Nonperturbative studies of supersymmetric matrix quantum mechanics with 4 and 8 supercharges at finite temperature

Masanori Hanada,[a,b] **So Matsuura,**[c] **Jun Nishimura**[d,e] **and Daniel Robles-Llana**[a]

[a] *Department of Particle Physics and Astrophysics,*
 Weizmann Institute of Science, Rehovot 76100, Israel
[b] *Department of Physics, University of Washington, Seattle, WA 98195-1560, USA*
[c] *Department of Physics, and Research and Education Center for Natural Science,*
 Keio University, 4-1-1 Hiyoshi, Yokohama, 223-8521, Japan
[d] *KEK Theory Center, High Energy Accelerator Research Organization (KEK),*
 Tsukuba, Ibaraki 305-0801, Japan
[e] *Department of Particle and Nuclear Physics, School of High Energy Accelerator Science,*
 Graduate University for Advanced Studies (SOKENDAI),
 Tsukuba, Ibaraki 305-0801, Japan

 `mhanada@u.washington.edu, s.matsu@phys-h.keio.ac.jp,`
 `jnishi@post.kek.jp, daniel.robles@weizmann.ac.il`

ABSTRACT: We investigate thermodynamic properties of one-dimensional $U(N)$ supersymmetric gauge theories with 4 and 8 supercharges in the planar large-N limit by Monte Carlo calculations. Unlike the 16 supercharge case, the threshold bound state with zero energy is widely believed not to exist in these models. This led A.V.Smilga to conjecture that the internal energy decreases exponentially at low temperature instead of decreasing with a power law. In the 16 supercharge case, the latter behavior was predicted from the dual black 0-brane geometry and confirmed recently by Monte Carlo calculations. Our results for the models with 4 and 8 supercharges indeed support the exponential behavior, revealing a qualitative difference from the 16 supercharge case.

Contents

1. Introduction

Large-N supersymmetric gauge theories in low dimensions are useful laboratories to test the AdS/CFT correspondence and its generalizations [1, 2]. Among these, the 1d case is particularly simple since the gauge theory is just a quantum mechanical system, and therefore one may hope to test the duality relations explicitly and to understand them more in depth. Indeed Monte Carlo studies of such a system with maximal supersymmetry (16 supercharges) have been performed by using non-lattice simulations [3], which reproduced various predictions from the dual string theory including α'-corrections in the planar large-N limit [4, 5, 6, 7].[1] Another reason for interest in the one-dimensional case is that the same model, which represents the worldvolume theory of N D0-branes, has been proposed as a fully non-perturbative formulation of uncompactified M-theory in the light-cone frame [11]. Interesting observations relevant in this direction are also obtained from Monte Carlo calculations of correlation functions [7].

    In this paper we apply the same Monte Carlo method to the study of non-maximally supersymmetric quantum mechanics with 4 and 8 supercharges.[2] As opposed to the model with 16 supercharges, there is no known gravity dual for these models. On the other hand, a property common to all these supersymmetric models is that there exist flat directions

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[1]A lattice simulation of the same model has been performed in refs. [8, 9] with qualitatively consistent results for thermodynamical quantities. See also refs. [10] for earlier calculations based on the Gaussian expansion method.

[2]In fact the 4 supercharge model has been studied [3, 8] prior to the 16 supercharge model for the purpose of testing the method. In order to address the issues given below, however, we need to study the system at much lower temperature, as we do in this paper.

in the potential, which are not lifted quantum mechanically due to supersymmetry unlike their bosonic counterparts. As a result, the theory contains not only the discrete states that the bosonic models have, but also the scattering states forming the continuous branch of the spectrum [12]. In the 16 supercharge model, on top of the states just mentioned,[3] it is known that there exists a threshold bound state, which is somewhat extended in the flat directions and yet has a finite norm [15, 16, 17]. Such a state, which is crucial for the M-theory interpretation [11], is considered not to exist in the non-maximally supersymmetric models [15, 16, 17].

Recently A.V.Smilga [18] conjectured that the above difference between the 16 supercharge model and the other supersymmetric models may lead to a qualitative difference in the temperature dependence of the internal energy. In the 16 supercharge case the power-law behavior $E \propto T^{14/5}$ at low T was predicted from the dual black 0-brane geometry [2], and was confirmed by the Monte Carlo simulation [4, 6, 9]. Smilga first gave a theoretical argument on the gauge theory side for the particular power "14/5". Here an important role is played by normalizable excitations around the threshold bound state with a new energy scale (proportional to $N^{-5/9}$) suggested from the effective Hamiltonian for the relevant O(N) degrees of freedom in the flat directions. For the 4 and 8 supercharge cases, he conjectured that the internal energy decreases exponentially $E \propto \exp(-c/T)$ at low T assuming the absence of normalizable states with a new energy scale. However, he also mentioned a possibility that there exist normalizable states associated with the effective Hamiltonian with a new energy scale proportional to N^{-1}. In that case one obtains $E \propto T^2$ at low T following the same argument as in the 16 supercharge case. Our Monte Carlo data support the exponential behavior rather than the power-law behavior.

As other basic properties of the supersymmetric matrix quantum mechanics, we also study the phase structure along the temperature axis, which turns out to be qualitatively the same for all the supersymmetric models. There is only one phase, in which the center symmetry is broken spontaneously.

The rest of this paper is organized as follows. In section 2 we define the supersymmetric matrix quantum mechanics and discuss their basic properties. In section 3 we briefly review Smilga's argument. In section 4 we present our Monte Carlo results for the internal energy and compare them with the low temperature behaviors suggested by Smilga. In section 5 we compare our results with the energy spectrum obtained in the 4 supercharge model for the SU(2) gauge group [19]. Section 6 is devoted to a summary and discussions. In appendix A we derive the expression we use to calculate the internal energy in actual Monte Carlo simulation.

