M-Theory and the Light Cone

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I discuss D0-brane quantum mechanics as a nonperturbative formulation of string theory, in particular the relation between the Banks–Fischler–Shenker–Susskind matrix model, the Maldacena conjecture for D0-branes, and type IIA/M-theory duality. Some features of the quantum mechanics of D0-branes are also discussed.

§1. Introduction

The main focus of this lecture is D0-brane quantum mechanics as a nonperturbative formulation of string theory. In particular I would like to discuss the relation between the Banks–Fischler–Shenker–Susskind matrix model,[1] the Maldacena conjecture for D0-branes[2,3], and type IIA/M-theory duality.[4] This has sometimes been a confusing subject, and it is one of which everyone has their own understanding.**) I hope that I will reduce rather than add to the confusion. I have certainly added to it in the past; the present version reflects my improved understanding.

Figure 1 is a schematic picture of the lessons of string duality, one that I have drawn perhaps a hundred times. It is intended to emphasize the limited range of conventional string theory, which only provides an asymptotic expansion at the five stringy cusps. This is true even when one includes D-branes — only asymptotically in the string coupling do these provide a quantitatively accurate description. To describe the complete physics at a finite distance from a cusp, and to reach the M-theory point with 11-dimensional Lorentz invariance, requires a leap, a conjecture, to a new idea with a greater range of validity. This new idea can only be guessed, not deduced from information already in string theory.

This has always been the intended message of figure 1, but as I will explain I now believe, rather surprising, that one can actually deduce a great deal from what is already known. The main outline of this talk is the description of four limits: the DKPS limit, the SSS limit, the BGLHKS limit, and the IMSY limit. Actually, these are all the same limit of type IIA string theory, but presented in different ways and with different emphases. I will then explain how these different descriptions illuminate the relations among the various ideas. At the end I discuss possible future directions in D0-brane quantum mechanics, and some properties of the D0-brane bound state.

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**) Other discussions of which I am aware, some of which overlap mine, are listed in ref.[5].

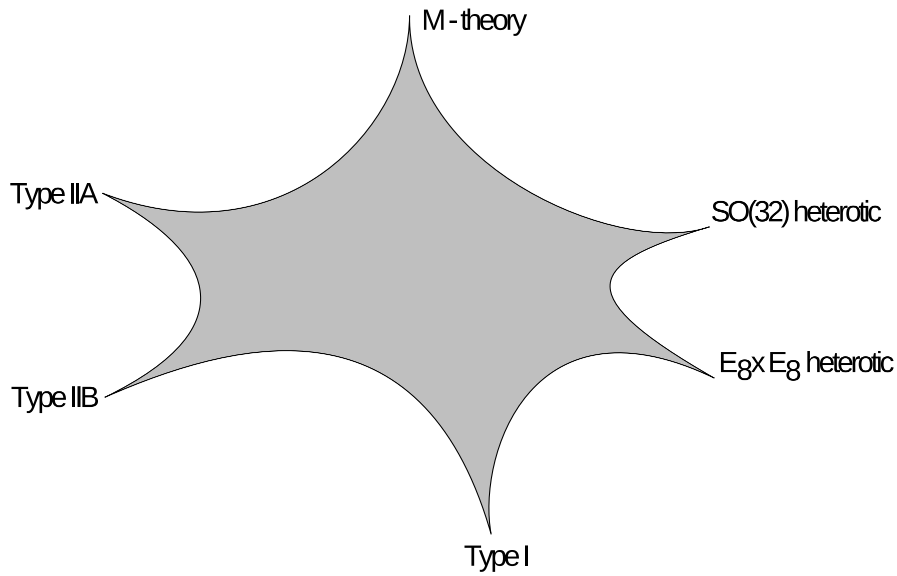

Fig. 1. Space of string vacua. The cusps are limits in which a weakly coupled string description is possible (except for the M-theory limit).

§2. The Douglas–Kabat–Pouliot–Shenker (DKPS) Limit

Consider a state in the IIA string theory containing N D0-branes plus any collection of open and closed strings.*) The energy of such a state is

$$\begin{aligned} H = {} & \frac{N}{g_{\mathrm{s}} l_{\mathrm{s}}} + O(l_{\mathrm{s}}^{-1}) + O(q) \\ & + \frac{g_{\mathrm{s}} l_{\mathrm{s}}}{2} \mathrm{Tr}(P^2) - \frac{1}{16\pi^2 g_{\mathrm{s}} l_{\mathrm{s}}^5} \sum_{m,n} \mathrm{Tr}([X^m, X^n]^2) + \text{spin term} + \text{higher derivs.} \end{aligned} \quad (2\cdot1)$$

Here g_{s} is the dimensionless string coupling and $l_{\mathrm{s}} = (\alpha')^{1/2}$ is the string length scale. The first term is the D0-brane rest energy. The second is the mass of excited open or closed strings. The third is from massless closed strings, with momenta of order q.

