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Statistical mechanics of D0-branes and black hole thermodynamics

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Abstract

We consider a system of D0-branes in toroidally compactified space with interactions described by a Born-Infeld-type generalisation of the leading $v^2 + v^4/r^{D-4}$ terms (D is the number of non-compact directions in M-theory, including the longitudinal one). This non-linear action can be interpreted as an all-loop large N super Yang-Mills effective action and has a remarkable scaling property. We first study the classical dynamics of a brane probe in the field of a central brane source and observe the interesting difference between the $D = 5$ and $D > 5$ cases: for $D > 5$ the center acts as a completely absorbing black hole of effective size proportional to a power of the probe energy, while for $D = 5$ there is no absorption for any impact parameter. A similar dependence on D is found in the behaviour of the Boltzmann partition function Z of an ensemble of D0-branes. For $D = 5$ (i.e. for compactification on 6-torus) Z is convergent at short distances and is analogous to the ideal gas one. For $D > 5$ the system has short-distance instability. For sufficiently low temperature Z is shown to describe the thermodynamics of a Schwarzschild black hole in $D > 5$ dimensions, supporting recent discussions of black holes in Matrix theory.

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1. Introduction

In this paper we shall explore some aspects of the classical mechanics and classical statistical mechanics of an ensemble of D0-branes in toroidally compactified string theory (with $d = D-1 = 10-p$ non-compact dimensions), or, equivalently, a system of Dp-branes wrapped over the dual torus. Our study is motivated, in particular, by recent works on black holes in Matrix theory [1,2,3,4,5,6].

D-branes are BPS objects: if N parallel branes of the same type are located at some points in space the interaction potential between them vanishes [7]. Given some kinetic energy, the branes are attracted to each other [8,9]. One may thus try to model a non-extremal D-brane with *large* charge N by a non-relativistic gas of interacting constituent unit-charge D-branes. In particular, one may try to reproduce the entropy of the corresponding near-extremal Dp-brane as that of a gas of classical extremal Dp-branes. This picture is 'dual' to some previous suggestions [10,11] of understanding the near-extremal Dp-brane entropy in terms of quantum gases of massless excitations on brane world volumes. The entropy of classical near-extremal R-R p-branes can be expressed in terms of the super Yang-Mills coupling constant $g_{\rm YM}$ of the world volume theory [2]

$$S \ \sim \ [E^{D-2}\ V_p^{D-6}\ N^{D-4}(g^2_{\rm YM})^{8-D}]^{\frac{1}{2(D-4)}} \ , \qquad D = 11-p \ , \tag{1.1}$$

suggesting a Yang-Mills interpretation [10,11,1,2].

One may expect that the two (massive D0-brane gas and massless YM gas) descriptions may be useful in the two different regions. Normally, a massless gas will have a much higher entropy than a non-relativistic gas of the same energy. Thus when the deviation E of the energy from the extremal value is large (but still small compared to the total energy of D-branes), the relevant excitations should be mostly the massless modes, i.e. the massless gas picture should be applicable. At the very low energy the massless degrees of freedom in compact dimensions can no longer be excited, so that the description in terms of a non-relativistic gas of D0-branes should be the adequate one.

Reinterpreted in the Matrix theory language (i.e. applied to a Schwarzschild black hole in one higher dimension $D = d+1$ with a compact light-like direction) the description in terms of a gas of D0-branes was recently discussed in [3,4,5]. In the systematic picture developed in [5] the D0-brane gas and the SYM gas on the dual torus should be equivalent provided one takes into account the effects due to off-diagonal gluons as short strings connecting D0-branes. As a result, D0-branes in the gas should actually be coupled to a an effective background which makes them distinguishable [5].

One may still hope to be able to describe some general features of this unusual gas by using mean field approximation and ignoring details of the background. In [5] it was suggested to view the black hole as a Boltzmann gas of distinguishable D0-branes with two-body interactions given by the one-loop 1+0 SYM effective action [9,12]

$$L_{\rm eff} = \sum_i \frac{v_i^2}{2R} + \frac{cG_D}{R^3} \sum_{i,j} \frac{v_{ij}^4}{r_{ij}^n} \ , \tag{1.2}$$

where $n = D - 4 = d - 3 = 7 - p$. Applying qualitative considerations based on the virial theorem, i.e. setting the first and the second term in (1.2) to be of the same order and assuming that $v \sim p/m \sim R/R_s$ (where R is the radius of the light-like direction and R_s is the size of D0-brane system), one obtains [3,5] the Schwarzschild black hole expressions for the mass and the entropy of the system.