2. The models and their basic properties

The supersymmetric matrix quantum mechanics are defined by the action

$$S = \frac{1}{g^2} \int_0^\beta dt \, \mathrm{tr} \left\{ \frac{1}{2}(D_t X_i)^2 + \frac{1}{2}\psi_\alpha D_t \psi_\alpha - \frac{1}{4}[X_i, X_j]^2 - \frac{1}{2}\psi_\alpha(\gamma_i)_{\alpha\beta}[X_i, \psi_\beta] \right\} , \quad (2.1)$$

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[3] A detailed analysis of the continuum spectrum and its implications on the physics of supermembranes have been given in refs. [13] and [14].

where $D_t \equiv \partial_t - i[A(t), \cdot]$ represents the covariant derivative. The bosonic matrices $A(t)$, $X_i(t)$ $(i = 1, 2, \cdots, d)$ and the fermionic matrices $\psi_\alpha(t)$ $(\alpha = 1, 2, \cdots, p)$ are $N \times N$ Hermitian matrices, where $p = 4, 8, 16$ for $d = 3, 5, 9$, respectively. The models can be obtained formally by dimensionally reducing $\mathcal{N} = 1$ super Yang-Mills theory in $D = d + 1$ dimensions to one dimension, and they can be viewed as a 1d U(N) gauge theory, where $A(t)$, $X_i(t)$ and $\psi_\alpha(t)$ are the gauge field, adjoint scalars and spinors, respectively.[4] The $p \times p$ symmetric matrices γ_i obey the Euclidean Clifford algebra $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. The models are supersymmetric, and the number of supercharges is given by p.

In order to study the thermodynamics, we impose periodic and anti-periodic boundary conditions on the bosonic and fermionic matrices, respectively, which breaks supersymmetry. The extent β in the Euclidean time direction t then corresponds to the inverse temperature $\beta = T^{-1}$.

The action is invariant under the shifts

$$A(t) \mapsto A(t) + \alpha(t)\mathbf{1} , \quad X_i(t) \mapsto X_i(t) + x_i\mathbf{1} , \tag{2.2}$$

where $\alpha(t)$ is an arbitrary periodic function, and x_i is an arbitrary constant. In order to remove the corresponding decoupled modes, we impose the conditions

$$\operatorname{tr} A(t) = 0 , \quad \int_0^\beta dt \operatorname{tr} X_i(t) = 0 \tag{2.3}$$

for all $i = 1, 2, \cdots, d$.

Since the coupling constant g can be absorbed by rescaling the matrices and t appropriately, we set the 't Hooft coupling constant $\lambda \equiv g^2 N$ to unity without loss of generality. This implies that we replace the prefactor $\frac{1}{g^2}$ in the action (2.1) by N in what follows. In order to put the theory on a computer [3], we first fix the gauge as $A(t) = \frac{1}{\beta}\operatorname{diag}(\alpha_1, \cdots, \alpha_N)$, where $-\pi < \alpha_i \le \pi$, include the corresponding Fadeev-Popov determinant, and then introduce a Fourier mode cutoff Λ as

$$X_i(t) = \sum_{n=-\Lambda}^{\Lambda} \tilde{X}_{in}e^{in\omega t} , \quad \psi_\alpha(t) = \sum_{r=-\Lambda'}^{\Lambda'} \tilde{\psi}_{\alpha n}e^{ir\omega t} , \tag{2.4}$$

where $\omega = \frac{2\pi}{\beta}$. The indices n and r run over integers and half integers, respectively, and we set $\Lambda' = \Lambda - \frac{1}{2}$. The breaking of supersymmetry due to finite Λ is shown to disappear quickly with increasing Λ [3]. The fermion determinant is positive semi-definite for the $D = 4$ model [5] even at finite Λ. However, it is generally complex for the $D = 6$ and $D = 10$ models. As is done in previous works for the $D = 10$ model,[6] we simply omit the phase of

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[4]Here we use the notation of Majorana spinors to describe the $D = 4, 6, 10$ cases in a unified manner. When we write a code for the $D = 4, 6$ models, we rewrite the action in terms of Weyl fermions, which has only $\frac{p}{2}$ components.

[5]The proof goes similarly to the case of the totally reduced model [20].

[6]Previous results for the $D = 10$ model obtained in this way agreed well with the predictions from the gauge-gravity duality [4, 5, 6, 7]. It is therefore expected that the fluctuation of the phase is totally decorrelated with all physical quantities. While there is some evidence that this is indeed the case, complete understanding is still missing.

the fermion determinant when we study the $D = 6$ model. Simulations has been performed with the Rational Hybrid Monte Carlo (RHMC) algorithm [21], which is quite standard in recent lattice QCD simulations.

As a fundamental quantity in thermodynamics, the free energy $\mathcal{F} = -\frac{1}{\beta} \ln Z(\beta)$ is defined in terms of the partition function

$$Z(\beta) = \int [\mathcal{D}A]_\beta [\mathcal{D}X]_\beta [\mathcal{D}\psi]_\beta \, e^{-S(\beta)} \, , \tag{2.5}$$

where the suffix of the measure $[\,\cdot\,]_\beta$ represents the period of the field to be path-integrated. However, the evaluation of the partition function $Z(\beta)$ is not straightforward in Monte Carlo simulation. We therefore study the internal energy defined by

$$E \equiv \frac{d}{d\beta}(\beta \mathcal{F}) = -\frac{d}{d\beta} \log Z(\beta) \, , \tag{2.6}$$

which has equivalent information as the free energy, given the boundary condition $\mathcal{F} = E$ at $T = 0$. In Appendix A we explain how we calculate the internal energy E in actual simulation.

