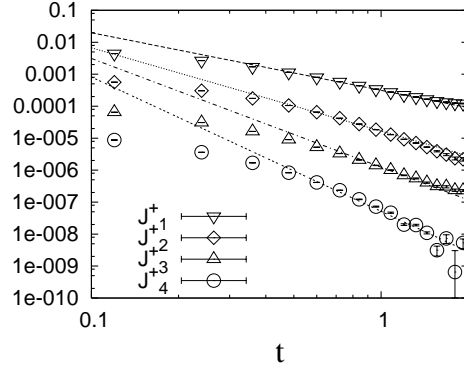
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[*] As a closely related work, Refs. 49) presented a numerical analysis based on the discrete light-cone quantization for the $(1 + 1)$-dimensional case in the large-N limit. The two-point correlation function of the stress-energy tensor has been calculated, and the expected power-law behavior has been confirmed.

in particular, enables us to perform Monte Carlo studies of the D3-brane case, which corresponds to the four-dimensional $\mathcal{N} = 4$ SYM.

Here we use a novel large-N reduction, which was proposed in Ref. 13) to study $\mathcal{N} = 4$ SYM on $R \times S^3$. The reduced model in this case is obtained by collapsing the S^3 to a point, and it is given by the action

$$S_{\text{PW}} = \frac{1}{g_{\text{PW}}^2} \int dt \, \text{tr} \left[\frac{1}{2}(D_t X_M)^2 - \frac{1}{4}[X_M, X_N]^2 + \frac{1}{2}\Psi D_t \Psi - \frac{1}{2}\Psi \gamma_M [X_M, \Psi] \right.$$
$$\left. + \frac{\mu^2}{2}(X_i)^2 + \frac{\mu^2}{8}(X_a)^2 + i\mu\epsilon_{ijk} X_i X_j X_k + i\frac{3\mu}{8}\Psi\gamma_{123}\Psi \right] , \qquad (2\cdot13)$$

where the parameter μ is related to the radius of S^3 as $R_{S^3} = \frac{2}{\mu}$, and the covariant derivative is defined by $D_t = \partial_t - i[A, \cdot \,]$, where $A(t)$ as well as $X_M(t)$ and $\Psi(t)$ is an $N \times N$ Hermitian matrix. The range of indices is given by $1 \leq M, N \leq 9$, $1 \leq i, j, k \leq 3$ and $4 \leq a \leq 9$, respectively. The model has the SU(2|4) symmetry with 16 supercharges.

The one-dimensional supersymmetric gauge theory (2·13) is called the plane wave matrix model (PWMM) or the BMN matrix model[51]*). It is nothing but the 1d SYM discussed in the previous sections, plus some mass deformation, which preserves 16 supersymmetries of the undeformed theory.

The PWMM possesses many discrete vacua representing multi fuzzy spheres, which are given explicitly by

$$X_i = \mu \bigoplus_{I=1}^{\nu} \left(L_i^{(n_I)} \otimes \mathbf{1}_{k_I} \right) \quad \text{with} \quad \sum_{I=1}^{\nu} n_I k_I = N , \qquad (2\cdot14)$$

where $L_i^{(r)}$ are the r-dimensional irreducible representation of the SU(2) algebra $[L_i^{(r)}, L_j^{(r)}] = i\,\epsilon_{ijk} L_k^{(r)}$. These vacua preserve the SU(2|4) symmetry, and are all degenerate.

In order to retrieve the planar $\mathcal{N} = 4$ SYM on $R \times S^3$, one has to pick up a particular background from (2·14), and consider the theory (2·13) around it. Let us consider the vacuum defined by

$$k_I = k , \quad n_I = n + I - \frac{\nu + 1}{2} \qquad \text{for} \quad I = 1, \cdots, \nu , \qquad (2\cdot15)$$

and take the large-N limit in such a way that

$$k \to \infty , \quad \frac{n}{\nu} \to \infty , \quad \nu \to \infty , \quad \text{with} \quad \lambda_{\text{PW}} \equiv \frac{g_{\text{PW}}^2 k}{n} \text{ fixed} . \qquad (2\cdot16)$$

Then the resulting theory is claimed[13] to be equivalent**) to the planar limit of

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*) Properties of this model at finite temperature are studied at weak coupling[52),53)] and at strong coupling.[54)]

**) See Refs. 55) for earlier studies that led to this proposal. This equivalence was checked at finite temperature in the weak coupling regime.[53)] It has also been extended to general group manifolds and coset spaces.[56)]

$\mathcal{N} = 4$ SYM on $R \times S^3$ with the 't Hooft coupling constant given by

$$\lambda_{\text{SYM}} = 2\pi^2 \lambda_{\text{PW}} (R_{S^3})^3 = \frac{16\pi^2 k}{n} \frac{g_{\text{PW}}^2}{\mu^3} \ . \tag{2·17}$$

In practice, we use (2·14) with (2·15) as the initial configuration and check that no transition to other vacua occurs during the simulation.

The above equivalence may be viewed as an extension of the large-N reduction,[12] which asserts that the large-N gauge theories can be studied by dimensionally reduced models. It is known that the original idea for theories compactified on a torus can fail due to the instability of the $\text{U}(1)^D$ symmetric vacuum of the reduced model.[14] This problem is avoided in the novel proposal since the PWMM is a massive theory and the vacuum preserves the maximal SUSY. This regularization respects 16 supersymmetries, which is half of the full superconformal symmetry of $\mathcal{N} = 4$ SYM on $R \times S^3$. Since any kind of UV regularization breaks the conformal symmetry, this regularization is optimal from the viewpoint of preserving SUSY.

Since the parameter g_{PW}^2 in the action (2·13) can be scaled out by appropriate redefinition of fields and parameters, we take $g_{\text{PW}}^2 N = 1$ without loss of generality as in Refs. 4),5),9)–11). In this convention one finds from eq. (2·17) that the small (large) μ region in the PWMM corresponds to the strong (weak) coupling region in the 4d $\mathcal{N} = 4$ SYM.

2.6. *Wilson loops in 4d $\mathcal{N} = 4$ SYM*

Let us consider the following type of Wilson loop

$$W(C) = \frac{1}{N} \text{tr} \, \mathcal{P} \exp \oint_C ds \left(i A_\mu^{R^4} \dot{x}^\mu(s) + |\dot{x}^\mu(s)| X_a^{R^4} \theta_a \right) , \tag{2·18}$$

where $\dot{x}^\mu(s) \equiv \frac{dx^\mu(s)}{ds}$ and θ_a is a constant which satisfies $\theta_a \theta_a = 1$. The fields $A_\mu^{R^4}$ and $X_a^{R^4}$ represent the gauge field and the six scalars, respectively, in 4d $\mathcal{N} = 4$ SYM on R^4. Due to the particular way in which the scalars appear, one can obtain predictions from the gravity side based on the AdS/CFT correspondence as[7]

$$\lim_{N \to \infty, \lambda_{\text{SYM}} \to \infty} \left\langle W(C) \right\rangle_{\text{SYM}} = e^{-S(C)} , \tag{2·19}$$

where $S(C)$ represents the area of the minimal surface spanning the loop C on the boundary of the AdS space.