The important terms are in the second line. The first is a nonrelativistic kinetic term, $P^2/2m$, except that it is a trace over N^2 terms rather than a sum over N terms. This is a reflection of the important non-Abelian geometry of D-branes. [7] The coordinates arise from the lightest open string state

$$\psi^m_{-1/2} |ij\rangle \ . \quad (2\cdot2)$$

This state has a 9-dimensional spatial index m and so naturally corresponds to a collective coordinate. The ket $|ij\rangle$ denotes a string that begins on the ith D0-brane and ends on the jth, so the coordinate X^m_{ij} is a *matrix* in the N-dimensional D0-brane index space. Then P^m_{ij} is the conjugate momentum. The potential term corrects

*) For reviews of D0-brane properties see refs. [6].

this overcounting. It is nonnegative (the minus sign canceling an i^2) and vanishes precisely when all the X^m commute and so can be diagonalized. Further, there is a $U(N)$ gauge invariance, so the state is independent of basis and the low energy physics reduces to the quantum mechanics of the $9N$ eigenvalues, as appropriate for N particles in $9+1$ dimensions.

The potential term has the same form as the 4-gluon Yang–Mills interaction $\mathrm{Tr}(F^{mn}F_{mn})$. This is no accident, the two being related by T-duality. The omitted spin terms are such that the system has 16 linearly realized supersymmetries, generating 256 states for each D0-brane.

Now let us take the DKPS limit,[8] which is $g_\mathrm{s} \to 0$ while holding fixed all of

$$N \ , \ P \ , \ q \ , \ M = g_\mathrm{s}^{-1/3} l_\mathrm{s}^{-1} \ , \ \mathcal{H} = \frac{1}{g_\mathrm{s} l_\mathrm{s} M}\left(H - \frac{N}{g_\mathrm{s} l_s} \right) . \tag{2·3}$$

It follows from the energy (2·1) that all closed strings decouple, as do excited open strings, leaving only massless open strings with

$$\mathcal{H} = \frac{1}{2M}\mathrm{Tr}(P^2) - \frac{M^5}{16\pi^2}\sum_{m,n}\mathrm{Tr}([X^m, X^n]^2) + \text{spin term} . \tag{2·4}$$

Thus in this limit the IIA string theory becomes very simple, just the matrix quantum mechanics[9] (2·4). The string coupling is taken to zero, but because of the scaling of the energy the most relevant interaction does survive.

Summary: The matrix Hamiltonian (2·4) describes the sector of IIA string theory with D0-brane-charge N, in the DKPS limit (2·3).

§3. The Susskind–Seiberg–Sen (SSS) Limit

The IIA string is M-theory compactified on a circle of radius $R_{10} = g_\mathrm{s} l_s$, with eleven-dimensional Planck length $M = g_\mathrm{s}^{-1/3} l_\mathrm{s}^{-1}$. The D0-brane charge is the number of units of compact momentum, $N = p_{10}R_{10}$. In terms of M-theory parameters, the DKPS limit takes $R_{10} \to 0$ while holding fixed all of

$$p_{10}R_{10} \ , \ P \ , \ q \ , \ M \ , \ \mathcal{H} = \frac{1}{MR_{10}}(H - p_{10}) . \tag{3·1}$$

This scaling of the energy has a simple pictorial interpretation, as in figure 2. For any finite R_{10} we can find a frame in which the spatial radius takes an arbitrary fixed value R'_{10}; the cost is that the periodicity is no longer at equal times but at nearly null separations. The Lorentz factor is $\gamma = R'_{10}/R_{10}$ and so

$$(H - p_{10})' = \frac{R'_{10}}{R_{10}}(H - p_{10}) = MR'_{10}\mathcal{H} . \tag{3·2}$$

Since $p'_{10} = N/R'_{10}$ is fixed, the DKPS limit is equivalent to holding fixed the energy in the boosted frame of figure 2b. Thus we have a new interpretation:[10,11,12]

Summary: The matrix Hamiltonian (2·4) describes M-theory in a space compactified in a lightlike direction, in the sector with N units of compact momentum.