A common property of the supersymmetric matrix quantum mechanics for all $D = 4, 6, 10$, which distinguishes them from the bosonic counterparts, is that they have a continuum branch of the energy spectrum associated with the flat directions $[X_i, X_j] = 0$ in the potential. Its existence can be seen in Monte Carlo simulations as a run-away behavior, which can be probed by the observable

$$R^2 \equiv \frac{1}{N\beta} \int_0^\beta dt \, \text{tr} \left(X_i(t) \right)^2 \, . \tag{2.7}$$

In the high-T limit the fermions decouple, and the system becomes essentially a bosonic model, which is well-defined for any $N \geq 2$. The expectation value $\langle R^2 \rangle$ at large but finite T can be obtained without any problem, and it can be nicely reproduced by the high temperature expansion including the subleading term [22]. As we lower T for a fixed N, the expectation value $\langle R^2 \rangle$ decreases in accord with the high temperature expansion. However, at some T, we observe some cases in which the value of R^2 starts to increase endlessly during the simulation, which represents the aforementioned run-away behavior. If we use larger values of N, we can go to lower T without seeing such a behavior. Theoretical understanding of this property is provided in ref. [4] based on the one-loop effective action. We therefore consider that a well-defined ensemble can be defined at any finite T in the large-N and large-Λ limits, and that the ensemble average corresponds to taking the expectation value within the Hilbert space restricted to the normalizable states.

In fig. 1 we plot the expectation value $\langle R^2 \rangle$ obtained by Monte Carlo simulation. At low T and for N not extremely large, we find some cases in which $\langle R^2 \rangle$ increases rapidly as the cutoff Λ is increased. This can be understood as finite Λ effects, which tend to lift the flat directions slightly [4], and hence to suppress the run-away behavior. One can also see from fig. 1 that larger N tends to suppress the run-away behavior.

Our data suggest that the value of $\langle R^2 \rangle$ in the above limit behaves differently for the $D = 6$ and $D = 4$ models. For $D = 6$, it decreases monotonically (or stays almost
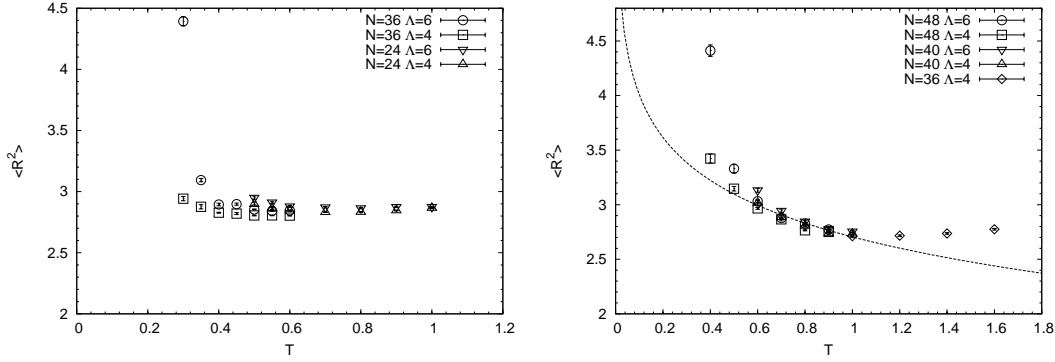
Figure 1: The extent of space is plotted against the temperature for the $D = 6$ model (left) and for the $D = 4$ model (right). The dashed line in the right panel represents a possible log-divergent behavior $\langle R^2 \rangle = a + b \log(1/T)$, where $a = 2.71$ and $b = 0.566$ obtained by fitting the data for $N = 48$, $\Lambda = 6$, $T = 0.6, \cdots, 0.9$.

constant) as T decreases similarly to what is observed for $D = 10$ [4]. For $D = 4$, on the other hand, it starts to increase as T is lowered below $T \sim 1$. This behavior is reminiscent of the divergence of the second moment in the totally reduced model of 4d $\mathcal{N} = 1$ super Yang-Mills theory [23, 20]. Since fermions obey anti-periodic boundary conditions in our 1d model, the temperature T plays a role of the SUSY breaking mass parameter for the fermions. This gives a regularization to the second moment $\langle R^2 \rangle$, which is logarithmically divergent with respect to the regularization parameter. Hence we obtain $\langle R^2 \rangle \sim \log \frac{1}{T}$. Our results in fig. 1 (right) are consistent with this behavior, but more data with larger N around $T \sim 0.4$ are needed to confirm it unambiguously.

As another property of the supersymmetric matrix quantum mechanics, let us discuss the phase structure along the temperature axis. For that we define the Polyakov line

$$P \equiv \frac{1}{N} \mathrm{tr}\, \mathcal{P} \exp \left(i \int_0^\beta dt\, A(t) \right) \ , \tag{2.8}$$

where the symbol "$\mathcal{P} \exp$" represents the path-ordered exponential. It serves as the order parameter of the spontaneous breaking of the center symmetry. In fig. 2 we plot the expectation value $\langle |P| \rangle$. In both $D = 6$ and $D = 4$ models, the results can be fitted by $\langle |P| \rangle \sim a \exp \left(-\frac{b}{T} \right)$, which is a characteristic low-T behavior of a "non-confining" theory. This implies that the center symmetry is broken at any finite temperature similarly to the case of maximal supersymmetry [4].

The bosonic models are known to undergo a "deconfining transition" at a critical temperature [24, 25, 26], below which the center symmetry remains unbroken.[7] Therefore, the Eguchi-Kawai equivalence [27] (or the "volume independence" [28]) holds above the critical β in the bosonic models. As a result, the internal energy does not depend on T below some critical value. The physical meaning of this behavior is discussed in the next section.