Our point of departure is the following observation: at the virial point where $\frac{G_D}{R^2}\frac{v^2}{r^{D-4}} \sim 1$ the higher-order $\frac{v^{2L+2}}{r^{nL}}$ terms in the interaction potential are all of the same order and thus cannot be ignored a priori. Our basic assumptions will be: (i) large number N of constituent branes and (ii) small and slowly changing velocities (negligible accelerations). We shall also assume, in the spirit of mean field approximation, that under the conditions (i) and (ii) the forces of all other $N-1$ branes acting on a given one can be approximated by a force exerted on a unit charge D-brane 'probe' by a static D-brane 'source' of charge $N - 1 \approx N$ placed at the center of the system. In our crude model the effect of the background will be included only in that the partition function of the classical D0-brane gas will not contain the usual $1/N!$ factor, reflecting the fact that constituent D0-branes should be treated as distinguishable.[1]

There exists a natural candidate for the all-order generalisation of the $v^2 + \frac{v^4}{r^n}$ action [14,15,16,17,18]. Consider the classical Born-Infeld action for a D0-brane probe moving in a supergravity background produced by a D0-brane source in T^p compactified space (or T-dual configuration of a wrapped Dp-brane probe in Dp-brane source background on $\tilde{T}^p$)

$$I = -m \int dt \ H_0^{-1} \big[\sqrt{1 - H_0 v^2} - 1 \big] \ , \tag{1.3}$$

$$H_0 = 1{+}K \ , \qquad K = \frac{Q}{r^n} \ , \quad n = 7{-}p = D{-}4 \ , \quad r = |x_i| \ , \quad v_i = \dot{x}_i \ , \ \ i = 1, ..., D{-}2 \ , \tag{1.4}$$

[1] A suggestion that quantum extreme black holes described by D-branes in the large N limit should obey Boltzmann infinite statistics was made in [13].

$$m = \frac{1}{g_s\sqrt{\alpha'}} = R^{-1} \ , \quad Q = N g_s \frac{(2\pi\sqrt{\alpha'})^7}{V_p \omega_{6-p}} \sim N g^2_{\rm YM} \sim N G_D R^{-2} \ , \tag{1.5}$$

where V_p is the volume of the torus T^p and $g^2_{\rm YM} = (2\pi)^{p-2}(\sqrt{\alpha'})^{p-3}\tilde{g}_s$ is the YM coupling on the dual torus ($\tilde{g}_s = g_s(2\pi\sqrt{\alpha'})^p V_p^{-1}$). The supergravity action (1.3) is supposed to apply only at *large* distances. If we take the charge N to be large (for fixed r) then we can ignore the asymptotic value 1 in the harmonic function, replacing H_0 by K, i.e. replacing (1.3) by[2]

$$S = \int dt \ L \ , \quad L = -mK^{-1}\big[\sqrt{1-Kv^2} - 1\big] = \frac{mv^2}{2} + \frac{mQv^4}{8r^n} + \frac{mQ^2v^6}{16r^{2n}} + \dots \ . \tag{1.6}$$

This action can be interpreted as the supergravity action (1.3) formally extrapolated to the short-distance (or near-horizon $r \approx 0$) region. It appears that this action captures some important information about D0-brane dynamics at short distances.[3]

It is *this* action that we shall use below, assuming that for *large* N it is the one that describes interactions of slowly moving D0-branes both at *short* and *large* distances. Indeed, (1.6) can be interpreted [15,16] as a sum of all gauge-theory (light open string) loop corrections to the low-energy interaction potential between a pair of moving Dp-branes, i.e. as the large N quantum effective action of $p+1$ dimensional super Yang-Mills theory with the coupling $g_{\rm YM}$. This identification was explicitly checked at the 2-loop SYM level in [21,14]. The miracle of maximal supersymmetry that relates the v^4 [9] and v^6 [14] terms in the long-distance (classical supergravity) and short distance (quantum SYM) string theory expression for the D-brane interaction potential is conjectured [15,16] to continue (for large N) to all orders in v.

Like its $v^2 + \frac{v^4}{r^n}$ truncation, the full action (1.6) has a remarkable ('holographic' [22,9,12]) scaling property: the dependence on the parameters m and Q ($\sim N$) can be

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[2] The same action can be obtained by a reduction from 11 dimensions along the 'null' [19] direction [14,15] or by taking the low-energy (YM theory) limit [16] $\alpha' \to 0$, $g^2_{\rm YM} \sim \tilde{g}_s(\sqrt{\alpha'})^{p-3}$ =fixed, $X_i = x_i/\alpha'$ =fixed.

[3] In general, the quantum SYM effective action (and its large N Born-Infeld-type part) contains also fermionic contributions leading to spin-dependent terms in the interaction potential [20,4]. We shall not include such terms in our model which is based on mean field approximation. We consider a field of a large number of branes acting on a on a single brane in the gas so it is natural to treat interactions as spherically symmetric (spin effects should average to zero).

completely absorbed into redefinitions $x_i \to \lambda_1 x_i,\ t \to \lambda_2 t$. The resulting parameter-independent action is then

$$S = -\int d\mathrm{t}\ \mathrm{x}^n\ [\sqrt{1 - \frac{\dot{\mathrm{x}}^2}{\mathrm{x}^n}} - 1]\ , \tag{1.7}$$

$$\mathrm{x}_i = m^{\frac{2}{n+2}} Q^{-\frac{1}{n+2}}\ x_i\ , \qquad \mathrm{t} = m^{\frac{n-2}{n+2}} Q^{-\frac{2}{n+2}}\ t\ , \qquad \dot{\mathrm{x}}_i = \frac{d\mathrm{x}_i}{d\mathrm{t}}\ . \tag{1.8}$$

The interaction terms in the Lagrangian in (1.6) are *attractive*, but the corresponding Hamiltonian is still positive

$$\mathrm{H} = p_i v_i - L = mK^{-1}[\sqrt{1 + K\frac{p^2}{m^2}} - 1]\ , \tag{1.9}$$

though this is not manifest if one truncates its expansion in powers of momenta,

$$\mathrm{H} = \frac{p^2}{2m} - \frac{Qp^4}{4m^3 r^n} + \frac{Q^2 p^6}{8m^5 r^{2n}} + \dots\ , \qquad p_i = \frac{mv_i}{\sqrt{1 - Kv^2}}\ . \tag{1.10}$$

While the square root structure of (1.6) and (1.9) is reminiscent of the relativistic particle expressions, it should be stressed again that (1.6) applies only for sufficiently small and slowly changing velocities (note that depending on r, the corresponding momenta may, however, be large).