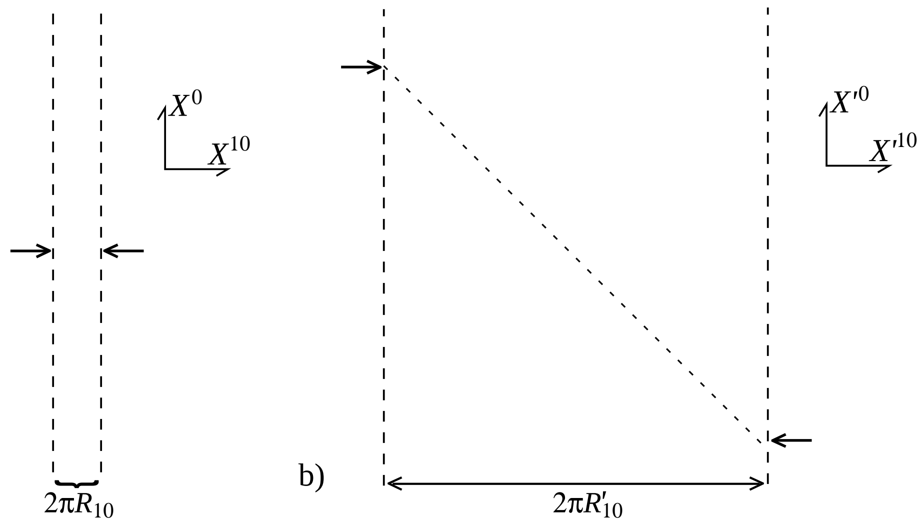

Fig. 2. The same space in two frames. a) Identified point (indicated by arrows) are at equal times and small spatial separation. b) Identified points are at fixed spatial distance and almost null separation.

This is defined as the limit of spacelike compactification, keeping all states that have finite energy in the limit. The eleven-dimensional Planck scale and the noncompact momenta are held fixed in the limit.

§4. The Balasubramanian–Gopakumar–Larsen–Hyun–Kiem–Shin (BGLHKS) Limit

An important refinement of the preceding picture comes when we consider that the particles carrying p_{10} must gravitate. With matter in the periodic space one would not expect that the compactified direction remain precisely null. A perturbation towards timelike compactification is implausible, so likely any gravitational effect is in the direction of making the periodicity spacelike. Indeed, as a function of distance r from the source, the radius of the compact dimension is given in terms of the black 0-brane dilaton solution [13)]

$$\begin{aligned} MR_{10} &= e^{2\Phi(r)/3} \\ &= g_{\mathrm{s}}^{2/3}\left(1+\frac{g_{\mathrm{s}}Nl_{\mathrm{s}}^7}{r^7}\right)^{1/2} \\ &= \left(g_{\mathrm{s}}^{4/3}+\frac{N}{M^7r^7}\right)^{1/2} \\ &\to \left(\frac{N}{M^7r^7}\right)^{1/2} . \end{aligned} \tag{4·1}$$

Here and in the metric below numerical constants have been ignored. In the last line we have taken the DKPS limit, $g_{\rm s} \to 0$ at fixed M. The striking feature is that the radius $R_{10}(r)$ has a nonvanishing limit, and only when $r \to \infty$ does the radius of the compact dimension go to zero. Thus the geometry is only *asymptotically* as depicted in figure 2b. [14), 15)]

Let us develop somewhat further the nature of the resulting space. The first obvious separation occurs at $r \approx N^{1/7}/M$, where $e^{\Phi(r)}$ is of order one. At larger r, $R_{10}(r)$ is less than M^{-1} and so the effective description is in terms of weakly coupled IIA string theory. At smaller r, $R_{10}(r)$ is greater than M^{-1} and so the effective description is in terms of M-theory.

To go further examine the M-theory metric, obtained by lifting the black D0-brane metric to M-theory in the coordinate system of figure 2b,

$$ds^2 = -dt'^2 + dx'^2_{10} + \frac{N}{M^9 r^7 R'^2_{10}}(dt' - dx'_{10})^2 + (dx^\perp)^2 \ . \tag{4·2}$$

The periodicity is

$$(t', x'_{10}) \cong (t' - \pi R'_{10}, x'_{10} + \pi R'_{10}) \ . \tag{4·3}$$

The metric appears flat at large r, but this deceptive because the radius of the periodic dimension and so the local string tension are going to zero. The latter is

$$T(r) = M^3 R_{10}(r) = N^{1/2} M^{-3/2} r^{-7/2} \ . \tag{4·4}$$

The corresponding length scale $l_{\rm s}(r) = N^{-1/4} M^{3/4} r^{7/4}$ exceeds r when $r > N^{1/3} M^{-1}$. At this point geometry can no longer make sense: the strings are larger than the system (in terms of the string metric, which is curved, the curvature becomes large in string units).