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[7]In fact there are two phase transitions along the temperature axis as shown by Monte Carlo simulation [25] and by $1/D$ expansion [26].
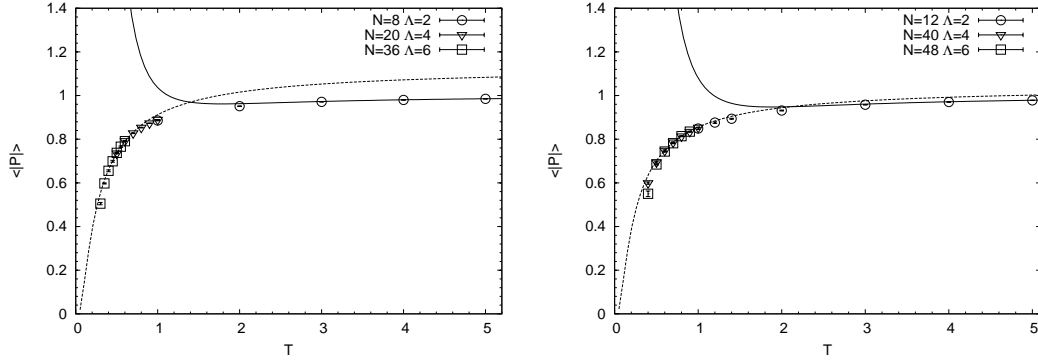
Figure 2: The expectation value $\langle |P| \rangle$ is plotted against the temperature for the $D = 6$ model (left) and for the $D = 4$ model (right). The solid line represents the results of the high temperature expansion [22] (up to the next leading order) at $N = 8$ for $D = 6$ and at $N = 12$ for $D = 4$, and the dashed line represents a fit to the behavior $\langle |P| \rangle = a \exp(-\frac{b}{T})$ characteristic to a "non-confining" theory. The parameters obtained by fitting are $a = 1.13(1)$ and $b = 0.215(6)$ for the $D = 6$ model and $a = 1.03(1)$ and $b = 0.19(1)$ for the $D = 4$ model.

3. Brief review of Smilga's argument

In this section we review Smilga's argument [18] for the low-T behavior of the internal energy. Note that the characteristic energy scale of the matrix quantum mechanics is $E_{\rm char} \sim \lambda^{\frac{1}{3}}$, which is O(1) in our convention $\lambda = 1$.

Let us first start with the bosonic models, which can be obtained by omitting the fermions. The large-N behavior of such models was studied by Monte Carlo simulation in ref. [25]. The normalized internal energy $\frac{1}{N^2} E$ as a function of T is found to stay constant $\frac{1}{N^2} E = \varepsilon_0$ below the critical temperature. This can be viewed as a consequence of the Eguchi-Kawai equivalence mentioned in the previous section. Here $E_{\rm vac} \equiv N^2 \varepsilon_0$ can be identified as the vacuum energy of the bosonic model.[8] Beyond the critical temperature, the normalized internal energy $\frac{1}{N^2} E$ starts to grow with T. This behavior of $E(T)$ can be understood from the behavior of the entropy $S(E)$ using the relations

$$Z = \int dE \, {\rm e}^{S(E)} {\rm e}^{-\beta E} \tag{3.1}$$

and $E = -\frac{\partial}{\partial \beta} \log Z$. The growth of the entropy $S(E)$ changes qualitatively[9] at some critical energy $E_{\rm cr}$, where $E_{\rm cr} - E_{\rm vac} = {\rm O}(1)$. Since the region $E < E_{\rm cr}$ corresponds to the confined phase, the entropy is $S(E) \sim {\rm O}(1)$. It is then clear that $\frac{1}{N^2} E = \varepsilon_0$ below some critical temperature. As the energy approaches the critical value $E_{\rm cr}$, the entropy $S(E)$ grows rapidly and becomes ${\rm O}(N^2)$. (This transition has been studied in refs. [29, 30].) Since the region $E > E_{\rm cr}$ corresponds to the deconfined phase, it is more appropriate to describe

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[8]For instance, $\varepsilon_0 = 6.695(5)$ is obtained for the $D = 10$ bosonic model by Monte Carlo simulation [25]. Ref. [18] gives an explanation for $E_{\rm vac} \sim {\rm O}(N^2)$ based on the variational method.

[9]The phase transition is found to be of second order for $D = 10$ [26]. This implies that $\frac{dS(E)}{dE}$ is continuous but $\frac{d^2 S(E)}{dE^2}$ is discontinuous at $E = E_{\rm cr}$.

the system at large N in terms of the normalized entropy $\sigma \equiv \frac{1}{N^2}S$ as a function of the normalized energy $\varepsilon \equiv \frac{1}{N^2}E$. In particular, the saddle-point approximation becomes exact in evaluating (3.1) at $N = \infty$, and one obtains

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = \beta \ , \tag{3.2}$$

which gives the normalized internal energy ε as a function of the temperature. This explains why ε has nontrivial dependence on β above some critical temperature.

Next we discuss the case of supersymmetric matrix quantum mechanics. Here we consider only the normalizable states, which are included in our Monte Carlo simulation as we discussed below eq. (2.7). Since the supersymmetric models are not confining, it is more appropriate to describe the system at large N in terms of the normalized entropy $\sigma \equiv \frac{1}{N^2}S$ as a function of the normalized energy $\varepsilon \equiv \frac{1}{N^2}E$.