For the circular Wilson loop $W(C_{\text{circ}})$, which is a (1/2-)BPS operator, there is an exact result on the gauge theory side, which is obtained by summing up planar ladder diagrams[57] or by using the localization method.[58] The result is given by

$$\lim_{N \to \infty} \left\langle W(C_{\text{circ}}) \right\rangle_{\text{SYM}} = \sqrt{\frac{2}{\lambda_{\text{SYM}}}} \, I_1 \left(\sqrt{2\lambda_{\text{SYM}}} \right) \tag{2·20}$$

$$\simeq \frac{e^{\sqrt{2\lambda_{\text{SYM}}}}}{\left(\frac{\pi}{2} \right)^{1/2} (2\lambda_{\text{SYM}})^{3/4}} \qquad \text{for } \lambda_{\text{SYM}} \gg 1 , \tag{2·21}$$

where $I_1(x)$ is the modified Bessel function of the first kind. The result is independent of the radius of the circle, which is a consequence of the scale invariance of $\mathcal{N} = 4$ SYM. At strong coupling it agrees with the result obtained from the dual geometry[59] $S(C_{\text{circ}}) = -\sqrt{2\lambda_{\text{SYM}}}$. This is an explicit example of the AdS/CFT correspondence. We use the exact result (2·20) for arbitrary λ_{SYM} to test our calculation method.

The Wilson loop in $\mathcal{N} = 4$ SYM can be calculated in PWMM in the following way. When we perform the conformal mapping from R^4 to $R \times S^3$, the radial and angular directions are mapped to the time and S^3-directions, respectively. Therefore, an arbitrary loop on a plane in R^4 is mapped to a loop on $R \times S^3$, which can be projected to a great circle on S^3. Such a Wilson loop can be represented in the large-N reduced model as

$$W_{\text{red}}(C) = \frac{1}{N}\text{tr}\,\mathcal{P}\exp\oint_C ds\left(iA_0\frac{dt}{ds} + iX_i e_\mu^i \dot{x}^\mu(s) + |\dot{x}_\mu(s)|\,X_a\theta_a\right),\qquad(2\cdot22)$$

where $e_j^i(x^\mu(s))$ is the dreibein on S^3.

The expectation value of this operator is related to the average of the original Wilson loop as[60]

$$\Big\langle W(C)\Big\rangle_{\text{SYM}} = \Big\langle W_{\text{red}}(C)\Big\rangle,\quad(2\cdot23)$$

where $\langle\cdots\rangle$ on the right-hand side denotes the expectation value in the large-N reduced model (PWMM). In the case of the circular Wilson loop, the relation (2·23) was confirmed by reproducing the SYM result (2·20) from the reduced model to all orders in perturbation theory assuming that non-ladder diagrams do not contribute.[61]

In Fig. 6 we present our preliminary results for the circular Wilson loop.[17] We have performed the $\Lambda \to \infty$ extrapolation using $\Lambda = 6,8,10,12$ assuming that
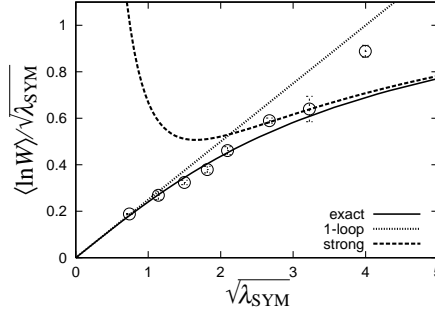

Fig. 6. The log of the circular Wilson loop normalized by $\sqrt{\lambda_{\text{SYM}}}$ is plotted against $\sqrt{\lambda_{\text{SYM}}}$. The solid line represents the exact result (2·20). The dashed line represents the behavior (2·21) at strong coupling, whereas the dotted line represents the leading perturbative behavior $\ln\langle W\rangle \simeq \frac{1}{4}\lambda_{\text{SYM}}$.

finite Λ effects are $O(1/\Lambda)$. The extent in the time direction is fixed to $\beta = 5$. The parameters describing the background (2·15) are chosen to be $(n,\nu) = (\frac{3}{2},2)$, and we performed an extrapolation to $k = \infty$ using the data for $k = 2,3,4,5$ assuming that the finite-k effects are $O(1/k^2)$. We also plot the exact result (2·20). Except for the data point at $\sqrt{\lambda_{\text{SYM}}} = 4$, the agreement with the exact result is promising. Note, in particular, that we already start to observe a bent from the weak coupling behavior towards the strong coupling behavior. This is remarkable considering the rather small matrix size. We consider this as a result of the fact that our formulation respects sixteen supersymmetries.

2.7. *Correlation functions in 4d $\mathcal{N} = 4$ SYM*

In this section we consider chiral primary operators (CPOs) as simple examples of 1/2 BPS operators in 4d $\mathcal{N} = 4$ SYM, and present Monte Carlo results for their correlation functions.[16] In particular, we find that the two-point and three-point functions agree with the free theory results up to overall constant factors even at fairly strong coupling. Moreover the ratio of the overall factors agrees with the prediction of the AdS/CFT correspondence.[*]
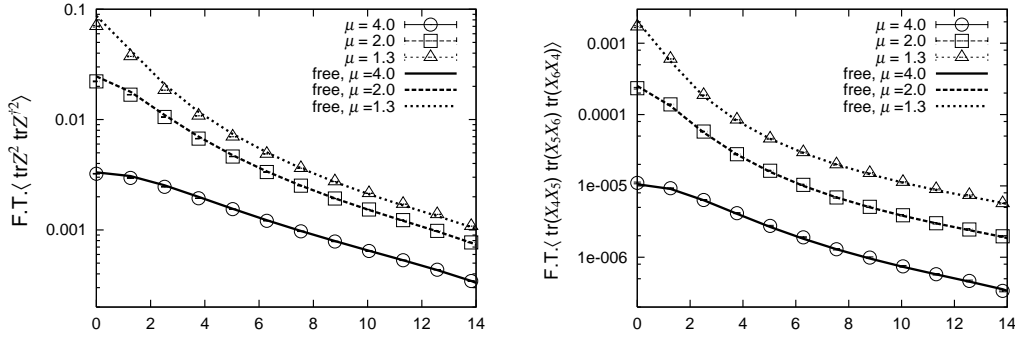

Fig. 7. (Left) The two-point function $\left\langle \mathrm{tr}\, \widetilde{Z^2}(p)\ \mathrm{tr}\, \widetilde{Z^{\dagger 2}}(-p) \right\rangle$ is plotted in the log scale. The curves represent the corresponding free theory results multiplied by 0.919, 0.799, 0.647 for $\mu = 4.0, 2.0, 1.3$, respectively. (Right) The three-point function $\left\langle \mathrm{tr}\left(\widetilde{X_4 X_5}(p)\right) \mathrm{tr}\left(\widetilde{X_5 X_6}(0)\right) \mathrm{tr}\left(\widetilde{X_6 X_4}(-p)\right) \right\rangle$ is plotted in the log scale. The curves represent the corresponding free theory results multiplied by 0.850, 0.716, 0.491 for $\mu = 4.0, 2.0, 1.3$, respectively.

