Example sheet 4

Answers

Cosmology and Particle Physics **Toby Wiseman**

Example sheet 4

Qu. 1 Assume flat FRW, then in the radiation era $a = a_{eq} (t/t_{eq})^{1/2}$ where t_{eq} is the time of matter-radiation equality. Show that assuming an instantaneous transition from radiation to matter era at $t = t_{eq}$ that in the subsequent matter era,

$$a = a_{eq} \left(\frac{3t + t_{eq}}{4t_{eq}} \right)^{2/3}$$

Assuming the radiation era extends back to a big bang at $t = 0$, compute the particle horizon size at time t in the matter era showing that the proper size at time $t > t_{eq}$ in the matter era is,

$$d_H(t) = \frac{2}{H} - \frac{a}{a_{eq} H_{eq}}$$

Ignore the presence of the dark energy today so that today the scale factor still evolves as for matter domination. Take t_{eq} to be at redshift $Z = Z_{eq} \sim 3600$ and last scattering to be at $Z = Z_{ls} \sim 1100$. Then show that the particle horizon at last scattering subtends an angle θ on the sky which is,

$$\theta \simeq \frac{360^o}{2\pi} \frac{1}{\sqrt{Z_{ls}}}$$

Hence at last scattering the scales that today are less than $\sim 1^o$ on the sky were in causal contact in the early radiation era, but larger scales were not.

Qu. 1 answer

In the radiation era $a = a_{eq} (t/t_{eq})^{1/2}$ so $H = 1/(2t)$. The general solution in the matter era is,

$$a = c\,(t + f)^{2/3}$$

for constants c and f, so $H = 2/3(t + f)$.

Assuming an instantaneous transition to the matter era, then for $t > t_{eq}$ we should have continuity of a and $\dot{a}$ or equivalently H at t_{eq}. Continuity of a implies,

$$c\,(t_{eq} + f)^{2/3} = a_{eq}$$

and for H we have,

$$\frac{2}{3(t_{eq}+f)} = \frac{1}{2t_{eq}}$$

Then we conclude that,

$$f = \frac{1}{3}t_{eq}$$

and

$$c = a_{eq}\left(\frac{4}{3}t_{eq}\right)^{-\frac{2}{3}}$$

and hence we obtain,

$$a = a_{eq}\left(\frac{4}{3}t_{eq}\right)^{-\frac{2}{3}}\left(t+\frac{1}{3}t_{eq}\right)^{2/3} = a_{eq}\left(\frac{3t+t_{eq}}{4t_{eq}}\right)^{2/3}$$

and,

$$H = \frac{2}{3t+t_{eq}}$$

so then,

$$\frac{1}{aH} = \frac{3t+t_{eq}}{2}\frac{1}{a_{eq}\left(\frac{3t+t_{eq}}{4t_{eq}}\right)^{2/3}} = \frac{(4t_{eq})^{2/3}}{2a_{eq}}\left(3t+t_{eq}\right)^{1/3}$$

Using FRW coordinates,

$$ds^2 = -dt^2 + a(t)^2\left(dr^2 + r^2 d\Omega^2\right)$$

then the comoving size of the particle horizon at time $t > t_{eq}$ is,

$$\begin{aligned} R(t) &= \int_0^t \frac{dt}{a(t)} = \int_{t_{eq}}^t \frac{dt}{a(t)} + \int_0^{t_{eq}} \frac{dt}{a(t)} \\ &= \int_{t_{eq}}^t \frac{dt}{a_{eq}}\left(\frac{3}{4t_{eq}}\right)^{-2/3}\left(t+\frac{1}{3}t_{eq}\right)^{-2/3} + \int_0^{t_{eq}} t_{eq}^{1/2}\frac{dt}{a_{eq}t^{1/2}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a_{eq}}\left(\frac{3}{4t_{eq}}\right)^{-2/3}\left[3(t+\frac{1}{3}t_{eq})^{1/3}\right]_{t_{eq}}^{t} + \frac{\sqrt{t_{eq}}}{a_{eq}}\left[2t^{1/2}\right]_0^{t_{eq}} \\
&= \frac{3}{a_{eq}}\left(\frac{4t_{eq}}{3}\right)^{2/3}\left((t+\frac{1}{3}t_{eq})^{1/3} - \left(\frac{4}{3}t_{eq}\right)^{1/3}\right) + \frac{2t_{eq}}{a_{eq}} \\
&= \frac{3}{a_{eq}}\left(\frac{4t_{eq}}{3}\right)^{2/3}(t+\frac{1}{3}t_{eq})^{1/3} - \frac{3}{a_{eq}}\left(\frac{4t_{eq}}{3}\right) + \frac{2t_{eq}}{a_{eq}} \\
&= \frac{1}{a_{eq}}(4t_{eq})^{2/3}(3t+t_{eq})^{1/3} - \frac{2t_{eq}}{a_{eq}}
\end{aligned}
$$