For the $D = 10$ model one obtains

$$\varepsilon \sim 7.41 \, T^{\frac{14}{5}} \tag{3.3}$$

at low T from the gauge-gravity duality, and this power-law behavior (including the coefficient) is confirmed accurately by Monte Carlo simulation[10] [6]. From eq. (3.2), one readily finds that

$$\sigma \sim \varepsilon^p \tag{3.4}$$

for small ε with the power

$$p = \frac{9}{14} \ . \tag{3.5}$$

Rewriting (3.4) in terms of unnormalized variables, we obtain

$$S \sim N^2 \left(\frac{E}{N^2}\right)^p \tag{3.6}$$

for small $\frac{1}{N^2}E$. If we assume, for simplicity, that this formula holds even at $E \sim \mathrm{O}(1)$, we obtain $S \sim \mathrm{O}(N^{2(1-p)})$, which suggests that there are many normalizable states with the energy $E \sim \mathrm{O}(1)$. (We will shortly make this argument more precise.) In ref. [18] it was speculated that these states should be related to the existence of the threshold bound state in the $D = 10$ model.

Indeed one can reproduce the power (3.5) by considering the low energy excitations around the threshold bound state. Let us first recall that the supersymmetric models can have states extended in the flat directions. Wavefunctions for such states can be written symbolically as

$$\Psi = \chi(x_{\text{slow}}) \, \psi(x_{\text{fast}}) \ , \tag{3.7}$$

using the Born-Oppenheimer approximation. Here x_{slow} parametrizes the slow oscillations along the flat directions corresponding to the Cartan subalgebra of the gauge group, whereas x_{fast} parametrizes the fast oscillations along the orthogonal directions. Typically

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[10]The power of the subleading term at low T was also derived in ref. [6] from the gravity side by considering the α' corrections. Monte Carlo results reproduced that power, too.

these states correspond to non-normalizable states, and the spectrum becomes continuous. In the $D = 10$ case, however, there also exists a threshold bound state, which is a normalizable state with zero energy that can be expressed in the form (3.7). The excitations around such a state is expected to have a new energy scale $E_{\text{char}}^{\text{new}} \sim N^{-\frac{5}{9}}$ [18] from the form of the effective Hamiltonian [31, 32] for the states (3.7). Considering that the degrees of freedom in the effective Hamiltonian is O(N), the entropy should be $S(E) = $ O(N) for the energy $E \sim N E_{\text{char}}^{\text{new}} = $ O($N^{\frac{4}{9}}$). Imposing this requirement[11] on the power law behavior (3.6), one can determine the power p and finds that it agrees precisely with (3.5).

Strictly speaking, the prediction (3.3) from the gauge-gravity duality is valid for $N^{-\frac{10}{21}} \ll T \ll 1$. On the other hand, here we are considering an energy scale $E \sim $ O($N^{\frac{4}{9}}$), which corresponds to the temperature $T \sim $ O($N^{-\frac{5}{9}}$) slightly below the lower bound of the validity region. The precise agreement therefore implies that the power law (3.3) actually holds for a range of T wider than naively expected. Similar observation was made in the comparison of correlation functions with predictions from the gauge-gravity duality [7]. The power-law behavior of the correlation functions obtained from the gauge-gravity duality was found to hold in the far infrared regime, in which the supergravity analysis is not valid naively.

For the $D = 4, 6$ cases, calculation of the Witten index suggests that the threshold bound state does not exist [15, 16, 17]. Therefore one may naively consider that there is no normalizable state written in the form (3.7). The lowest energy level[12] in the discrete spectrum is considered to have the energy $E_{\text{vac}} = $ O(1), and hence we have $\varepsilon \to 0$ as $T \to 0$ in the large-N limit. Then the entropy is $S(E) \sim $ O(1) for $E \sim $ O(1), and it is expected to increase almost linearly with the energy E as it approaches $S(E) \sim $ O(N^2) at $E \sim $ O(N^2). Therefore a more plausible behavior for the normalized entropy $\sigma(\varepsilon)$ at small ε is

$$\sigma \sim \varepsilon \log \left(\frac{1}{\varepsilon} \right) , \tag{3.8}$$

where the logarithmic factor is introduced to ensure $\varepsilon \to 0$ as $T \to 0$. Then from eq. (3.2) we obtain

$$\varepsilon \sim \mathrm{e}^{-\frac{c}{T}} . \tag{3.9}$$

On the other hand, the form of the effective Hamiltonian for the continuum states (3.7) suggests a new scale $E_{\text{char}}^{\text{new}} \sim N^{-1}$ [33]. If there exist normalizable states with this new scale, the same argument[13] as the $D = 10$ case suggests (3.6) with $p = \frac{1}{2}$, and therefore

$$\varepsilon \sim T^2 . \tag{3.10}$$

We will see that our Monte Carlo data support (3.9) rather than (3.10).

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[11]In ref. [18] the power was determined by matching the energy at temperature $T \sim E_{\text{char}}^{\text{new}}$ with $E \sim N E_{\text{char}}^{\text{new}}$, which is equivalent to the argument given here.

[12]In the $D = 4$ model with $N = 2$, it was shown explicitly that these states form a supermultiplet [19] and has a non-zero energy. See section 5 for more details.