Let us consider the CPOs given by

$$\mathcal{O}_\Delta^{R^4}(x) = T_{a_1 \cdots a_\Delta}\ \mathrm{tr}\left(X_{a_1}^{R^4} X_{a_2}^{R^4} \cdots X_{a_\Delta}^{R^4}(x) \right)\ , \tag{2.24}$$

where $T_{a_1 \cdots a_\Delta}$ is a symmetric traceless tensor and $X_a^{R^4}$ represents the six scalars in 4d $\mathcal{N} = 4$ SYM on R^4. Thanks to the conformal symmetry, the forms of two-point and three-point functions of the CPOs are determined as

$$\left\langle \mathcal{O}_\Delta^{R^4}(x_1) \mathcal{O}_\Delta^{R^4}(x_2) \right\rangle = c_\Delta \left\langle \mathcal{O}_\Delta^{R^4}(x_1) \mathcal{O}_\Delta^{R^4}(x_2) \right\rangle_{\mathrm{free}}\ ,$$

$$\left\langle \mathcal{O}_{\Delta_1}^{R^4}(x_1) \mathcal{O}_{\Delta_2}^{R^4}(x_2) \mathcal{O}_{\Delta_3}^{R^4}(x_3) \right\rangle = c_{\Delta_1 \Delta_2 \Delta_3} \left\langle \mathcal{O}_{\Delta_1}^{R^4}(x_1) \mathcal{O}_{\Delta_2}^{R^4}(x_2) \mathcal{O}_{\Delta_3}^{R^4}(x_3) \right\rangle_{\mathrm{free}}\ , \tag{2.25}$$

where c_Δ and $c_{\Delta_1 \Delta_2 \Delta_3}$ are over-all constants depending on λ_{SYM} in general, and $\langle \cdots \rangle_{\mathrm{free}}$ denotes the results of free theory. The analysis on the gravity side suggests[63]

$$\left. \frac{c_{\Delta_1 \Delta_2 \Delta_3}}{\sqrt{c_{\Delta_1} c_{\Delta_2} c_{\Delta_3}}} \right|_{N \to \infty, \lambda_{\mathrm{SYM}} \to \infty} = \left. \frac{c_{\Delta_1 \Delta_2 \Delta_3}}{\sqrt{c_{\Delta_1} c_{\Delta_2} c_{\Delta_3}}} \right|_{N \to \infty, \lambda_{\mathrm{SYM}} \to 0} = 1 \quad \text{for}\ ^\forall \Delta_i\ . \tag{2.26}$$

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[*] There are also Monte Carlo studies of the 4d $\mathcal{N} = 4$ SYM based on matrix quantum mechanics of 6 bosonic commuting matrices,[62] which give results consistent with the AdS/CFT for the three-point functions of CPOs.

In order to relate the above operators to those in the PWMM, we first perform the conformal mapping[*] from R^4 to $R \times S^3$. Then the M-point functions of the CPO $\mathcal{O}_{\Delta_i}^{R \times S^3}$ on $R \times S^3$ are related to those in PWMM as

$$\int \frac{d\Omega_3^{(1)}}{2\pi^2} \cdots \int \frac{d\Omega_3^{(M)}}{2\pi^2} \left\langle \mathcal{O}_{\Delta_1}^{R \times S^3}(t_1, \Omega_3^{(1)}) \cdots \mathcal{O}_{\Delta_M}^{R \times S^3}(t_M, \Omega_3^{(M)}) \right\rangle \quad (2\cdot27)$$

$$= \frac{1}{n^M \nu} \left\langle \mathcal{O}_{\Delta_1}^{\mathrm{PW}}(t_1) \cdots \mathcal{O}_{\Delta_M}^{\mathrm{PW}}(t_M) \right\rangle , \quad (2\cdot28)$$

where we have defined[13] $\mathcal{O}_{\Delta}^{\mathrm{PW}}(t) = T_{a_1 \cdots a_\Delta} \, \mathrm{tr} \left(X_{a_1} X_{a_2} \cdots X_{a_\Delta}(t) \right)$.

We calculate the two-point functions $\left\langle \mathrm{tr} \, Z^2(t_1) \, \mathrm{tr} \, Z^{\dagger 2}(t_2) \right\rangle$, where $Z = \frac{1}{\sqrt{2}}(X_4 + iX_5)$, and the three-point functions $\left\langle \mathrm{tr} \left(X_4 X_5(t_1) \right) \mathrm{tr} \left(X_5 X_6(t_2) \right) \mathrm{tr} \left(X_6 X_4(t_3) \right) \right\rangle$. The CPOs we consider here have $\Delta = 2$, and the AdS/CFT predicts $c_{222} = c_2^{3/2}$, which we test by Monte Carlo calculations.

The parameters describing the background $(2\cdot15)$ are chosen as $n = \frac{3}{2}, \nu = 2, k = 2$, which corresponds to the matrix size $N = 6$. The values of μ we use are $\mu = 4.0, 2.0, 1.3$, which correspond to $\lambda_{\mathrm{SYM}} \simeq 0.55, 4.39, 16.0$, respectively, in the chosen background. Thus we cover a wide range of the coupling constant. The regularization parameters in the t-direction are taken as $\beta = 5.0$ and $\Lambda = 12$ for all cases.

In Fig. 7 (Left) we plot the two-point function[**] $\left\langle \mathrm{tr} \, \widetilde{Z^2}(p) \, \mathrm{tr} \, \widetilde{Z^{\dagger 2}}(-p) \right\rangle$. We find that the results agree well — up to overall constants depending on μ — with the corresponding free the-

Fig. 8. The overall constants corresponding to c_2 and c_{222} in eq. $(2\cdot25)$ are plotted in the log-log scale. The straight line presents the relation $c_{222} = c_2^{3/2}$ predicted by the AdS/CFT.

ory results, which are obtained analytically by just switching off the interaction terms in the reduced model with the same regularization parameters. In Fig. 7 (Right) we show similar results for the three-point function defined by $\left\langle \mathrm{tr} \left(\widetilde{X_4 X_5}(p) \right) \mathrm{tr} \left(\widetilde{X_5 X_6}(0) \right) \mathrm{tr} \left(\widetilde{X_6 X_4}(-p) \right) \right\rangle$.

We can extract the the overall constants corresponding to c_2 and c_{222} in eq. $(2\cdot25)$ from Fig. 7. In Fig. 8 we plot the overall constants obtained in this way for three values of μ. The data points represent the mean value of the upper and lower bounds. We find that our results for various coupling constants lie on the straight line which represents the prediction $c_{222} = c_2^{3/2}$ from the AdS/CFT. Our
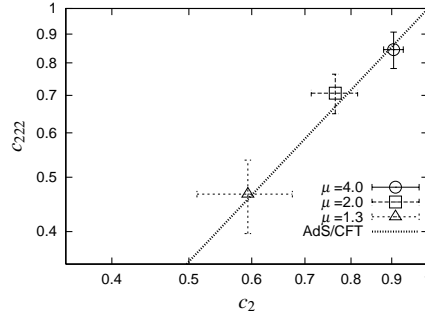
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[*] The metrics of R^4 and $R \times S^3$ are related as $ds_{R^4}^2 = dr^2 + r^2 d\Omega_3^2 = e^{\mu t} ds_{R \times S^3}^2$, where $r = \frac{2}{\mu} e^{\frac{\mu}{2} t}$. The transformation of the CPOs is given by $\mathcal{O}_{\Delta}^{R \times S^3} = e^{\frac{\Delta}{2} \mu t} \mathcal{O}_{\Delta}^{R^4}$.

[**] The Fourier transform of an operator $\mathcal{O}(t)$ is defined as $\tilde{\mathcal{O}}(p) = \frac{1}{\beta} \int_0^\beta dt \, \mathcal{O}(t) \, e^{-ipt}$.

results therefore suggest that the relation (2·26) holds also at intermediate coupling constants.