Another key distance is $r = N^{1/9} M^{-1}$, where $R_{10}(r) = r$. At distances below this the metric (4·2) is no longer correct. Since generic sources will not be uniformly distributed in x^{10}, the resulting metric will not be uniform either. This effect damps away when $r > R_{10}(r)$ (the space is then long in the r-direction and narrow in the x^{10}-direction). Finally, when $r < M^{-1}$ we would not expect any classical metric to be a good description. Collecting together these results gives [14), 15), 3)]

$N^{1/3} M^{-1} < r$	no geometry
$N^{1/7} M^{-1} < r < N^{1/3} M^{-1}$	IIA string in D0-brane metric
$N^{1/9} M^{-1} < r < N^{1/7} M^{-1}$	M-theory in lifted D0-brane metric
$M^{-1} < r < N^{1/9} M^{-1}$	M-theory in inhomogeneous x^{10}
$r < M^{-1}$	no geometry .

We have been treating the sources as pointlike. This is true, for example, for the ground state, which is a single graviton with N units of compact momentum. [4)] For more distributed sources, the geometry below a given r would be smoothed out.

SUMMARY: The matrix Hamiltonian (2·4) describes M-theory in the sector with N units of compact momentum, where the compactified dimension is asymptotically lightlike. That is, it describes all states whose metric is asymptotically of the form (4·2).

§5. The Itzhaki–Maldacena–Sonnenschein–Yankielowicz (IMSY) Limit

Now let us think about the matrix Hamiltonian (2·4) directly. If we rescale to new canonical variables

$$P \to M^{1/2}\tilde{P} \ , \quad X \to M^{-1/2}\tilde{X} \ , \tag{5·1}$$

then the kinetic term is canonically normalized and the $\tilde{X}^4$ interaction has a coefficient M^3. The latter is the correct dimension for a Yang–Mills coupling-squared in $0+1$ dimensions. The effective dimensionless coupling at an energy scale E is then

$$g^2_{\rm eff} = NM^3/E^3 \ , \tag{5·2}$$

the N being from the usual 't Hooft counting.[16] At $E > N^{1/3}M$, the effective coupling is small and quantum mechanical perturbation theory is good. When $E < N^{1/3}M$, perturbation theory breaks down — we have a strongly coupled problem. In the past this might have been a difficulty, but now that we are in the Age of Duality it presents an opportunity, to find a weakly coupled description. Indeed, this is what IMSY do.[3] By essentially the DKPS plus BGLHKS arguments, they show that the low energy physics of the matrix Hamiltonian is given by IIA/M-theory in the supergravity background (4·2).

In making the connection, there are two points that should be clarified. First, the IMSY limit sounds different, in that it holds fixed[3]

$$U = \frac{r}{l_{\rm s}^2} \ , \quad g^2_{\rm YM} = \frac{g_{\rm s}}{l_{\rm s}^3} \ , \quad H - p_{10} = H - \frac{N}{g_{\rm s} l_{\rm s}} \ . \tag{5·3}$$

However the same dimensionless quantities,

$$Mr = U/g^{2/3}_{\rm YM} \ , \quad \mathcal{H}/M = (H - p_{10})/g^{2/3}_{\rm YM} \tag{5·4}$$

are held fixed in the two limits.*)

Second, the IMSY limit is a statement about the physics as a function of energy scale. To relate this to the previous discussion, consider a standard observable, the expectation value of a product of local operators with some characteristic external energy E. If the reader is more comfortable in Euclidean space, he/she is free to analytically continue, since the quantum mechanics and the supergravity have a common global time (which in the supergravity is defined by the asymptotic translation invariance). A local operator in the QM is equivalent to some local disturbance in the supergravity. The only coordinate invariant local objects live at the boundary (coordinate transformations at the boundary are frozen by the specification of the asymptotic behavior), so by logic similar to the AdS/CFT case[17] we are led to the conclusion that these correspond to deformations of the boundary conditions. In this sense the QM 'lives' at the boundary at infinity, but it should be emphasized that it contains *all* states of the bulk theory.

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*) Note that the IMSY $g^2_{\rm YM}$ differs from the value $M^3 = g_{\rm s}^{-1} l_{\rm s}^3$ given above because it is a dimensional quantity and the Hamiltonian has been rescaled.