[13]Here it is assumed that the form (3.6) holds even at the energy scale $E \sim N E_{\text{char}}^{\text{new}} = $ O(1).
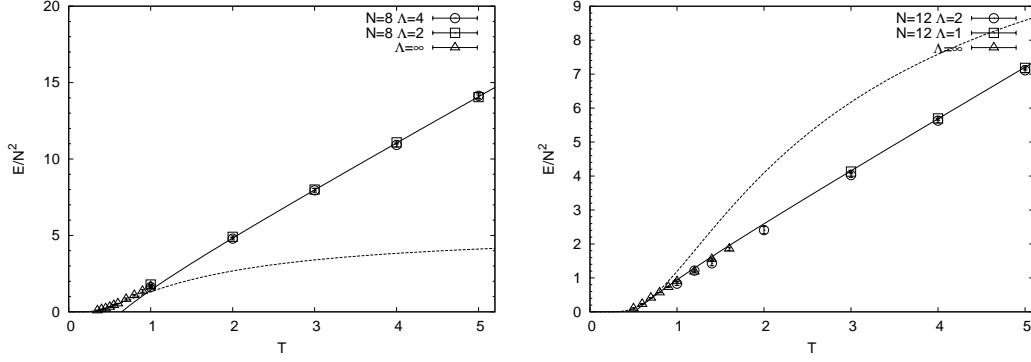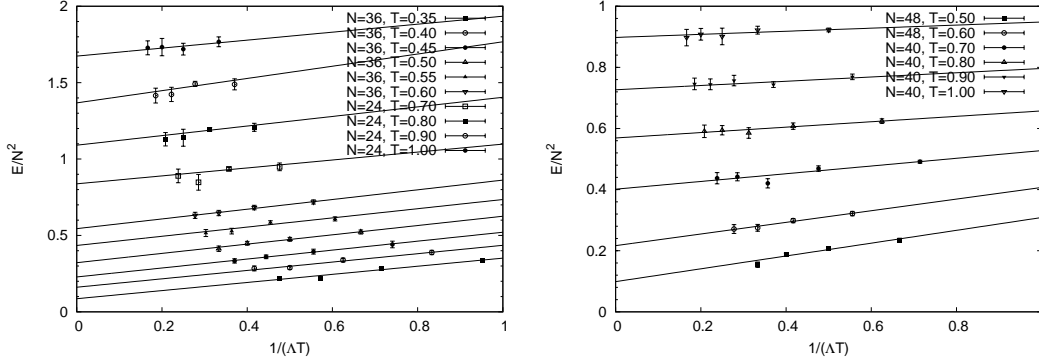
Figure 3: The normalized internal energy $\frac{1}{N^2}E$ is plotted against temperature for the $D = 6$ model (left) and the $D = 4$ model (right). The solid line represents the results of the high temperature expansion [22] (up to the next leading order) at $N = 8$ for $D = 6$ and at $N = 12$ for $D = 4$. The triangles represent the results obtained at low T by the large-Λ extrapolation in fig. 4. The dashed lines represent the behavior (4.1) obtained by fitting the data at low T.

Figure 4: The normalized energy $\frac{1}{N^2}E$ is plotted against $\frac{1}{\Lambda T}$ for the $D = 6$ model (left) and for the $D = 4$ model (right). The data points for each T can be fitted nicely to a straight line, which enables a sensible large Λ extrapolation.

4. Results for the internal energy

In fig. 3 we plot the internal energy at high temperature. The results agree nicely with the results of the high temperature expansion obtained in ref. [22].

Next we consider the low-T behavior. Since the physical cutoff scale is given by $\omega\Lambda = 2\pi T\Lambda$, which is proportional to ΛT, one needs to increase Λ as one goes to lower T. In fig. 4 we plot the internal energy against $\frac{1}{\Lambda T}$ for various T. We see that the data points for fixed T can be fitted nicely by a straight line. Based on this plot, we make an extrapolation to $\Lambda = \infty$.

Figure 5 shows the log plot of the normalized internal energy extrapolated to $\Lambda = \infty$ as a function of the temperature for the $D = 6, 4$ models. A straight line in this figure represents the behavior

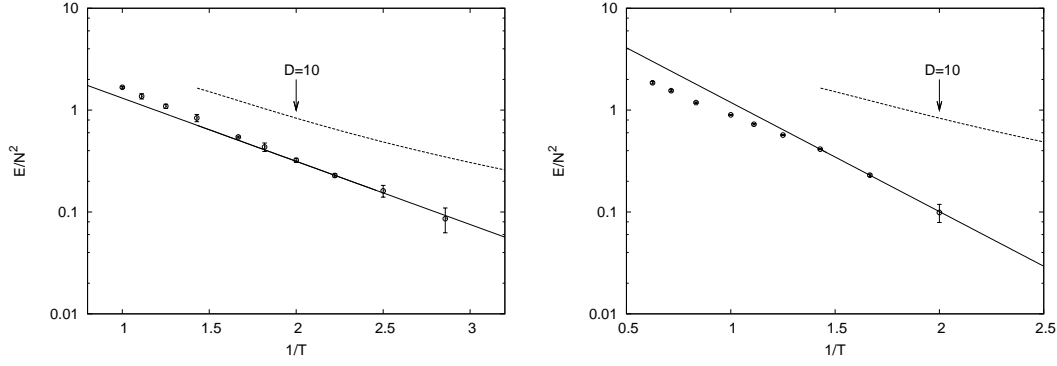$$\frac{1}{N^2}E = a\exp\left(-\frac{b}{T}\right) \ . \tag{4.1}$$