§3.   Nonperturbative formulation of superstring theory

Superstring theory not only provides a most natural candidate for a consistent theory of quantum gravity but also enables unified description of all the interactions and the matters. A crucial problem is that we do not yet have a well-established nonperturbative formulation, which would be needed in addressing dynamical issues such as the determination of space-time dimensionality.[*)]

In the 1990s, there was a remarkable progress in understanding the nonperturbative aspects of superstring theory based on D-branes. Most importantly, it was noticed that large-N matrices are the appropriate microscopic degrees of freedom which are useful in formulating superstring theory in a nonperturbative manner.[20),65),66)] In particular, the type IIB matrix model was proposed as a nonperturbative formulation of type IIB superstring theory in ten-dimensional space-time.[20)] It was also realized that the five types of superstring theory in ten dimensions are just different descriptions of the same theory. Therefore, it was speculated that the type IIB matrix model actually describes the unique underlying theory, although it takes the form that has explicit connection to perturbative type IIB superstring theory.[20),67)]

In the type IIB matrix model, the space-time is represented *dynamically* by the eigenvalue distribution of ten bosonic $N \times N$ traceless Hermitian matrices.[23)] We first discuss the Euclidean model, which is obtained by making a Wick rotation. Then we discuss a recent work on the Lorentzian model, which shows that (3+1)-dimensional expanding universe emerges dynamically.[19)]

3.1.   *Definition of the Euclidean model*

The type IIB matrix model can be obtained formally by the zero-volume limit of $D = 10$ SU(N) pure super Yang-Mills theory. The partition function of the Euclidean model is given by

$$Z = \int \mathrm{d}A \, \mathrm{d}\Psi \, e^{-S_\mathrm{b} - S_\mathrm{f}} \, , \tag{3·1}$$

$$S_\mathrm{b} = -\frac{1}{4g^2} \mathrm{tr} \, [A_\mu, A_\nu]^2 \, , \tag{3·2}$$

$$S_\mathrm{f} = -\frac{1}{2g^2} \mathrm{tr} \, (\Psi_\alpha (\mathcal{C}\Gamma^\mu)_{\alpha\beta} [A_\mu, \Psi_\beta]) \, . \tag{3·3}$$

Here A_μ ($\mu = 1, \cdots, 10$) are traceless $N \times N$ Hermitian matrices, whereas Ψ_α ($\alpha = 1, \cdots, 16$) are traceless $N \times N$ matrices with Grassmannian entries. The parameter g can be scaled out by appropriate redefinition of the matrices, and hence it is just a scale parameter. We therefore set $g^2 N = 1$ from now on without loss of generality.

[*)] See Ref. 64) for a well-known scenario based on string-gas cosmology.

The integration measure for A_μ and Ψ_α is given by

$$\mathrm{d}A = \prod_{a=1}^{N^2-1} \prod_{\mu=1}^{10} \frac{\mathrm{d}A_\mu^a}{\sqrt{2\pi}} , \qquad \mathrm{d}\Psi = \prod_{a=1}^{N^2-1} \prod_{\alpha=1}^{16} \mathrm{d}\Psi_\alpha^a , \tag{3·4}$$

where A_μ^a and Ψ_α^a are the coefficients in the expansion $A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a T^a$ *etc.* with respect to the SU(N) generators T^a normalized as $\mathrm{tr}\,(T^a T^b) = \frac{1}{2}\delta^{ab}$.

   The model has an SO(10) symmetry, under which A_μ and Ψ_α transform as a vector and a Majorana-Weyl spinor, respectively. The 16×16 matrices Γ_μ are the gamma matrices after the Weyl projection, and $\mathcal{C}$ is the charge conjugation matrix, which satisfies $(\Gamma_\mu)^T = \mathcal{C}\Gamma_\mu \mathcal{C}^\dagger$ and $\mathcal{C}^T = \mathcal{C}$.

   In general, one can obtain supersymmetric matrix models by taking the zero-volume limit of pure super Yang-Mills theories in $D = 3$, 4, 6 and 10 dimensions, where the $D = 10$ case corresponds to the type IIB matrix model. The convergence of the partition function for general D was investigated both numerically[21] and analytically.[22] The $D = 3$ model is ill-defined since the partition function is divergent. The $D = 4$ model has a real positive fermion determinant, and Monte Carlo simulation suggested the absence of the SSB of rotational symmetry.[68] (See also Refs. 69),70).) The $D = 6$ model and the $D = 10$ model both have a complex fermion determinant, whose phase is expected to play a crucial role[25)–27),71),72] in the SSB of SO(D).

   In order to discuss the spontaneous symmetry breaking (SSB) of SO(10) in the large-N limit, we consider the "moment of inertia" tensor[23),73]

$$T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu) , \tag{3·5}$$

which is a 10×10 real symmetric tensor. We denote its eigenvalues as λ_j ($j = 1, \cdots, 10$) with the specific order

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{10} . \tag{3·6}$$

If the SO(10) is not spontaneously broken, the expectation values $\langle \lambda_j \rangle$ ($j = 1, \cdots, 10$) should be all equal in the large-N limit. Therefore, if we find that they are not equal, it implies that the SO(10) symmetry is spontaneously broken. Thus the expectation values $\langle \lambda_j \rangle$ serve as an order parameter of the SSB.

3.2. *The Gaussian expansion method*

   Since there are no quadratic terms in the actions (3·2) and (3·3), we cannot perform a perturbative expansion in the ordinary sense. Finding the vacuum of this model is therefore a problem of solving a strongly coupled system. Here we review the results obtained recently by the Gaussian expansion method.[29] See also Refs. 28),74) for earlier works. The application of such a method to large-N matrix quantum mechanics was advocated by Kabat and Lifschytz,[75] and various black hole physics of the dual geometry has been discussed.[45]

The starting point of the Gaussian expansion method is to introduce a Gaussian term S_0 and to rewrite the action $S = S_{\rm b} + S_{\rm f}$ as

$$S = (S_0 + S) - S_0 . \tag{3.7}$$

Then we can perform a perturbative expansion regarding the first term $(S_0 + S)$ as the "classical action" and the second term $(-S_0)$ as the "one-loop counter term". The results at finite order depend, of course, on the choice of the Gaussian term S_0, which contains many free parameters in general. However, it is known in various examples that there exists a region of parameters, in which the results obtained at finite order are almost constant. Therefore, if we can identify this "plateau region", we can make concrete predictions. It should be emphasized that the method enables us to obtain genuinely nonperturbative results, although most of the tasks involved are nothing more than perturbative calculations.[76] There are some cases in which one finds more than one plateau regions in the parameter space. In that case, each of them is considered to correspond to a local minimum of the effective action, and the plateau which gives the smallest free energy corresponds to the true vacuum. These statements have been confirmed explicitly in simpler matrix models.[77]

As the Gaussian action for the present model, we consider the most general one that preserves the SU(N) symmetry. Note, in particular, that we have to allow the Gaussian action to break the SO(10) symmetry so that we can study the SSB of SO(10). Making use of the SO(10) symmetry of the model, we can always bring the Gaussian action into the form

$$S_0 = \frac{N}{2} \sum_{\mu=1}^{10} M_\mu {\rm tr}\, (A_\mu)^2 + \frac{N}{2} \sum_{\alpha,\beta=1}^{16} \mathcal{A}_{\alpha\beta} {\rm tr}\, (\Psi_\alpha \Psi_\beta) , \tag{3.8}$$

where M_μ and $\mathcal{A}_{\alpha\beta}$ are arbitrary parameters. The 16×16 complex matrix $\mathcal{A}_{\alpha\beta}$ can be expanded in term of the gamma matrices as

$$\mathcal{A}_{\alpha\beta} = \sum_{\mu,\nu,\rho=1}^{10} \frac{i}{3!} m_{\mu\nu\rho} (\mathcal{C}\Gamma_\mu \Gamma_\nu^\dagger \Gamma_\rho)_{\alpha\beta} , \tag{3.9}$$

using a 3-form $m_{\mu\nu\rho}$.