Now,

$$H_{eq} = \frac{1}{2t_{eq}}$$

so,

$$R(t) = \frac{2}{aH} - \frac{1}{a_{eq}H_{eq}}$$

Then the proper size of this comoving particle horizon size R at time t is,

$$d_H(t) = a(t)R(t) = \frac{2}{H} - \frac{a}{a_{eq}H_{eq}}$$

The comoving size of the particle horizon at last scattering is then,

$$R_{ls} = \frac{2}{a_{ls}H_{ls}} - \frac{1}{a_{eq}H_{eq}}$$

and the comoving size of the particle horizon today, ignoring dark energy so that the universe is matter dominated today, is,

$$R_0 = \frac{2}{a_0H_0} - \frac{1}{a_{eq}H_{eq}}$$

Then the angle in radians subtended by a patch in causal contact (i.e. the size of the particle horizon) at last scattering is,

$$\theta = \frac{R_{ls}}{R_0} = \left(\frac{\frac{2}{a_{ls}H_{ls}} - \frac{1}{a_{eq}H_{eq}}}{\frac{2}{a_0H_0} - \frac{1}{a_{eq}H_{eq}}}\right)$$

Matter radiation equality t_{eq} is at $Z_{eq} = 3600$, and ignoring dark energy this implies,

$$Z_{eq} = 1 + \frac{a_0}{a_{eq}} \sim \frac{a_0}{a_{eq}} = \left(\frac{3t_0 + t_{eq}}{4t_{eq}}\right)^{2/3}$$

and then,

$$\frac{1}{a_0 H_0} = \frac{(4t_{eq})^{2/3}}{2a_{eq}} (3t_0 + t_{eq})^{1/3} = \frac{2t_{eq}}{a_{eq}} \sqrt{Z_{eq}} = \frac{1}{a_{eq} H_{eq}} \sqrt{Z_{eq}}$$

For $Z_{ls} = 1100$ so,

$$Z_{ls} = 1 + \frac{a_0}{a_{ls}} \sim \frac{a_0}{a_{ls}}$$

and hence,

$$a_{eq} \frac{Z_{eq}}{Z_{ls}} = a_{ls} = a_{eq} \left(\frac{3t_{ls} + t_{eq}}{4t_{eq}}\right)^{2/3}$$

and then,

$$\frac{Z_{eq}}{Z_{ls}} = \left(\frac{3t_{ls} + t_{eq}}{4t_{eq}}\right)^{2/3}$$

Now we can compute,

$$\frac{1}{a_{ls} H_{ls}} = \frac{(4t_{eq})^{2/3}}{2a_{eq}} (3t_{ls} + t_{eq})^{1/3} = \frac{2t_{eq}}{a_{eq}} \sqrt{\frac{Z_{eq}}{Z_{ls}}} = \frac{1}{a_{eq} H_{eq}} \sqrt{\frac{Z_{eq}}{Z_{ls}}}$$

Then we find,

$$\begin{aligned} \theta &= \left(\frac{\frac{2}{a_{ls} H_{ls}} - \frac{1}{a_{eq} H_{eq}}}{\frac{2}{a_0 H_0} - \frac{1}{a_{eq} H_{eq}}}\right) \\ &= \left(\frac{2 \frac{1}{a_{eq} H_{eq}} \sqrt{\frac{Z_{eq}}{Z_{ls}}} - \frac{1}{a_{eq} H_{eq}}}{2 \frac{1}{a_{eq} H_{eq}} \sqrt{Z_{eq}} - \frac{1}{a_{eq} H_{eq}}}\right) \\ &= \left(\frac{\sqrt{\frac{Z_{eq}}{Z_{ls}}} - \frac{1}{2}}{\sqrt{Z_{eq}} - \frac{1}{2}}\right) \end{aligned}$$

Now for $Z_{ls}, Z_{eq} \gg 1$ then,

$$\theta \;\; \simeq \;\; \left(\frac{\sqrt{\frac{Z_{eq}}{Z_{ls}}}}{\sqrt{Z_{eq}}} \right) \simeq \frac{1}{\sqrt{Z_{ls}}} \simeq \frac{1}{\sqrt{1100}}$$

in radians. This corresponds to an angle in degrees,

$$\theta \simeq \frac{360^o}{2\pi} \frac{1}{\sqrt{1100}} \simeq 1.7^o$$

Note that this number is slightly smaller if one includes the dark energy today, more like $\sim 1^o$.