Our aim will be to study some consequences of the action (1.6) and apply it to a description of a statistical ensemble of D0-branes in order to clarify the relation to black hole thermodynamics. The non-linear Born-Infeld-type actions are known to have non-trivial classical dynamics with novel short-distance properties and the present case will not be an exception. Indeed, the structure of (1.6),(1.7) suggest the presence of a dynamical bound $\frac{Qv^2}{r^n} \leq 1$, which implies, in particular, that v must be small at small distances, in agreement with the non-relativistic nature of (1.6). While at large distances (1.6) is well-approximated by the $v^2 + \frac{v^4}{r^n}$ terms, in the short distance region (and the region of characteristic distances and velocities satisfying $\frac{Qv^2}{r^n} \sim 1$ or $p \gg 1$) its predictions will be very different from what would follow simply from the $v^2 + \frac{v^4}{r^n}$ action.

We shall first consider the classical motion of a 0-brane probe in the field of a D0-brane source described by the action (1.6) (Section 2). We shall find that the behaviour of the classical trajectories has surprising dependence on the number of compactified dimensions, i.e. on the power n in K: the case of $n = 1$ (corresponding to D0-branes on T^6 or to wrapped 6-branes on $\tilde{T}^6$, i.e. to the near-extremal black hole in $d = 4$ or the Schwarschild black hole in $D = 5$) is quite different from $n > 1$ ($D > 5$) cases.[4] For $n = 1$ the probe with a non-zero angular momentum J will always scatter away from the center, while for $n > 1$ it will always fall at the center if it has a sufficiently low J (but it takes an infinite time for it to reach the core $r = 0$). For larger J (or larger impact parameters) the particle scatters away from the core.[5] This suggest a 'black hole' interpretation of the attractive center and implies the existence of a characteristic short-distance scale (the critical value of the impact parameter) $b \sim (\frac{EQ}{m})^{\frac{1}{D-4}}$. A peculiar feature of the action (1.6) (related to its $D = 11$ fixed p_- graviton eikonal scattering interpretation [14]) is that this 'horizon radius' depends on the energy E of the incoming particle.[6] This translates into the temperature dependence in the statistical mechanics context and is crucial for establishing the relation to black hole thermodynamics.

In Section 3 we shall consider the classical statistical mechanics of an ensemble of a large number N of point particles whose interactions can be effectively described by a 'mean field' Hamiltonian (1.9). Equivalently, the Boltzmann partition function we shall compute can be interpreted as describing a gas of non-interacting classical D0 branes in the external field produced by a central extremal D0-brane source (with the 'bootstrap' condition that the charge of the source is chosen to be equal to the number of particles N in the surrounding cloud).

Since the interaction potential in (1.6),(1.9) vanishes at large distances, the large-volume behaviour of the partition function Z is the same as for an ideal gas. Remarkably, for $n = 1$ ($d = 4$ or $D = 5$) Z is completely finite (in spite of an apparent singularity of the interaction part of H at $r \to 0$). In higher dimensions $n > 1$ ($d > 4$ or $D > 5$)

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[4] The special nature of the $n = 1$ (and $n = 2$) case is suggested by the fact that in the case of the radial motion $v^2 = \dot{\mathrm{x}}^2$, (1.7) can be put into the form $S = -\int dt\ \mu(\rho)[\sqrt{1-\dot{\rho}^2} - 1]$, where $\mu = m\rho^{-\frac{2n}{n-2}}$, $\rho = \frac{2}{n-2}\mathrm{x}^{-\frac{1}{2}(n-2)}$ for $n > 2$ while for $n = 1$ ($p = 6$) $\mu = \frac{1}{2}m\rho^2$, $\rho = \sqrt{2\mathrm{x}}$ and for $n = 2$ ($p = 5$) $\mu = me^{2\rho}$, $\rho = \ln \mathrm{x}$.

[5] The classical scattering we discuss can be viewed as a WKB approximation to scattering of a quantum D0-brane on an extremal D0-brane black hole in $D = 10$, or as a scattering of a graviton (described by a curved space Laplace operator) on a pp-wave (or infinitely boosted Schwarzschild) background in $D = 11$. Similar dimension-dependent feature ($p = 6$, $p = 5$ and $p < 4$ as special cases) of the scattering of classical minimally coupled scalar fields on Dp-brane backgrounds which is implied by interplay between the attractive 'Newtonian' $1/r^{7-p}$ and repulsive centrifugal $1/r^2$ potential was pointed out in [23].