In practice, we truncate the series expansion at some finite order. Then the free energy and the observable depend on the free parameters M_μ and $\mathcal{A}_{\alpha\beta}$ in the Gaussian action. We search for the values of parameters, at which the free energy becomes stationary by solving the "self-consistency equations"

$$\frac{\partial}{\partial M_\mu} F = 0 , \qquad \frac{\partial}{\partial m_{\mu\nu\rho}} F = 0 , \tag{3.10}$$

and estimate F and observables at the solutions. As we increase the order of the expansion, the number of solutions increases. If we find that there are many solutions close to each other in the parameter space which give similar results for the free energy and the observables, we may identify the region as a plateau. In fact we restrict the parameter space by imposing the SO(d) symmetry with $2 \le d \le 7$. The
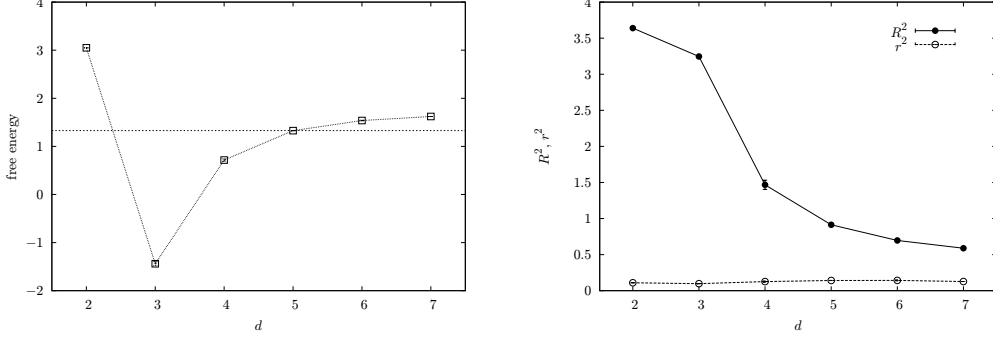
Fig. 9. (Left) The free energy density averaged over the "physical solutions" for each d is plotted
against d. The horizontal line represents the KNS value $f = \log 8 - \frac{3}{4} = 1.32944\ldots$, and the
dotted line connecting the data points is drawn to guide the eye. (Right) The extent of space-
time R^2 and r^2 in the extended and shrunken directions, respectively, are plotted against d
after taking the average over the "physical solutions" for each d. The solid and dashed lines
connecting the data points are drawn to guide the eye.

plateau region identified for each d corresponds to a local minimum which breaks
the SO(10) symmetry spontaneously down to SO(d). By comparing the free energy,
we can determine which local minimum is actually the true vacuum.

For each d, we first obtain the free energy up to the third order as a function
of the free parameters in the Gaussian action. More precisely, we actually calculate
"the free energy density" defined as

$$f = \lim_{N \to \infty} \left\{ \frac{F}{N^2 - 1} - (-3 \log N) \right\} , \quad \text{where } F = -\log Z . \qquad (3 \cdot 11)$$

By differentiating the free energy density with respect to the free parameters, we
obtain the self-consistency equations, which we solve numerically by Mathematica.

In Fig. 9 (Left) we show the free energy density obtained by averaging over the
"physical solutions" (See Ref. 29) for more detail.) for each d at order 3. We put
error bars representing the mean square error when there are more than one physical
solutions. The horizontal line $\log 8 - \frac{3}{4} = 1.32944...$ represents the result obtained
from the analytic formula of the partition function conjectured by Krauth, Nicolai
and Staudacher (KNS).[21]

The result decreases monotonically as d decreases from 7 to 3, and it becomes
much larger for $d = 2$. Thus, the SO(3) symmetric vacuum gives the smallest free
energy density. The d-dependence of the free energy density is quite analogous to
the one observed in the six-dimensional case.[78] There the value of the free energy
tends to decrease slightly as one goes from order 3 to order 5. Considering such
artifacts due to truncation, we speculate that the KNS conjecture actually refers to
the partition function for the SO(10) symmetric vacuum.

Let us discuss the results for the extent of space-time represented by the eigen-
values (3·6). For each of the SO(d) symmetric vacua, the d large eigenvalues $\langle \lambda_j \rangle$
($1 \leq j \leq d$) are equal due to the imposed SO(d) symmetry, and we denote the corre-
sponding value as R^2. The remaining $(10 - d)$ eigenvalues for each solution turn out

to be quite close to each other and we denote the mean value as r^2. In Fig. 9 (Right) we plot the result for R^2 and r^2 averaged over all the physical solutions for each d. We put error bars representing the mean square error when there are more than one physical solutions. We find that r^2 stays almost constant at $r^2 = 0.1 \sim 0.15$, which seems to be universal for all the SO(d) symmetric vacua with $2 \le d \le 7$. On the other hand, the results for R^2 are found to be larger for smaller d. It turned out that this behavior is consistent with the constant-volume property,[78] which is given by $R^d\, r^{10-d} \approx \ell^{10}$ with $\ell^2 \approx 0.38$, except for $d = 2$.

3.3. *The mechanism of SSB in the Euclidean model*

In this subsection, we review the arguments in Ref. 25) which show that the phase of the Pfaffian favors $d(\ge 3)$-dimensional configurations with λ_j ($j = d + 1, \cdots, 10$) much smaller than the others. This suggests the possibility that the SO(10) symmetry is broken down to SO(d) with $d \ge 3$.

Going back to the definition of the model (3·1), let us first integrate over the fermionic matrices Ψ_α, which yields

$$\int \mathrm{d}\Psi\, \mathrm{e}^{-S_\mathrm{f}} = \mathrm{Pf}\,\mathcal{M}\ , \tag{3·12}$$

where

$$\mathcal{M}_{a\alpha,b\beta} = -i\, f_{abc}(\mathcal{C}\,\Gamma_\mu)_{\alpha\beta} A_\mu^c \tag{3·13}$$

is a $16\,(N^2 - 1) \times 16\,(N^2 - 1)$ anti-symmetric matrix, regarding each of $(a\alpha)$ and $(b\beta)$ as a single index. The real totally-antisymmetric tensor f_{abc} gives the structure constants of SU(N). In what follows, it proves convenient to work with an explicit representation of the gamma matrices given by

$$\Gamma_1 = i\,\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2\ ,\ \ \Gamma_2 = i\,\sigma_2 \otimes \sigma_2 \otimes \mathbf{1} \otimes \sigma_1\ ,\ \ \Gamma_3 = i\,\sigma_2 \otimes \sigma_2 \otimes \mathbf{1} \otimes \sigma_3\ ,$$

$$\Gamma_4 = i\,\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \mathbf{1}\ ,\ \ \Gamma_5 = i\,\sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbf{1}\ ,\ \ \Gamma_6 = i\,\sigma_2 \otimes \mathbf{1} \otimes \sigma_1 \otimes \sigma_2\ ,$$

$$\Gamma_7 = i\,\sigma_2 \otimes \mathbf{1} \otimes \sigma_3 \otimes \sigma_2\ ,\ \ \Gamma_8 = i\,\sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\ ,\ \ \Gamma_9 = i\,\sigma_3 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\ ,$$

$$\Gamma_{10} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\ , \tag{3·14}$$

for which the charge conjugation matrix $\mathcal{C}$ becomes a unit matrix. Note that Γ_μ is pure imaginary except for Γ_{10} in this representation. Therefore, the Pfaffian is real for $A_{10} = 0$ (or for the Lorentzian case $A_{10} = -iA_0$).