Qu. 2 As in the previous question, neglect dark energy today and assume instantaneous transitions from matter to radiation, and from radiation to an inflationary era which you may approximate as de Sitter. Using this, compute the number of e-folds required to solve the horizon problem if the universe reheated just above nucleosynthesis temperatures, $\sim 10^{10}K$ (ie. $\sim 1MeV$), or at an intermediate scale $\sim 10^{23}$ (ie. $\sim 10^{10}GeV$), or at the GUT scale $10^{29}K$ (ie. $\sim 10^{16}GeV$).

Qu. 2 answer Consider the behaviour in the radiation and matter eras as in the previous question so,

$$
\begin{aligned}
a &= a_{eq}\left(\frac{3t+t_{eq}}{4t_{eq}}\right)^{2/3} , \qquad t > t_{eq} \\
a &= a_{eq}\left(t/t_{eq}\right)^{1/2}
\end{aligned}
$$

However, now we start the radiation era at t_{rad}, and before that we have an approximate de Sitter expansion with constant H, so for $t_{rad} > t$, then,

$$a = a_i e^{H_i(t-t_{rad})}$$

for constant $H = H_i$ and a_i. Now by continuity of a and H at the transition at $t = t_{rad}$ from radiation to inflation we have,

$$a_i = a_{rad}$$

and,

$$H_i = H_{rad}$$

so that,

$$a = a_{rad} e^{H_{rad}(t-t_{rad})}$$

In the matter era we have,

$$a = a_{eq}\left(\frac{3t+t_{eq}}{4t_{eq}}\right)^{2/3}$$

as in the last question.

Consider a null geodesic starting at $r = 0$ at $t = t_{rad}$. Then today it will have travelled to $r = R(t_0)$

$$R_0 = R(t_0) \quad = \quad \int_{t_{rad}}^{t_0} \frac{dt}{a(t)}$$

Likewise a ray travelling back from t_{rad} into the previous inflationary era to an early time t_{inf} will travel to,

$$R_{inf} = R(t_{inf}) \quad = \quad \int_{t_{inf}}^{t_{rad}} \frac{dt}{a(t)}$$

and in order to solve the horizon problem we require that t_{inf} is early enough such that $R_{inf} > R_0$.

Let us compute R_0, basically following the previous question;

$$\begin{aligned}
R_0 &= \int_{t_{rad}}^{t_0} \frac{dt}{a(t)} = \int_{t_{eq}}^{t_0} \frac{dt}{a(t)} + \int_{t_{rad}}^{t_{eq}} \frac{dt}{a(t)} \\
&= \int_{t_{eq}}^{t_0} \frac{dt}{a_{eq}} \left(\frac{3}{4t_{eq}}\right)^{-2/3} \left(t + \frac{1}{3}t_{eq}\right)^{-2/3} + \int_{t_{rad}}^{t_{eq}} t_{eq}^{1/2} \frac{dt}{a_{eq} t^{1/2}} \\
&= \frac{1}{a_{eq}} \left(\frac{3}{4t_{eq}}\right)^{-2/3} \left[3(t_0 + \frac{1}{3}t_{eq})^{1/3}\right]_{t_{eq}}^{t_0} + \frac{\sqrt{t_{eq}}}{a_{eq}} \left[2t^{1/2}\right]_{t_{rad}}^{t_{eq}} \\
&= \frac{3}{a_{eq}} \left(\frac{4t_{eq}}{3}\right)^{2/3} \left((t_0 + \frac{1}{3}t_{eq})^{1/3} - \left(\frac{4}{3}t_{eq}\right)^{1/3} \right) + \frac{2t_{eq}}{a_{eq}} - \frac{2\sqrt{t_{eq}t_{rad}}}{a_{eq}} \\
&= \frac{3}{a_{eq}} \left(\frac{4t_{eq}}{3}\right)^{2/3} (t_0 + \frac{1}{3}t_{eq})^{1/3} - \frac{3}{a_{eq}} \left(\frac{4t_{eq}}{3}\right) + \frac{2t_{eq}}{a_{eq}} - \frac{2\sqrt{t_{eq}t_{rad}}}{a_{eq}} \\
&= \frac{1}{a_{eq}} (4t_{eq})^{2/3} (3t_0 + t_{eq})^{1/3} - \frac{2t_{eq}}{a_{eq}} - \frac{2\sqrt{t_{eq}t_{rad}}}{a_{eq}}
\end{aligned}$$