[6] It actually looks as if the incoming particle acts as a black hole with mass related to its kinetic energy.

the partition function is divergent at $r = 0$. Surprisingly, this is analogous to a similar dimension dependence observed in the properties of the classical dynamics in Section 2.[7] The $n = 1$ case does not admit a black hole interpretation, i.e. there is no phase transition and the system looks like an ideal gas even for small temperatures. We shall, therefore, concentrate on the $n > 1$ case and suggest that the instability at small distances is related to the existence of the critical absorption scale b in the classical scattering problem. We shall assume that the region of sufficiently low temperatures when the thermal length scale $\lambda \sim \beta^{-1/2}$ is of order of b is described by the partition function where the integral over x is computed with a lower (short-distance) cutoff of order b. The upper (volume) cutoff (taken to be $\sim b + \lambda$) is then also of order b.

As a result, the partition function becomes a simple product of powers of β and N and it can be interpreted as that of a free gas of particles in a $(D-2)$-dimensional *transverse* space (or 'horizon sphere') of size $R_s = b \sim (N/\beta)^{\frac{1}{D-4}}$. Treating the particles as distinguishable ones as suggested in [5], we will show in Section 4, that the corresponding entropy is proportional to N. Interpreting the energy of the system as the light-cone energy [1] we obtain the same entropy-mass relation as for a Schwarzschild black hole in D dimensions, supporting the previous arguments [3,5].

2. Classical dynamics

To gain some intuition about dynamical properties of a system of D0-branes which will be useful in the statistical mechanics context we shall first discuss the classical dynamics governed by the action (1.6). Since this action describes a single particle with mass m moving in a velocity-dependent central force potential, the angular momentum vector and the energy of the particle are conserved. As in the standard central force problem, the particle motion will be planar. The Lagrangian for the coordinates in the plane of motion (r, θ) is given by (1.6) with

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \ ,$$

so that the conserved angular momentum and the energy are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{mr^2\dot{\theta}}{\sqrt{1-kv^2}} = J \ , \qquad (2.1)$$

[7] Other arguments indicating that the cases of $D = 5$ and $D = 6$ are special were given in [1,2,5].

$$\mathrm{H} = p_i v_i - L = mK^{-1}(\frac{1}{\sqrt{1-Kv^2}} - 1) = E \quad . \tag{2.2}$$

Solving for v we get

$$v^2 = \frac{2E}{m}\frac{1+\frac{E}{2m}K}{(1+\frac{E}{m}K)^2} \ , \tag{2.3}$$

and thus

$$\dot r^2 = \frac{2E}{m}\frac{1+\frac{EQ}{2mr^n}}{(1+\frac{EQ}{mr^n})^2} - \frac{J^2}{m^2r^2}\frac{1}{(1+\frac{EQ}{mr^n})^2} \ . \tag{2.4}$$

As is clear from (2.2),(2.3), v^2 and thus also $\dot r^2$ are always bound by $K^{-1} = r^n/Q$, i.e. $\dot r^2 < r^n/Q$. The dependence on Q and m in all of the above relations can be eliminated by the scaling transformation (1.8) (which implies also a rescaling of E but not of J).

To analyse the classical trajectories it is useful to interpret (2.4) in terms of a particle moving in an effective velocity-independent central force potential $\mathcal{V}(r)$, rewriting (2.4) as

$$\tfrac{1}{2}m\dot r^2 + \mathcal{V}(r) = E \ , \tag{2.5}$$

$$\mathcal{V} = E\big[1 - \frac{1+\frac{b^n}{2r^n}}{(1+\frac{b^n}{r^n})^2}\big] + \frac{J^2}{2mr^2}\frac{1}{(1+\frac{b^n}{r^n})^2} \ , \tag{2.6}$$

where we have introduced the characteristic scale b,

$$b = (\frac{EQ}{m})^{\frac{1}{n}} \ . \tag{2.7}$$

At large distances $r \gg b$ this effective potential reduced to the sum of a *repulsive* $\frac{1}{r^n}$ potential[8] and the standard centrifugal term,

$$\mathcal{V}_{r\gg b} \to \ \frac{3E^2Q}{2mr^n} + \frac{J^2}{2mr^2} \ . \tag{2.8}$$

Since $\mathcal{V} \to 0$ for $r \to \infty$, the velocity approaches the constant asymptotic value $v_\infty = \sqrt{\frac{E}{2m}}$.

At short distances $r \ll b$ the potential $\mathcal{V}$ reduces to a sum of a *universal* (i.e. E-independent) *attractive* r^n term and the repulsive centrifugal term suppressed by the r^{2n},

$$\mathcal{V}_{r\ll b} \ \to \ E - \frac{m}{2Q}r^n + \frac{J^2}{2mb^{2n}}r^{2(n-1)} \ . \tag{2.9}$$

[8] The intuition that the large-distance potential should be attractive which is based on the sign of the second term in the expansion in (1.6) is misleading since for a velocity-dependent potential one should first fix the energy and then solve for velocity as was done above (cf. also (1.10)).