Next, when $A_3 = A_4 = \cdots = A_{10} = 0$, one finds that $\mathrm{Pf}\,\mathcal{M}(A) = 0$.[21] Note first that $(\mathrm{Pf}\,\mathcal{M})^2 = \det\mathcal{M} = |\det U|^{16}$, where U is an $(N^2 - 1) \times (N^2 - 1)$ matrix defined as $U_{ab} = f_{abc}X^c$, where $X^c = A_1^c + i\, A_2^c$. Since $U_{ab}X^b = 0$, the matrix U has a zero-eigenvalue, and therefore $\det U = 0$.

Let us denote the phase of the Pfaffian by Γ. When the configuration is $d(\ge 3)$-dimensional, we find that

$$\frac{\partial^n\,\Gamma}{\partial A_{\mu_1}^{a_1} \partial A_{\mu_2}^{a_2} \cdots \partial A_{\mu_n}^{a_n}} = \frac{1}{2}\frac{\partial^n}{\partial A_{\mu_1}^{a_1} \partial A_{\mu_2}^{a_2} \cdots \partial A_{\mu_n}^{a_n}}\,\mathrm{Im}\,\ln\det\mathcal{M} = 0 \tag{3·15}$$

for $n = 1, \cdots, (9 - d)$. This is because, up to $(9 - d)$-th order of perturbations, the configuration stays within 9d configuration and therefore $\det\mathcal{M}$ remains to be

real positive. This means that the phase of the Pfaffian becomes more and more stationary as the configuration becomes lower dimensional until one reaches $d = 3$.

3.4. *Monte Carlo studies of the Euclidean model*

The phase of the Pfaffian makes Monte Carlo studies difficult because of the sign problem. However, a new method termed the factorization method[26), 27)] is expected to give a definite conclusion on the SSB of SO(10) in the Euclidean model.

Let us begin by defining the distribution functions for the eigenvalues $\{\lambda_k\}$ as

$$\rho(x_1, \cdots, x_{10}) = \left\langle \prod_k \delta(x_k - \lambda_k) \right\rangle , \tag{3.16}$$

$$\rho^{(0)}(x_1, \cdots, x_{10}) = \left\langle \prod_k \delta(x_k - \lambda_k) \right\rangle_0 \tag{3.17}$$

for the full model and the phase-quenched model, respectively. By definition (3·6), these functions vanish unless $x_1 \geq \cdots \geq x_{10}$. Applying the reweighting formula to the right-hand side of (3·16), one finds that it factorizes as

$$\rho(x_1, \cdots, x_{10}) = \frac{1}{C} \rho^{(0)}(x_1, \cdots, x_{10}) \, w(x_1, \cdots, x_{10}) . \tag{3.18}$$

The real parameter C is a normalization constant given by[*)]

$$C \stackrel{\text{def}}{=} \langle e^{i\Gamma} \rangle_0 = \langle \cos \Gamma \rangle_0 , \tag{3.19}$$

which need *not* be calculated in the present method. The function $w(x_1, \cdots, x_{10})$ is defined by

$$w(x_1, \cdots, x_{10}) \stackrel{\text{def}}{=} \langle e^{i\Gamma} \rangle_{x_1, \cdots, x_{10}} = \langle \cos \Gamma \rangle_{x_1, \cdots, x_{10}} , \tag{3.20}$$

where $\langle \cdot \rangle_{x_1, \cdots, x_{10}}$ denotes a VEV with respect to the partition function

$$Z_{x_1, \cdots, x_{10}} = \int dA \, e^{-S_0} \prod_k \delta(x_k - \lambda_k) . \tag{3.21}$$

At large N, the expectation values $\langle \lambda_k \rangle$ are obtained by maximizing the distribution function $\rho(x_1, \cdots, x_{10})$ with respect to $x_1, \cdots, x_{10}$. This leads to the coupled equations

$$\frac{\partial}{\partial x_n} \log \rho^{(0)}(x_1, \cdots, x_{10}) = -\frac{\partial}{\partial x_n} \log w(x_1, \cdots, x_{10}) \quad \text{for } n = 1, \cdots, 10 . \tag{3.22}$$

The function on the left-hand side and $w(x_1, \cdots, x_{10})$ defined by (3·20) can be obtained by simulating the constrained model (3·21).

An important observation now[26)] is that $\log w(x_1, \cdots, x_{10}) \sim \mathrm{O}(N^2)$ as naturally expected from the number of degrees of freedom. On the other hand, we speculate[78)] that the left-hand side (3·22) is $\mathrm{O}(N)$ if $\prod x_k = (\ell^2)^{10}$ and $x_n \geq r^2$,

---

[*)] In the second equality, we have used the fact that the phase Γ flips its sign under the parity transformation $A_{10} \mapsto -A_{10}$, which is a symmetry of the phase-quenched model.

where ℓ and r are some scale determined by the phase-quenched model. Thus the maximum is essentially given by $x_1 = \cdots = x_d = R^2$ and $x_{d+1} = \cdots = x_{10} = r^2$ with $R^d r^{10-d} = \ell^{10}$, where $3 \leq d \leq 9$. In order to determine which d gives the absolute maximum of $\rho(x_1, \cdots, x_{10})$, one can neglect the effect of $\rho^{(0)}(x_1, \cdots, x_{10})$ since it is suppressed by $1/N$. Due to the property in the previous section, it is expected that $\log w(x_1, \cdots, x_{10})$ becomes maximum for the solution with $d = 3$. Therefore, $\rho(x_1, \cdots, x_{10})$ becomes maximum also for $d = 3$. It is anticipated that the results of the Gaussian expansion method can be reproduced by Monte Carlo simulation in this way.

3.5. *Definition of the Lorentzian model*

In this section we discuss the Lorentzian matrix model as a nonperturbative formulation of type IIB superstring theory in (9+1)-dimensional space-time.[19]

Our starting point is the action[20] $S = S_{\mathrm{b}} + S_{\mathrm{f}}$, where

$$
S_{\mathrm{b}} = -\frac{1}{4g^2} \operatorname{tr}\left([A_\mu, A_\nu][A^\mu, A^\nu]\right) ,
$$
$$
S_{\mathrm{f}} = -\frac{1}{2g^2} \operatorname{tr}\left(\Psi_\alpha (\mathcal{C}\, \Gamma^\mu)_{\alpha\beta}[A_\mu, \Psi_\beta]\right) , \tag{3.23}
$$

with A_μ ($\mu = 0, \cdots, 9$) and Ψ_α ($\alpha = 1, \cdots, 16$) being $N \times N$ traceless Hermitian matrices. The Lorentz indices μ and ν are raised and lowered using the metric $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$. The 16×16 matrices Γ^μ are ten-dimensional gamma matrices after the Weyl projection, and the unitary matrix $\mathcal{C}$ is the charge conjugation matrix. The action has manifest SO(9,1) symmetry, where A_μ and Ψ_α transform as a vector and a Majorana-Weyl spinor, respectively. The Euclidean model, which has SO(10) symmetry, can be obtained from this action by the Wick rotation $A_0 = iA_{10}$, $\Gamma^0 = -i\Gamma_{10}$. A crucial difference is that the bosonic part of the action in the Euclidean model is positive definite, whereas in the Lorentzian model it is

$$
\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = -2\operatorname{tr}\left(F_{0i}\right)^2 + \operatorname{tr}\left(F_{ij}\right)^2 , \tag{3.24}
$$

where $F_{\mu\nu} = -i[A_\mu, A_\nu]$ are Hermitian matrices, and hence the two terms in (3.24) have opposite signs.