Figure 5: The log plot of the internal energy $\frac{1}{N^2}E$ against the temperature for the $D = 6$ model (left) and for the $D = 4$ model (right). The solid line represents a fit to the exponential behavior (4.1) using the 3 data points with lowest T. The dashed line represents the results for the $D = 10$ model [6] obtained by fitting the Monte Carlo data to the form $\frac{1}{N^2}E = 7.41T^{\frac{14}{5}} - cT^{\frac{23}{5}}$, where $c = 5.58$.
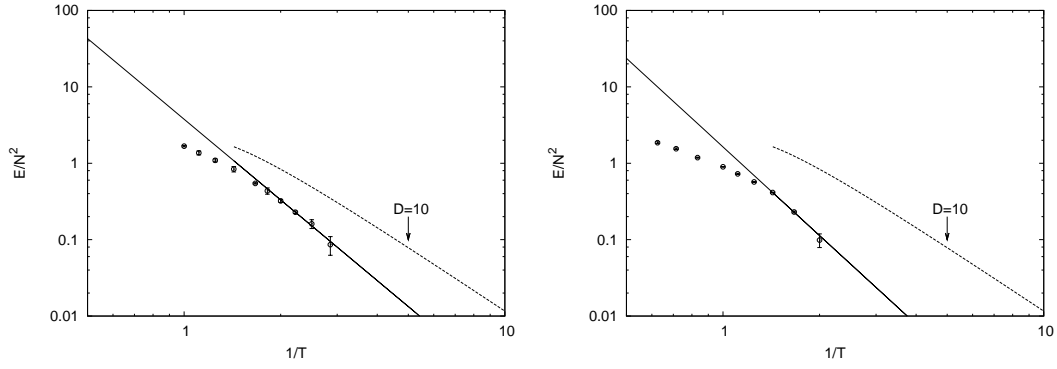

Figure 6: The log-log plot of the internal energy $\frac{1}{N^2}E$ against the temperature for the $D = 6$ model (left) and for the $D = 4$ model (right). The solid line represents a fit to the power law behavior (4.2) using the 3 data points with lowest T. The dashed line represents the results for the $D = 10$ model [6] obtained by fitting the Monte Carlo data to the form $\frac{1}{N^2}E = 7.41T^{\frac{14}{5}} - cT^{\frac{23}{5}}$, where $c = 5.58$.

Fitting the data points with 3 lowest T, we obtain $a = 5.5 \pm 1.8$, $b = 1.4 \pm 0.1$ for the $D = 6$ model, and $a = 14.1 \pm 0.3$, $b = 2.47 \pm 0.01$ for the $D = 4$ model. The dashed line represents the results for the $D = 10$ model, which cannot be fitted by a straight line at low T. Note also that the internal energy decreases much more rapidly in the $D = 6, 4$ models than in the $D = 10$ model.

In fig. 6 we show the log-log plot of the same data. A straight line in this figure represents

$$\frac{1}{N^2}E = cT^q , \tag{4.2}$$

and the slope is given by $-q$. Fitting the data points with 3 lowest T, we obtain $c = 3.8 \pm 1.4$, $q = 3.5 \pm 0.5$ for the $D = 6$ model and $c = 1.63 \pm 0.09$, $q = 3.9 \pm 0.1$ for the $D = 4$ model. The obtained values of q are much larger than 2, and we do not see any tendency that the

slope becomes closer to (-2) at lower T. Clearly our data are more consistent with (3.9) than with (3.10).
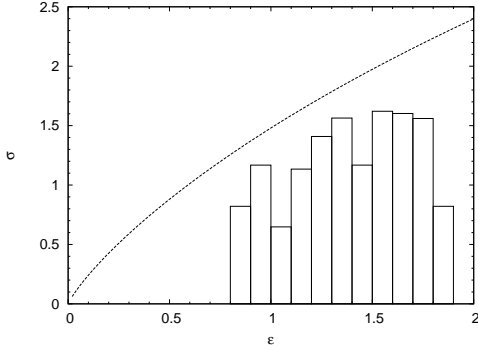
5. Comparison with the energy spectrum for SU(2)

Figure 7: The histogram represents the normalized entropy $\sigma(\varepsilon)$ as a function of the normalized energy ε in the $D = 4$ SU(2) supersymmetric model [19]. The dashed line represents the result (5.1) obtained by fitting the Monte Carlo data for $E(T)$ to the behavior (4.1).

In ref. [19] the energy spectrum of the $D = 4$ model for the gauge group SU(2) was obtained by diagonalizing the Hamiltonian directly. The Hilbert space was truncated by the number of bosons. The discrete states and the continuum states can be clearly identified by the convergence behavior as the cutoff on the number of bosons is increased. The energy levels for the discrete states were seen to converge, whereas those for the continuum states were seen to decrease slowly without convergence. The full list of the discrete energy levels obtained by this method is given by tables 9 and 10 of ref. [19]. In particular, the lowest level in the discrete spectrum is eight-fold degenerate.[14]

On the other hand, from (4.1), we obtain

$$\sigma = A\varepsilon \log\left(\frac{1}{\varepsilon}\right) + B\varepsilon \ ,$$
$$A = \frac{1}{b} \ , \quad B = \frac{\log a + 1}{b} \ , \tag{5.1}$$

where $A = 0.405$ and $B = 1.48$ plugging in the values of a and b obtained by fitting the Monte Carlo data for $E(T)$ to the behavior (4.1).

In order to compare the results for $N = 2$ with the result (5.1) for $N = \infty$, we should first note that the coupling constant g is set to unity in ref. [19]. The internal energy for arbitrary g can be obtained by $E(g) = g^{2/3}E(g = 1)$. On the other hand, when we take the large-N limit, we fix $g^2 N = 1$. Therefore, we first rescale the energy spectrum obtained in ref. [19] by a factor of $(\frac{1}{2})^{1/3}$. Then we count the number of states (taking into account the degeneracy) within the interval $E \sim E + \delta E$. Dividing the number by the bin size δE, we obtain the density of state $\rho(E)$, and hence the entropy $S(E) = \log \rho(E)$. The normalized quantities[15] $\sigma \equiv \frac{1}{N^2}S(E)$ and $\varepsilon \equiv \frac{1}{N^2}E$ are defined by setting $N = 2$. Let us

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[14]Four states form a supermultiplet, and there are two of them due to the symmetry $n_F \mapsto 6 - n_F$, where n_F represents the number of fermions.