For $n > 2$ the centrifugal term is subleading at $r \to 0$, while for $n = 2$ both terms are proportional to r^2. That means that for $n \geq 2$ (and sufficiently large J) the potential has a maximum at some finite r, decreasing towards E at $r = 0$ and 0 at $r \to \infty$. For $n = 1$ the centrifugal term does not vanish at $r = 0$ (it reduces to a constant up to higher order r^{3n} term) and the potential has a maximum at $r = 0$.

The details of motion of an incoming particle in the central force depend on relative values of E and J as well as on the value of n. To determine this dependence it is useful to analyse the possible turning points ($\dot{r} = 0$) solving the equation (2.5) or $\mathcal{V}(r) = E$. The latter is equivalent to

$$1 + \frac{b^n}{2r^n} = \frac{\mathrm{b}^2}{r^2} , \qquad \mathrm{b} \equiv \frac{J}{mv_\infty} = \frac{J}{\sqrt{2mE}} , \tag{2.10}$$

where b is the impact parameter. One special solution is $r = 0$. Since for $n > 1$ the derivative of the potential is zero at $r = 0$, the particle that can reach the center stops there. This is not true for $n = 1$ where the force at $r = 0$ is non-zero and the particle with $J = 0$ reaches the center in finite time and bounces back.

For $n = 1$ (2.10) becomes a quadratic equation which always has a (positive) solution for any non-vanishing b. That means that the particle with $J \neq 0$ is always scattered away, i.e. it can reach the center only when $J = 0$.

For $n = 2$ (2.10) reduces to $r^2 = \mathrm{b}^2 - \frac{1}{2}b^2$ so that for large enough angular momentum such that $\mathrm{b}^2 > \frac{1}{2}b^2$ there is a turning point at finite r while for smaller J (for fixed E) corresponding to $\mathrm{b}^2 < \frac{1}{2}b^2$ the incoming particle is eventually absorbed by the center. That means that the effective size of the absorbing center (corresponding to the critical case of $\mathrm{b}^2 = \frac{1}{2}b^2$) is $\mathrm{b}_* = \frac{1}{\sqrt{2}}b = \sqrt{\frac{EQ}{2m}}$.

For $n > 2$ one finds (first differentiating (2.10) to determine that the critical value of r is equal to $(\frac{n}{4}\frac{b^n}{\mathrm{b}^2})^{\frac{1}{n-2}}$) that for

$$\mathrm{b} \ > \ \mathrm{b}_* \equiv (2^{n-6}n^2)^{\frac{1}{2n}} \ b \tag{2.11}$$

there are two solutions $r_1, \ r_2$ of (2.10) (in addition to the $r = 0$ one). The potential $\mathcal{V}$ is equal to E at $r = 0$, then decreases to a local minimum, then has a maximum between r_1 and r_2 and finally decreases to zero at large r (see Figure).