These boundary deformations do not change the D0-brane charge, and so correspond to boundary conditions that are invariant under translations in the periodic dimension: they depend on t' and x'^{10} only in the combination $t' + x'^{10}$, of the form $\exp iE'(t' + x'^{10})$. Acting on these the wave operator reduces to

$$-\frac{N}{M^9 r^7 R'^2_{10}}\partial^2_{t'} + \partial^2_\perp = -\frac{NE^2}{M^7 r^7} + \partial^2_\perp \ . \tag{5·5}$$

Note that the arbitrary R'_{10} drops out after converting the SSS energy E' to the matrix QM energy E via eq. (3·2). From the relative scaling of the two terms in the wave operator it follows that the supergravity amplitude is dominated by the radial scale [18)]

$$r \approx N^{1/5} E^{2/5} M^{-7/5} \ \leftrightarrow \ E \approx N^{-1/2} r^{5/2} M^{7/2} \ . \tag{5·6}$$

This is the holographic energy-radius relation. [19)] In particular, the energy $E = N^{1/3}M$ corresponds to $r = N^{1/3}M^{-1}$. At higher energies/larger radii the geometric picture breaks down but matrix QM perturbation theory is valid. At lower energies/smaller radii, QM perturbation theory breaks down but the supergravity description is valid. At successively lower energies the physics is described by the successive supergravity pictures of the previous section. It is not clear how to understand $E < N^{-1/2}M$, where the supergravity breaks down at short distance, but we will argue in the next section that things become simple again at $E \sim N^{-1}M$. Note that the only threshold visible in the 't Hooft (spherical) limit is $E \approx N^{1/3}M$ corresponding to $r \approx N^{1/3}M^{-1}$.

Summary: The matrix Hamiltonian (2·4) is equivalent to M-theory in the sector with N units of compact momentum, where the compactified dimension is asymptotically lightlike. In the preceding discussion we were using this to define M-theory, but we can also use it in the other direction: low energy supergravity is the effective theory for the strongly coupled QM at $E < N^{1/3}M$.

§6. Discussion

Let us now consider the implications of the preceding results. One thing that I find particularly striking is the BGLHKS picture. We are conditioned to think of the $g_s \to 0$ limit as being strictly 10-dimensional. Now we see that in a sector of large but fixed D0-brane charge N, a finite bubble of 11-dimensional spacetime survives at $N^{1/9}M^{-1} < r < N^{1/7}M^{-1}$. This grows arbitrarily large at large N. Thus, *the large-N matrix QM describes 11-dimensional bulk physics.* The derivation appear to be quite reliable. On the QM side we have string theory with the coupling and the energy (in string units, above the BPS minimum) going to zero; we surely understand string theory in this regime. On the M-theory side, the geometry is quite smooth as $g_s \to 0$ in the indicated range of r. This is completely opposite to the mindset expressed in the introduction and figure 1, in that the nature of the 11-dimensional theory has been deduced from string perturbation theory. This is shown schematically in figure 3: the N-axis, rather than being perpendicular to the old parameter space, actually runs from the IIA vacuum toward the M-theory

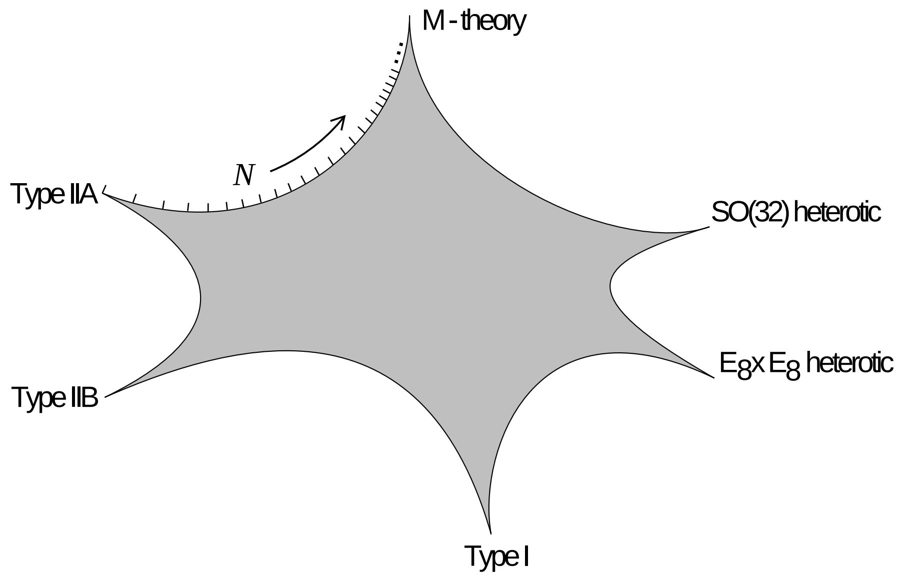

Fig. 3. Space of string vacua. At $g_s = 0$ but large N the IIA string has 11-dimensional physics.

vacuum. Of course the key input is IIA/M-theory duality — this turns out to have a great deal more information than was initially evident.