We define the partition function of the Lorentzian model by

$$
Z = \int dA\, d\Psi\, e^{iS} = \int dA\, e^{iS_{\mathrm{b}}} \mathrm{Pf}\mathcal{M}(A) , \tag{3.25}
$$

where the Pfaffian $\mathrm{Pf}\mathcal{M}(A)$ appears from integrating out the fermionic matrices Ψ_α. Note that in the Euclidean model, the Pfaffian is complex in general, and its phase plays a crucial role in the SSB of SO(10) symmetry.[25)–27),71),72)] On the other hand, the Pfaffian in the Lorentzian model is *real*. Therefore, the mechanism of SSB that was identified in the Euclidean model is absent in the Lorentzian model.

In the definition (3.25), we have replaced the "Boltzmann weight" e^{-S} used in the Euclidean model by e^{iS}. This is theoretically motivated from the connection to the worldsheet theory.[20] The partition function (3.25) can also be obtained

formally from pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in $(9 + 1)$ dimensions by dimensional reduction. Note, however, that the expression (3·25) is ill-defined and requires appropriate regularization in order to make any sense out of it. It turns out that the integration over A_μ is divergent, and we need to introduce two constraints

$$\frac{1}{N}\mathrm{tr}\,(A_0)^2 \leq \kappa \frac{1}{N}\mathrm{tr}\,(A_i)^2 \ , \tag{3·26}$$

$$\frac{1}{N}\mathrm{tr}\,(A_i)^2 \leq L^2 \ . \tag{3·27}$$

This is in striking contrast to the Euclidean model, in which the partition function is shown to be finite without any regularization.[21),22)]

Note that $e^{iS_{\mathrm{b}}}$ in the partition function (3·25) is a phase factor just as in the path-integral formulation of quantum field theories in Minkowski space. However, we can circumvent the sign problem by integrating out the scale factor of A_μ, which essentially replaces the phase $e^{iS_{\mathrm{b}}}$ by the constraint $S_{\mathrm{b}} \approx 0$. (Such a constraint is analogous to the one that appeared in the model inspired by space-time uncertainty principle.[79)]) Without loss of generality, we set $L = 1$ in (3·27), and thus we arrive at the model

$$Z = \int dA\,\delta\left(\frac{1}{N}\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})\right)\mathrm{Pf}\mathcal{M}(A)\,\delta\left(\frac{1}{N}\mathrm{tr}\,(A_i)^2 - 1\right)\theta\left(\kappa - \frac{1}{N}\mathrm{tr}\,(A_0)^2\right) \ , \tag{3·28}$$

where $\theta(x)$ is the Heaviside step function. Since the Pfaffian $\mathrm{Pf}\mathcal{M}(A)$ is real in the present Lorentzian case, the model (3·28) can be studied by Monte Carlo simulation without the sign problem.[*)] Note that this is usually not the case for quantum field theories in Minkowski space.

3.6. *Monte Carlo studies of the Lorentzian model*

We perform Monte Carlo simulation of the model (3·28) by using the RHMC algorithm[38)] as in the 1d SYM case discussed in section 2.

In order to extract the "time evolution", we diagonalize A_0, and define the eigenvectors $|t_a\rangle$ corresponding to the eigenvalues t_a of A_0 ($a = 1, \cdots, N$) with the specific order $t_1 < \cdots < t_N$. The spatial matrix in this basis $\langle t_a|A_i|t_b\rangle$ is not diagonal, but it turns out that the off-diagonal elements decrease rapidly as one goes away from a diagonal element. This motivates us to define $n \times n$ matrices $\bar{A}_i^{(ab)}(t) \equiv \langle t_{\nu+a}|A_i|t_{\nu+b}\rangle$ with $1 \leq a, b \leq n$ and $t = \frac{1}{n}\sum_{a=1}^{n} t_{\nu+a}$ for $\nu = 0, \cdots, (N - n)$. These matrices represent the 9d space structure at fixed time t. (This point of view can be justified in the large-N limit, in which more and more eigenvalues of A_0 appear around some value t within a fixed interval δt.) The block size n should be large enough to include non-negligible off-diagonal elements. In Fig. 10 (Left) we plot the extent of space $R(t)^2 \equiv \frac{1}{n}\mathrm{tr}\,\bar{A}_i(t)^2$ for $N = 16$ and $n = 4$. Since the result is symmetric under the time reflection $t \to -t$ as a consequence of the symmetry

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[*)] Strictly speaking, the Pfaffian can flip its sign, but we find that the configurations with positive Pfaffian dominates as N is increased. Hence, we just take the absolute value of the Pfaffian in actual simulation.
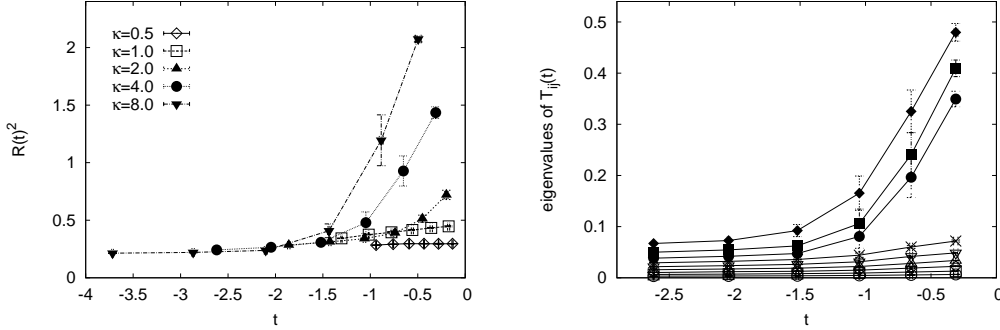
Fig. 10. (Left) The extent of space $R(t)^2$ with $N = 16$ and $n = 4$ is plotted as a function of t for five values of κ. The peak at $t = 0$ starts to grow at some critical κ. (Right) The 9 eigenvalues of $T_{ij}(t)$ with $N = 16$ and $n = 4$ are plotted as a function of t for $\kappa = 4.0$. After the critical time t_c, 3 eigenvalues become larger, suggesting that the SO(9) symmetry is spontaneously broken down to SO(3).

$A_0 \to -A_0$, we only show the results for $t < 0$. There is a critical κ, beyond which the peak at $t = 0$ starts to grow.

Next we study the spontaneous breaking of the SO(9) symmetry. As an order parameter, we define the 9×9 (positive definite) real symmetric tensor

$$T_{ij}(t) = \frac{1}{n}\mathrm{tr}\left\{\bar{A}_i(t)\bar{A}_j(t)\right\} , \qquad (3\cdot29)$$

which is an analog of (3·5) in the Euclidean model. The 9 eigenvalues of $T_{ij}(t)$ are plotted against t in Fig. 10 (Right) for $\kappa = 4.0$. We find that 3 largest eigenvalues of $T_{ij}(t)$ start to grow at the critical time t_c, which suggests that the SO(9) symmetry is spontaneously broken down to SO(3) after t_c. Note that $R(t)^2$ is given by the sum of 9 eigenvalues of $T_{ij}(t)$.

It turned out that one can remove the infrared cutoffs κ and L in the large-N limit in such a way that $R(t)$ scales. This can be done in two steps. (i) First we send κ to ∞ with N as $\kappa = \beta N^p$ ($p \simeq \frac{1}{4}$).[80] The scaling behavior is clearly seen in Fig. 11 (Left). The scaling curve of $R(t)$ one obtains in this way depends on β. (ii) Next we send β to ∞ with L. The two limits correspond to the continuum limit and the infinite volume limit, respectively, in quantum field theory. Thus the two constraints (3·26), (3·27) can be removed in the large-N limit, and the resulting theory has no parameter other than one scale parameter.