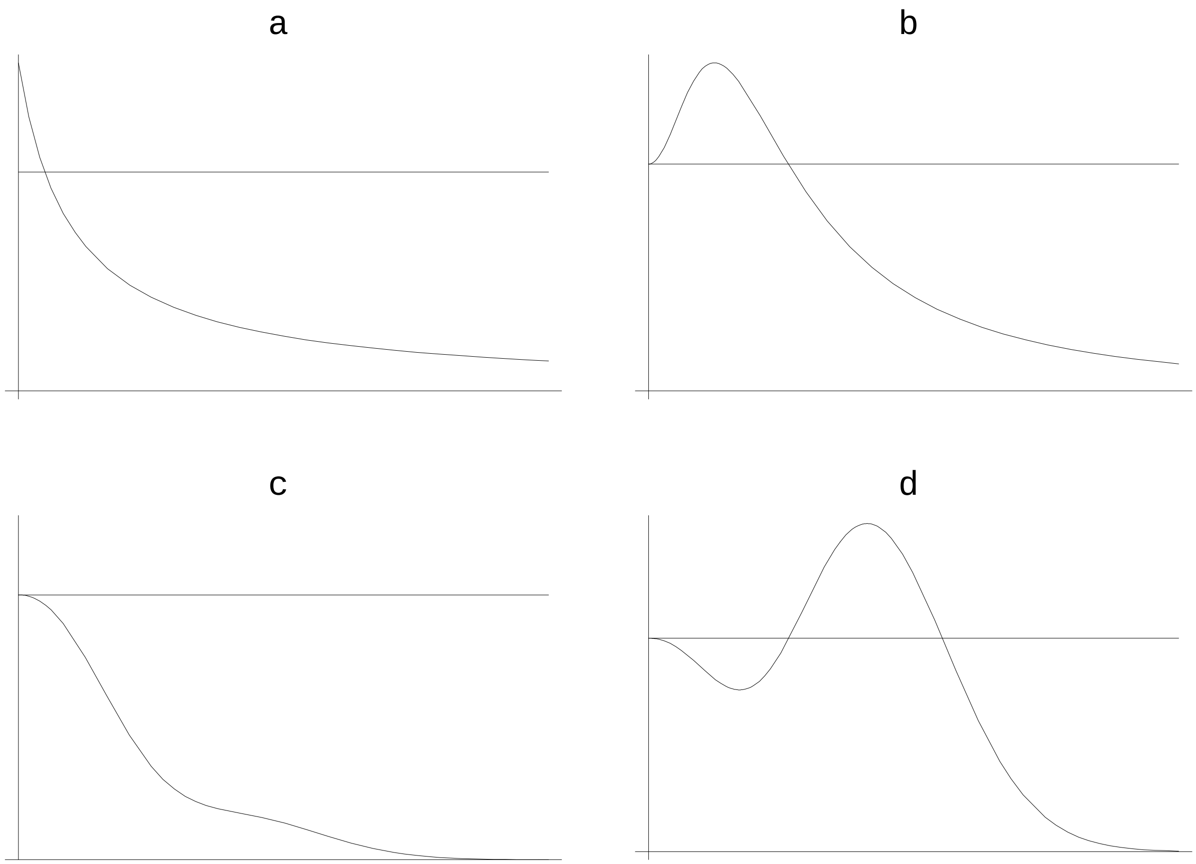

FIGURE: Examples of effective potentials. Plot (a) illustrates the form of the effective potential $\mathcal{V}$ for $n = 1$ and $J \neq 0$, plot (b) for $n = 2$ and $\mathrm{b} > \mathrm{b}_*$, (c) for $n > 1$ and $\mathrm{b} < \mathrm{b}_*$, (d) for $n > 2$ and $\mathrm{b} > \mathrm{b}_*$. The upper horizontal lines on the plots correspond to the energy of the particle E.

The particle approaching the center from infinity will reach the minimal distance r_2 and then scatter away. A particle that at $t = 0$ is put in the region $0 < r < r_1$ with some initial velocity directed out of the center will bounce once against the potential wall at $r = r_1$ and then fall at the center.

For $\mathrm{b} < \mathrm{b}_*$ there are no turning points (the potential has its maximum E at $r = 0$, see Figure) and the incoming particle is always absorbed by the center. The critical case $\mathrm{b} = \mathrm{b}_*$ defines[9] the effective size of the absorbing center (or 'black hole' radius) to be again of order b,

$$\mathrm{b}_* \sim b \sim (\frac{E}{m} Q)^{\frac{1}{n}} \sim (NE)^{\frac{1}{D-4}} \ . \tag{2.12}$$

This characteristic size of the system is consistent with the expectation (based on the square root form of (1.6),(2.2)) that the critical scale should be such that $Q\frac{v^2}{r^n} \sim 1$, or, for

[9] The relation (2.11) is true also for $n = 2$.

$v = v_\infty = \sqrt{\frac{E}{2m}}$, $r \sim b$. This is the same scale which appeared in the discussions [12,3,5] based on virial theorem considerations applied to the leading-order $v^2 + \frac{v^4}{r^n}$ action. It can be interpreted as the radius of a Schwarschild black hole in one higher ($D = d+1$) dimension (see Section 4). Surprisingly, this interpretation does not apply for $n = 1$ ($D = 5$).

3. Statistical mechanics

In general, the Boltzmann partition function of a system of N particles in $D-2 = n+2$ spatial ('transverse') dimensions is

$$Z = \frac{1}{N!} \prod_{s=1}^{N} \int d^{D-2}p^{(s)} \ d^{D-2}x^{(s)} \ \exp\{-\beta H_{tot}(p^{(1)}, x^{(1)}, \cdots, p^{(N)}, x^{(N)})\} \ . \qquad (3.1)$$

As was already mentioned in the Introduction, we shall apply a 'mean field' approximation, i.e. make the assumption that the effect of interactions of each constituent D0-brane with all others can be represented by the effective single-particle Hamiltonian (1.9), i.e.

$$H_{tot}(p^{(1)}, x^{(1)}, \cdots, p^{(N)}, x^{(N)}) = \sum_{s=1}^{N} \mathrm{H}(p^{(s)}, x^{(s)}) \ , \qquad (3.2)$$

$$\mathrm{H}(p, x) = \frac{m}{Q} r^{D-4} \big[\sqrt{1 + \frac{Q}{m^2} \frac{p^2}{r^{D-4}}} - 1 \big] \ , \qquad r = |x| \ . \qquad (3.3)$$

We shall follow also the suggestion of [5] and make the drastic assumption that the constituents of this unusual gas are to be treated as distinguishable so that one should drop the standard $\frac{1}{N!}$ factor in front of Z. As we shall see, this is crucial for reproducing the standard Schwarschild black hole entropy-mass relation.

The partition function then takes the form

$$Z = W^N \ , \qquad W = \int d^{D-2}p \ d^{D-2}x \ \ \exp\{-\beta \mathrm{H}(p, x)\} \ . \qquad (3.4)$$

Notice that Z depends on N explicitly, as well as implicitly via Q (1.5) in H.

The integral over p is convergent (it is actually the same as in the case of a relativistic ideal gas [24] in a general number of dimensions).[10] The integral over x diverges at large

[10] Note that if one would formally keep only the p^2 and p^4 terms in the expansion (1.10) of H in (3.3) the integral over p in (3.4) would be divergent.

r (since the potential part of H goes to zero at large distances) and, as for an ideal gas, is to be cut off at the size of the box

$$r < \Lambda \ \sim \ V^{\frac{1}{D-2}} \ , \qquad V = \omega_{D-3} L^{D-2} \ . \tag{3.5}$$

One finds also that (for any value of β) the integral is *convergent* at $r \to 0$ if $D = 5$ ($n = 1$) but is *divergent* if $D > 5$ ($n > 1$).