[15]We have also tried the normalization factor $\frac{1}{N^2-1}$ (instead of $\frac{1}{N^2}$) motivated by the fact that the results in ref. [19] are obtained for the gauge group SU(2) instead of U(2). The agreement with the large-N result does not change drastically, however.

note that σ cannot be defined as a continuous function of ε at finite N. In fig. 7 we plot σ against ε as a histogram with the bin size $\delta\varepsilon = 0.1$. At larger N, it is expected that the histogram becomes smoother even for smaller bin size, and that one obtains σ as a continuous function of ε in the large-N limit. The dashed line represents (5.1) for $N = \infty$ derived from our Monte Carlo results. It would be interesting to extend the method of ref. [19] to the $N = 3$ case, and to see whether the histogram comes closer to the $N = \infty$ curve (5.1).

6. Summary and discussions

In this paper we have performed Monte Carlo studies of supersymmetric matrix quantum mechanics with 4 and 8 supercharges at finite temperature. Similarly to the 16 supercharge case studied previously, the potential for the bosonic matrices has a flat direction, which is not lifted quantum mechanically due to supersymmetry. This is seen in our Monte Carlo simulation as the run-away behavior. However, it is possible to suppress this behavior and to define a well-defined ensemble by using sufficiently large N. We consider that the ensemble obtained by our simulation corresponds to restricting the Hilbert space to normalizable states.

While the phase diagram turns out to be similar to the 16 supercharge case, we observe a notable difference in the behavior of the internal energy. It decreases much faster as the temperature decreases than in the 16 supercharge case. In fact, our Monte Carlo data for the internal energy are consistent with exponential decrease at low temperature in striking contrast to the power-law behavior in the 16 supercharge case. This behavior was predicted by Smilga assuming the absence of normalizable states with a new energy scale unlike in the $D = 10$ model. Thus our results provide independent evidence for the peculiarity of the $D = 10$ model suggested previously in the literature [15, 16, 17].

To our knowledge, there are no concrete proposals for a gravity dual of the non-maximally supersymmetric models studied in the present work. We hope that our explicit results in the planar large-N limit would provide a useful guide for constructing such an example. That would give us new insights into the gauge-gravity duality in non-maximally supersymmetric cases.

A. Derivation of the formula for the internal energy

In this Appendix we explain how we calculate the internal energy defined by (2.6). In ref. [22] it is shown that the internal energy E is related to the expectation values

$$\frac{E}{N^2} = \langle \mathcal{E}_{\mathrm{b}} \rangle + \langle \mathcal{E}_{\mathrm{f}} \rangle \, , \tag{A.1}$$

where the operators $\mathcal{E}_{\mathrm{b}}$ and $\mathcal{E}_{\mathrm{f}}$ are defined as

$$\mathcal{E}_{\mathrm{b}} \equiv -\frac{3}{4} \frac{1}{N\beta} \int_0^\beta dt \, \mathrm{tr} \left([X_i, X_j]^2 \right) \, , \tag{A.2}$$

$$\mathcal{E}_{\mathrm{f}} \equiv -\frac{3}{4} \frac{1}{N\beta} \int_0^\beta dt \, \mathrm{tr} \left(\psi_\alpha (\gamma_i)_{\alpha\beta} [X_i, \psi_\beta] \right) \, . \tag{A.3}$$

However, the calculation of $\langle \mathcal{E}_{\mathrm{f}} \rangle$ is time-consuming since it requires the construction of the fermion matrix and inverting it. In fact we can use an identity to trade it off with a quantity written solely in terms of bosonic variables. This trick has been first proposed in ref. [8] and it was used also in refs. [4, 6].

Let us consider the change of variables

$$X_i \mapsto \mathrm{e}^\epsilon \, X_i(t) \tag{A.4}$$

in the path integral representation of the partition function (2.5). One finds that the path-integral measure transforms as[16]

$$\mathcal{D}X \mapsto \mathrm{e}^{\epsilon d \{N^2(2\Lambda+1)-1\}} \mathcal{D}X \, , \tag{A.5}$$

and each term in the action transforms according to the order of X_i. Since this is just a change of variables, the value of the partition function should not change. This implies that the $\mathrm{O}(\epsilon)$ terms should cancel. Thus we obtain an identity

$$\frac{1}{g^2} \left\langle \int_0^\beta dt \, \mathrm{tr} \left\{ 2\frac{1}{2}(D_t X_i)^2 - 4\frac{1}{4}[X_i, X_j]^2 - \frac{1}{2}\psi_\alpha(\gamma_i)_{\alpha\beta}[X_i, \psi_\beta] \right\} \right\rangle = d\{N^2(2\Lambda+1) - 1\} \, . \tag{A.6}$$

Note that the derivative term on the left-hand side has linear divergence at $\Lambda \to \infty$, which is given by the $\mathrm{O}(\Lambda)$ term on the right-hand side. Using this identity, we obtain

$$E = -\frac{3}{\beta} \left[\langle S_{\mathrm{b}} \rangle - \frac{1}{2}d \left\{ (2\Lambda+1)N^2 - 1 \right\} \right] \, . \tag{A.7}$$

The advantage of using this formula instead of (A.1) is that the evaluation of S_{b} does not take much time. The disadvantage is that the fluctuation of S_{b} is $\mathrm{O}(\sqrt{\Lambda})$, and therefore one needs to increase the statistics linearly in Λ to keep the statistical error constant. This is related to the fact that the relation (A.7) involves the cancellation of the $\mathrm{O}(\Lambda)$ terms. We do not encounter any problem, however, in the parameter regime investigated in this work.

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[16]The "-1" in the exponent is due to the constraint (2.3)

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