The IMSY result, or Maldacena duality for D0-branes, also appears to be a great deal of information from very little. It tells us about the behavior of a quantum mechanical system at strong coupling. Again, the key input is IIA/M-theory duality — with little additional information this implies the IMSY result.

Now consider the BFSS matrix theory. [1] The statement above, that the large-N matrix QM describes 11-dimensional physics, would appear to be morally the same thing. However, the BFSS theory is more specific.[*] Given the above discussion, the natural way to define the 11-dimensional S-matrix would be to begin with N source D0-branes, to create the 11-dimensional bubble, and then to consider scattering of supergravity fields corresponding to local operators in the QM, aimed to intersect in the bubble. This is parallel to recent constructions in the AdS_5 case, with D3-branes replaced by D0-branes. [20], [21]

The BFSS proposal is different, and more direct. Namely, scattering of matrix theory bound states is supposed to go directly over to the M-theory S-matrix in the large-N limit. The SSS picture suggests a derivation of this. [12] Consider gravitons (bound states) with D0-brane charges N_1, N_2, and so on. Scale the N_i to infinity together, and take R'_{10} to infinity at the same rate. The momenta $p'_{10,i} = N_i/R'_{10}$ are then constant. The size of the box is going to infinity, in a frame in which the scattering event is held fixed. One might then expect that the limit would give the noncompact theory.

For a limit of spacelike compactifications this would be expected, but lightlike

[*] I would like to thank various members of the IAS seminar audience, particularly T. Banks, M. Douglas, H. Verlinde, and E. Witten, for making this point to me.

compactification is surely more subtle, because the invariant radius of the compact dimension remains at zero. Indeed, the null compactification leads to specific effects that might complicate the limit. The first is zero tension strings, from membranes wrapping the periodic dimension.[22] The second is divergent fluctuations of modes with compact momentum zero. These arise because the effective 10-dimensional coupling in field theory would be g_{11}^2/R_{10}, which diverges in the lightlike limit.[*]

In some field theory examples these effects dominate and the large-N limit does not give the noncompact theory. However, in the present case the BGLHKS effect seems to do away with potential problems. Note that the relevant distance scale p^{-1} and relevant energy $E \sim p^2/N_i$ are both quite small, where the graviton transverse momentum p is held fixed as $N \to \infty$. (These do not satisfy the holographic relation (5·6) because they correspond to a different quantity in the QM theory.) At this distance the periodic dimension is very far from lightlike due to the gravitational effect of the D0-branes. Thus the wrapped membrane tension is large compared to the scale of the scattering, and so is the 10-dimensional Planck scale.

One might still have the following concern. At very large r (the 't Hooft scale) the spacetime interpretation does break down as we have argued earlier, due to a string tension that vanishes asymptotically and also to a 10-dimensional coupling that diverges asymptotically. The gravitons come in from infinite r, and so begin and end in this 'non-spacetime' region. How, then, can we get the physics of flat 11-dimensional spacetime? The point is that at these asymptotic distances the gravitons are propagating freely, at energies small compared to the characteristic energies of the QM. All that matters then is the metric on moduli space, and this is simply flat as a consequence of supersymmetry — it does not matter that at the 't Hooft radius we stop interpreting it as the moduli space of particles in spacetime and begin interpreting as the moduli space of the quantum mechanics.[**]

Thus the potential objections to the Seiberg–Sen argument seem to be red herrings in this case. The BFSS matrix theory, like the IMSY duality, can indeed be deduced from perturbative string theory and IIA/M-theory duality. For the zero modes we can say that they cure their own problem, in that the BGLHKS effect is the expectation value of the gravitational zero mode.