This is in striking analogy with the properties of the classical motion described by the action (1.6) or Hamiltonian H (1.9): as we have seen in the previous section, for $n = 1$ the particle always scatters away from the center, while for $n > 1$ there exists a special value of the impact parameter b (2.12) below which the center acts as a completely absorbing 'black hole'. This suggest that for $n > 1$ ($D > 5$) the system is unstable and to describe the 'black-hole phase' one should introduce a short-distance cutoff $r > b$ where b (2.7) plays the role of an intrinsic scale of the system. In the thermodynamical context the energy of the individual particle E in (2.7),(2.12) should be proportional to the temperature, $E \sim \beta^{-1}$, so that

$$b(E) \ \to \ b(\beta) = c_1 (\frac{Q}{m\beta})^{\frac{1}{D-4}} \sim (\frac{N}{\beta})^{\frac{1}{D-4}} \ , \tag{3.6}$$

where the coefficient c_1 is a numerical constant of order 1 (cf. (2.11)). Like for a Schwarzschild black hole, the characteristic scale thus depends on the temperature.

In what follows we shall consider only the case of $D > 5$ which admits a black-hole interpretation. We shall further assume that for sufficiently low temperature corresponding to the black hole phase the characteristic size of the system or the upper cutoff Λ is of the same order as the lower cutoff b, i.e.

$$\Lambda = c_2 (\frac{Q}{m\beta})^{\frac{1}{D-4}} = \ k \ b \ , \qquad k = \frac{c_2}{c_1} > 1 \ , \tag{3.7}$$

so that the particles are distributed in a layer near the horizon, $b < r < kb$. This seems a natural assumption since one may expect that the difference between the higher and lower cutoffs should be of the order of the thermal wave length $\lambda = \sqrt{\beta/m}$, but, as we shall argue below, at the critical temperature λ should be of order b.

It is the additional dependence of Z on β and N through the cutoffs Λ and b that makes the statistical mechanics of this system quite different from the usual classical ideal gas one, but, at the same time, related to the black hole thermodynamics.

Integrating over the angles, one finds that W in (3.4) is given by

$$W = (\omega_{D-3})^2 \int_b^\Lambda dr \ r^{D-3} \int_0^\infty dp \ p^{D-3} \exp\{-\frac{\beta m}{Q} r^{D-4} [\sqrt{1 + \frac{Q}{m^2} \frac{p^2}{r^{D-4}}} - 1]\}. \tag{3.8}$$

Using the remarkable scaling symmetry of $D0$-brane system as described by (1.6) (cf.(1.8)), we can rescale the coordinate and the momentum

$$\mathrm{r} = (\frac{Q}{\beta m})^{\frac{1}{D-4}}\ r\ , \qquad \mathrm{p} = (\frac{m}{\beta})^{\frac{1}{2}} p\ , \tag{3.9}$$

to find the explicit dependence on the parameters

$$W = (\omega_{D-3})^2 (\frac{Q}{\beta m})^{\frac{D-2}{D-4}} (\frac{m}{\beta})^{\frac{D-2}{2}}\ \mathcal{W}(c_1, c_2)\ , \tag{3.10}$$

where

$$\mathcal{W} = \int_{c_1}^{c_2} d\mathrm{r}\ \mathrm{r}^{D-3} \int_0^{\infty} d\mathrm{p}\ \mathrm{p}^{D-3}\ \exp\{-\mathrm{r}^{D-4} [\sqrt{1 + \frac{\mathrm{p}^2}{\mathrm{r}^{D-4}}} - 1]\}\ . \tag{3.11}$$

We have used the crucial observation that the rescaled values of the lower (3.6) and upper (3.7) cutoffs are simply numerical constants

$$b \to (\frac{Q}{\beta m})^{\frac{1}{D-4}} b = c_1\ , \qquad \Lambda \to (\frac{Q}{\beta m})^{\frac{1}{D-4}} \Lambda = c_2\ . \tag{3.12}$$

Thus $\mathcal{W}$ is just a finite constant which does not depend on β and N.

Note that integrating over p in (3.11) we get

$$\mathcal{W} = c_0 \int_{c_1^{D-4}}^{c_2^{D-4}} dy\ y^{\frac{D}{2(D-4)}}\ e^y\ K_{\frac{D-1}{2}}(y)\ , \qquad y \equiv \mathrm{r}^{D-4}\ , \tag{3.13}$$

where c_0 is a numerical constant and K_ν is the modified Bessel function. Since for $y \to 0$ $K_\nu(y) \to y^{-\nu}$ we have

$$y^{\frac{D}{2(D-4)}}\ e^y\ K_{\frac{D-1}{2}}(y)\ \to\ y^{-\frac{D^2+6D-4}{2(D-4)}}\ ,$$

so that, as claimed above, for $D > 5$ the integral (3.11) blows up in the limit when the lower cutoff goes to zero ($c_1 \to 0$).