Let us discuss the second limit (ii) in more detail. We find that the inequality (3·27) is actually saturated for the dominant configurations. Therefore, one only has to make the rescaling $A_\mu \mapsto LA_\mu$ in order to translate the configurations in the model (3·28) as those in the original partition function. It turns out that $R(t)$ for the rescaled configurations scales in β by tuning L and shifting t appropriately. In order to see this, it is convenient to choose L so that $R(t)$ at the critical time $t = t_c$ becomes unity, and to shift t so that the critical time comes to the origin. Then $R(t)$ with increasing β extends in t in such a way that the results at smaller $|t|$ scale. This is demonstrated in Fig. 11 (Right), where we find a reasonable scaling behavior
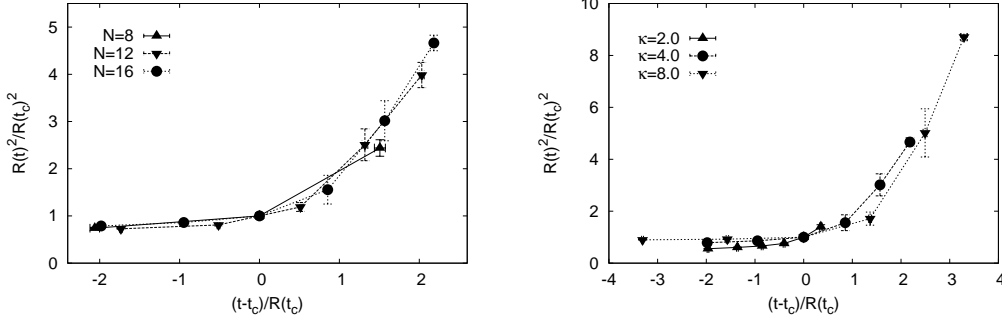
Fig. 11.   (Left) The extent of space $R(t)^2$ for $\kappa = \beta N^{1/4}$ is plotted for $N = 8, 12, 16$ with $\beta = 2$.
We plot the results against the shifted time $t - t_c$ in units of the size of the universe $R(t_c)$ at
the critical time. (Right) Similar plot for fixed $N = 16$ with $\kappa = 2, 4, 8$.

for $N = 16$ with $\kappa = 2.0, 4.0, 8.0$. Note, in particular, that the extent of "time"
increases as κ is increased, which is not the case in the bosonic model.[80]   Thus,
supersymmetry is crucial for the "emergent time" in the Lorentzian matrix model.

3.7.   *The mechanism of SSB in the Lorentzian model*

The SSB of SO(9) looks mysterious at first sight, but we can actually understand
the mechanism quite intuitively. Let us consider the case in which κ is large. Then
the first term of (3·24) becomes a large negative value, and therefore the second term
has to become large in order to make (3·24) zero as required in (3·28). Due to the
constraint $\frac{1}{N}\mathrm{tr}\,(A_i)^2 = 1$, however, it is more efficient to maximize the second term
of (3·24) at some fixed time. The system actually chooses the middle point $t = 0$,
where the suppression on A_i from the first term of (3·24) becomes the least. This
explains why the peak of $R(t)$ at $t = 0$ grows as we increase κ.

Let us then consider a simplified question: what is the configuration of A_i which
gives the maximum $\frac{1}{N}\mathrm{tr}\,(F_{ij})^2$ with fixed $\frac{1}{N}\mathrm{tr}\,(A_i)^2 = 1$. Using the Lagrange multi-
plier λ, we maximize the function $G = \mathrm{tr}\,(F_{ij})^2 - \lambda\,\mathrm{tr}\,(A_i)^2$. Taking the derivative
with respect to A_i, we obtain $2\,[A_j, [A_j, A_i]] - \lambda A_i = 0$. This equation can be solved
if $A_i = \chi L_i$ for $i \leq d$, and $A_i = 0$ for $d < i \leq 9$, where L_i are the representation
matrices of a compact semi-simple Lie algebra with d generators. Clearly d should
be less than or equal to 9. It turns out that the maximum of $\frac{1}{N}\mathrm{tr}\,(F_{ij})^2$ is achieved
for the SU(2) algebra, which has $d = 3$, with L_i being the direct sum of the spin-$\frac{1}{2}$
representation and $(N - 2)$ copies of the trivial representation. This implies the
SSB of SO(9) down to SO(3). The SSB can thus be understood as a classical ef-
fect in the $\kappa \to \infty$ limit. When we tune κ with increasing N as described above,
quantum effects become important. We have confirmed[80] that the $n \times n$ matrix
$Q = \sum_{i=1}^{9} \bar{A}_i(t)^2$ has quite a continuous eigenvalue distribution, which implies that
the space is not like a two-dimensional sphere as one might suspect from the classical
picture.

§4. Summary

We have discussed the origin of space-time from the viewpoint that matrices are the fundamental degrees of freedom in superstring theory. We have seen that matrices indeed provide an appropriate description of the quantum space-time at the singularities that appear inside a black hole or at the beginning of the Universe.

In particular, Monte Carlo studies of $U(N)$ SYM have deepened our understanding on the gauge/gravity duality considerably. In the D0-brane case, it would be interesting to investigate whether the duality holds including the $1/N$ corrections, which should be compared with the string loop corrections on the gravity side. The D3-brane case, which corresponds to the 4d $\mathcal{N} = 4$ SYM, is challenging, but we hope that more nontrivial tests of the AdS/CFT correspondence are possible by measuring non-circular Wilson loops[17] and four-point functions.[81] As a new direction, Monte Carlo simulation of matrix models obtained by applying the localization technique[58] to supersymmetric gauge theories is expected to be useful in extending the first-principle studies of the gauge/gravity duality to many more examples.[82]

The results for the Lorentzian matrix model, on the other hand, suggest that (3+1)-dimensional expanding universe emerges dynamically from type IIB superstring theory if the theory is treated nonperturbatively. This may be contrasted with the quantum cosmology in the early 80s that aimed at describing the birth of the universe[83] within the mini-superspace approximation.[*) Note also that the picture suggested here is quite different from that in (perturbative) superstring theory, where space-time with various dimensions can be obtained by compactification or by using D-brane backgrounds.

The rapid expansion of the three-dimensional space observed in Monte Carlo simulation may be interpreted as the beginning of inflation. It would be interesting to investigate the microscopic origin of the inflation along this line. Since the mechanism of the SSB relies crucially on the noncommutativity of space, it is important to see how a commutative space-time appears at later times. Furthermore, it might be possible to understand the origin of dark energy found in the present cosmological observations and to predict the fate of our Universe by studying the late time physics of the Lorentzian model.

Since superstring theory is not only a theory of quantum gravity but also a theory of all the matters and the fundamental interactions among them, it would be interesting to see how the Standard Model appears at later times in the Lorentzian matrix model. Finding solutions to the classical equation of motions[30],[86]–[88] and performing perturbative expansion around them would be an important direction as an approach complementary to Monte Carlo simulation.

We hope that all the ideas and technologies that we learned in Monte Carlo studies of QCD would be useful in developing the Monte Carlo studies of superstring theory further.

[*) More recently, a nonperturbative approach to quantum gravity has been pursued using the causal dynamical triangulation.[31] For earlier works that put forward the idea to use matrices for cosmology, see Refs. 84). See also Refs. 85) for related works on emergent gravity.

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