Let me conclude with some comments about the relation between perturbative matrix theory amplitudes and supergravity or M-theory amplitudes. From the literature one might get the impression that these are expected to agree in general: that if an amplitude has the same dependence on the parameters in the two theories (M, N, r, and v) then its form and coefficient must be the same. I can confess to sloppy thinking in this regard in the past. But in fact there is no general reason that there should be agreement. The ranges of validity are entirely different, $E/M < N^{-1}$ vs $E/M > N^{1/3}$, and as in any effective field theory there is no general reason to expect amplitudes at different scales to have any simple relation. For example, the exact amplitude could involve a function of E^3/M^3N which asymptotes to one value at

[*] Zero mode divergences in light-cone field theory are a long story. For discussions in the context of the lightlike limit see refs.[23),24)].

[**] Again I would like to thank the IAS seminar audience for straightening me out.

high energy and to a different one at low energy. Of course some amplitudes are protected by supersymmetry nonrenormalization theorems and so must agree: if they did not it would falsify matrix theory and possibly IIA/M-theory duality as well. It is still interesting to compare calculations in the two limits, both as a way to infer new supersymmetric nonrenormalization theorems (since supersymmetry appears to be a strong restriction) and also because this is the route by which new dualities, such as the Maldacena duality, are sometimes discovered.

§7. What Next?

The conclusion is that the large-N matrix QM gives the M-theory limit of string theory. It is one of many nonperturbative constructions of string theory, since we can take a similar large-N limit in any of the Maldacena dualities. Certainly these constructions are not yet in simplest form: string theory is constructed in states with specific, and unusual, boundary conditions. There should be some general framework describing all states of string theory — spacetimes without boundaries are especially puzzling from the present viewpoint. This will likely involve some new ideas, and look rather different from the current descriptions.

Still, one way to try to proceed is to develop more fully the existing constructions. Much attention goes to the $AdS_5 \times S^5$ and $AdS_3 \times S^3$ cases because of their large symmetries, but the matrix QM/D0-brane case would seem to have advantages as well, due to its smaller number of degrees of freedom.

As an example, it would be nice to have a direct test of the Maldacena duality, comparing two calculations at *the same point* in parameter space without the aid of nonrenormalization results. The simplest place to look is the D0-brane QM, just below the 't Hooft energy. The duality predicts that the gauge theory entropy is given by the nonextremal D0-brane entropy as

$$S = cN^2 T^{9/5} (M^3 N)^{-3/5} \tag{7·1}$$

with c a known constant.[25] The exponent 9/5 indicates the nontriviality of the dynamics. As the temperature is reduced the effect of the commutator potential is to freeze out degrees of freedom in a gradual way, but the precise exponent appears to be difficult to get by analytic means. However, this entropy should be accessible by Monte Carlo (I should emphasize that this is very different from the index-like path integrals that have been calculated, in that the thermal boundary conditions break supersymmetry.). In field theory we would need to renormalize to maintain the supersymmetry in the cutoff theory, but QM is superrenormalizable. Also, at this relatively high temperature precise supersymmetric cancellations are not so relevant, and high temperature methods may also simplify the fermionic determinant. There is unlikely to be enough dynamic range at reasonable values of N to pick out the 9/5 accurately, but for the Monte Carlo calculation to come close to the value of c obtained in a black hole calculation would be very striking.

Of course the IKKT matrix theory[26] should be even simpler than the BFSS model from this point of view, and it has been studied numerically. The question here is one of interpretation. The assumption in much of the literature appears to be that the IKKT model is a background-independent formulation of string theory, but an interpretation parallel to IMSY [3] seems more likely. The portion of the IKKT matrix integral where the eigenvalues are well-separated can be calculated perturbatively, but this breaks down when the eigenvalues become close. The natural extension of the IMSY analysis would be that there is an effective description of the latter part of the matrix integral in terms of supergravity in an asymptotically D-instanton space. For a numerical test one would need to find a simple prediction parallel to the entropy (7·1).

Another advantage of being in QM is that renormalization is not needed, so we can add arbitrary perturbations to the Hamiltonian and so study more general spaces.

A related advantage is that the wavefunction and operators have a direct meaning. Consider for example the size R of the D0-brane bound state.[*] In the BFSS paper [1], two estimates are given,

$$R \sim N^{1/3} M^{-1} \tag{7·2}$$

from 't Hooft scaling and

$$R \sim N^{1/9} M^{-1} \tag{7·3}$$

from the quantum mechanics of the v^4/r^7 potential. The relation between these was not resolved (though at least one of the authors [B] came to the conclusion that we will reach below). It was suggested that the 't Hooft result might be invalidated by IR divergences, but in fact one can show that it is rigorously correct. Start with the uncertainty principle, applied to one matrix element X^1_{11}:

$$\langle 0|(X^1_{11})^2|0\rangle\langle 0|(P^1_{11})^2|0\rangle \geq 1 \ . \tag{7·4}$$