4. Relation to black hole thermodynamics

Eq. (3.10) implies that the resulting partition function (3.4) is thus given by

$$Z = \ C\ [(\frac{Q}{\beta m})^{\frac{D-2}{D-4}} (\frac{m}{\beta})^{\frac{D-2}{2}}]^N\ \sim\ (\frac{N}{\beta^{\frac{D-2}{2}}})^{\frac{D-2}{D-4}N}\ , \tag{4.1}$$

where C is a β, N -independent numerical constant. The factor $(\frac{Q}{\beta m})^{\frac{D-2}{D-4}} \sim b^{D-2} \sim V$ is recognized (cf. (3.5)–(3.7)) as the $(D-2)$-dimensional volume of the D0-brane gas, while the second is the familiar factor of a power of the mean thermal wave length

$$\lambda = \sqrt{\frac{\beta}{m}} \ . \tag{4.2}$$

The partition function (4.1) is thus the same as that of a free gas of (*distinguishable*) particles living on a $(D-2)$-dimensional transverse space of radius $R_s \sim b \sim (\frac{Q}{m\beta})^{\frac{1}{D-4}}$ (which can be interpreted as the event horizon of a D-dimensional black hole), i.e.[11]

$$Z = \ C' \ (\frac{V}{\lambda^{D-2}})^N \ . \tag{4.3}$$

At the temperature so low that the thermal wave length λ of the particles is of the order of the size of the box b, i.e. $\lambda^{D-2} \sim V$, one has $Z \sim 1$ and the above expression for the partition function is no longer valid. It is natural to expect that in this limit the gas becomes strongly correlated (degenerate) and our classical approximation breaks down. Note that for an ideal gas this is the limit[12] in which one should apply quantum statistics (for a Bose gas, the corresponding critical temperature is the one at which the Bose-Einstein condensation occurs).

We expect that the point $\lambda \sim b$ should correspond to a phase transition also in the present case. This is supported by the resulting compelling interpretation in terms of the D-dimensional Schwarzschild black hole thermodynamics. Indeed, the limit when

$$\lambda \sim b \ , \qquad \text{i.e.} \qquad (\frac{Q}{\beta m})^{\frac{D-2}{D-4}} \sim (\frac{\beta}{m})^{\frac{D-2}{2}} \ , \tag{4.4}$$

corresponds to the critical temperature

$$T = \beta^{-1} \sim (G_D N)^{-\frac{2}{D-2}} R \ , \tag{4.5}$$

and the size

$$R_s \sim b \sim (\frac{Q}{\beta m})^{\frac{1}{D-4}} \sim (G_D N)^{\frac{1}{D-2}} \ , \tag{4.6}$$

[11] This interpretation in terms of a gas on a $(D-2)$-dimensional sphere may be related to the holographic principle [25,26]: the partons should live in a space with a non-trivial topology like a sphere. The restriction to the sphere may be related to the presence of a YM background discussed in [5].

[12] For the usual ideal gas with $1/N!$ included in Z, the limit is actually $\lambda^{D-2} \sim V/N$.

where we used (1.5). The energy of the system is then given by

$$\mathrm{E} = -\frac{\partial \ln Z}{\partial \beta} \sim \; NT \; \sim G_D^{-\frac{2}{D-2}} N^{\frac{D-4}{D-2}} R \ , \tag{4.7}$$

and the entropy is found to be proportional to N,

$$S = \beta \mathrm{E} + \ln Z \; \sim \; N \ , \tag{4.8}$$

where we have used that $\ln Z \ll N$ in the limit (4.4).

If we follow [12,1] and interpret the energy E as the light-cone energy related to the mass M of a boosted object by

$$\mathrm{E} = \frac{R}{N} M^2 \ , \tag{4.9}$$

then using the expressions (4.7) and (4.8) derived above[13] we find that

$$M \sim G_D^{-\frac{1}{D-2}} S^{\frac{D-3}{D-2}} \ , \quad \text{i.e.} \qquad S \sim G_D^{\frac{1}{D-3}} M^{\frac{D-2}{D-3}} \ . \tag{4.10}$$

This is precisely the relation characteristic to a Schwarzschild black hole in D dimensions.[14]

One obvious generalisation of the discussion of this paper is to replace the effective action (1.6) by that of a D0-brane probe moving in a background of *near-extremal* D0-brane source. Indeed, it seems to be more consistent to treat the mean field as that of a non-extremal charge N D0-brane with excess energy being related to the kinetic energy of constituent unit-charge D0-branes. Another interesting problem is to consider instead of the classical partition function (3.1),(3.4) the partition function of the quantum gas, i.e. the gas of quantum D0-branes (or gravitons in $D = 11$). It should be given by the path integral of the exponent of the euclidean continuation of (1.6) (or its non-extremal version) over periodic trajectories $x(\beta) = x(0)$ (the naive classical limit $\beta \to 0$ gives back the classical Z). It may be that the quantum partition function is the proper starting point for trying to explain the necessity to introduce the short-distance cutoff (for $D > 5$) in the classical Z and to drop the overall $1/N!$ factor.

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[13] Eq. (4.8) is equivalent to the condition $N \sim MR_s \sim S$ assumed in [1].

[14] Note that keeping the standard $\frac{1}{N!}$ factor (or N^{-N} for large N) in Z (3.1),(3.4),(4.1) would give the entropy-mass relation that is different from the Schwarzschild one.

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