Applying this to all $9N^2$ elements gives[**]

$$\langle 0|\mathrm{Tr}(X^1X^1)|0\rangle\langle 0|\mathrm{Tr}(P^nP^n)|0\rangle \geq 9N^4 \ . \tag{7·5}$$

Now apply the virial theorem, relating the expectation value of the kinetic term to that of the potential. From $\langle 0|[X^nP^n, H]|0\rangle = 0$ one obtains for the expectation values $-2K + 4P + F = 0$, where K is the kinetic term, P is the potential, and F is the fermionic term (linear in X). On the other hand $K + P + F = 0$ by supersymmetry. Thus $K : P : F = 1 : 1 : -2$, and so

$$\langle 0|\mathrm{Tr}(P^nP^n)|0\rangle = \frac{M^6}{8\pi^2}\sum_{m,n}|\langle 0|\mathrm{Tr}([X^m, X^n]^2)|0\rangle| \ . \tag{7·6}$$

and

$$\langle 0|\mathrm{Tr}(X^1X^1)|0\rangle\sum_{m,n}|\langle 0|\mathrm{Tr}([X^m, X^n]^2)|0\rangle| \geq \frac{72N^4\pi^2}{M^6} \ . \tag{7·7}$$

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[*] For other discussions see ref. [27] and in particular the second of refs. [20].

[**] The $U(1)$ piece requires separate treatment. This is irrelevant at large N, but in fact the bound (7·5) holds for finite N also.

For Hermitean matrices the Schwarz inequality gives $|\mathrm{Tr}(ABAB)| < \mathrm{Tr}(AABB) < \frac{1}{2}[\mathrm{Tr}(A^4) + \mathrm{Tr}(B^4)]$, and so

$$\langle 0|\mathrm{Tr}(X^1X^1)|0\rangle\langle 0|\mathrm{Tr}(X^1X^1X^1X^1)|0\rangle \geq \frac{N^4\pi^2}{2M^6} \ . \tag{7·8}$$

For X^1 having an eigenvalue distribution of width R, the left-hand side is of order N^2R^6 and so

$$R \geq O(N^{1/3}M^{-1}) \ . \tag{7·9}$$

This assumes that the expectation value of $(X^1)^4$ converges, as is indeed the case. This result, obtained from a Hamiltonian Born-Oppenheimer approximation in refs. [28], can be nicely explained by an effective Hamiltonian analysis (this is adapted from an unpublished argument of L. Susskind, with comments from G. M. Graf). The dangerous direction is when the wavefunction separates into two blocks, along the flat directions. Here the relative coordinate is described by a free $U(1)$ theory, so the Hamiltonian is just the Laplacian and the wavefunction is harmonic. For $l = 0$ the wavefunction falls as r^{-7}. The bound state has $l = 2$, because the relative spin plus motion gives an invariant, and so falls as r^{-9}. Then

$$\langle 0|\mathrm{Tr}[(X^1)^L]|0\rangle \sim \int d^9X\,|X|^{-18+L} \tag{7·10}$$

converges for $L < 9$.

Note that (7·9) is exactly the maximum radius in which the supergravity picture is valid. I therefore strongly expect that $N^{1/3}M^{-1}$ is the actual value, not just a lower bound.

How does this relate to the smaller estimate (7·3)? The bound (7·9) comes primarily from the quantum fluctuations of the off-diagonal modes of X^n, while the smaller estimate refers to the diagonal motion on the moduli space. For the purpose of the BFSS theory, one is interested only in very small energies. The off-diagonal modes have much higher excitation energies and so are frozen into their ground states; the associated zero-point fluctuations are then not observable. Roughly speaking,

$$\mathrm{Tr}(X^nX^n) = \mathrm{Tr}(X^nX^n)_{\mathrm{off-diag./high\ energy}} + \mathrm{Tr}(X^nX^n)_{\mathrm{diag./low\ energy}} \ . \tag{7·11}$$

The off-diagonal part is much larger but is state-independent at low energy. In other words, $\mathrm{Tr}(X^nX^n)$ in the low energy theory differs from the full $\mathrm{Tr}(X^nX^n)$ by an additive renormalization. It seems hard to give a precise definition of the low energy part in terms of the underlying QM variables.

In conclusion, to find a description of the flat 11-dimensional M-theory (or the 10-dimensional IIB theory) we somehow need, in the large-N limit, to identify a subtheory contained in the intersection of all the field theory constructions. To find a description of the general background we need something which contains the union of all the field theory constructions.

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