Recall from the previous question that in the matter era,

$$\frac{1}{aH} = \frac{3t + t_{eq}}{2} \frac{1}{a_{eq} \left(\frac{3t+t_{eq}}{4t_{eq}}\right)^{2/3}} = \frac{(4t_{eq})^{2/3}}{2a_{eq}} (3t + t_{eq})^{1/3}$$

and for radiation $H = 1/(2t)$ and $a = a_{eq} (t/t_{eq})^{1/2}$, so,

$$\begin{aligned}
R_0 &= \frac{2}{a_0 H_0} - \frac{1}{a_{eq} H_{eq}} - \frac{2}{a_{eq}} \sqrt{t_{eq} t_{rad}} \\
&= \frac{2}{a_0 H_0} - \frac{1}{a_{eq} H_{eq}} - \frac{1}{a_{rad} H_{rad}}
\end{aligned}$$

As in the previous question,

$$Z_{eq} = 1 + \frac{a_0}{a_{eq}} \simeq \frac{a_0}{a_{eq}} = \left(\frac{3t_0 + t_{eq}}{4t_{eq}}\right)^{2/3}$$

so then,

$$H_0 = \frac{2}{3t_0 + t_{eq}} = \frac{1}{2t_{eq}{Z_{eq}}^{3/2}}$$

so,

$$\frac{1}{a_0 H_0} = \frac{2t_{eq}{Z_{eq}}^{3/2}}{a_0} = \frac{2t_{eq}{Z_{eq}}^{3/2}}{a_{eq}Z_{eq}} = \frac{\sqrt{Z_{eq}}}{H_{eq}a_{eq}}$$

Now also in the radiation era at $t = t_{rad}$, where $a \propto t^{1/2}$ and $H \propto 1/t$, so then $1/aH \propto a$ then,

$$\frac{1}{a_{rad}H_{rad}} = \frac{1}{a_{eq}H_{eq}}\frac{a_{rad}}{a_{eq}} \simeq \frac{1}{a_{eq}H_{eq}}\frac{Z_{eq}}{Z_{rad}}$$

So then,

$$\begin{aligned} R_0 &= \frac{2}{a_0 H_0} - \frac{1}{a_{eq}H_{eq}} - \frac{1}{a_{rad}H_{rad}} \\ &= \frac{1}{a_{eq}H_{eq}}\left(2\sqrt{Z_{eq}} - 1 - \frac{Z_{eq}}{Z_{rad}}\right) \end{aligned}$$

Now for $Z_{rad} \gg Z_{eq} \gg 1$ then,

$$R_0 \simeq 2\sqrt{Z_{eq}}\frac{1}{a_{eq}H_{eq}}$$

Now to compute R_{inf},

$$\begin{aligned} R_{inf} &= \int_{t_{inf}}^{t_{rad}} \frac{dt}{a(t)} = \int_{t_{inf}}^{t_{rad}} \frac{dt}{a_{rad}e^{H_{rad}(t-t_{rad})}} \\ &= \frac{1}{a_{rad}}\int_{t_{inf}}^{t_{rad}} dt e^{-H_{rad}(t-t_{rad})} = \frac{1}{a_{rad}}\left[-\frac{1}{H_{rad}}e^{-H_{rad}(t-t_{rad})}\right]_{t_{inf}}^{t_{rad}} \\ &= \frac{1}{a_{rad}H_{rad}}\left(e^{H_{rad}(t_{rad}-t_{inf})} - 1\right) \\ &= \frac{1}{a_{rad}H_{rad}}\left(\frac{a_{rad}}{a_{inf}} - 1\right) \simeq \frac{1}{a_{rad}H_{rad}}e^N \simeq \frac{1}{a_{eq}H_{eq}}\frac{Z_{eq}}{Z_{rad}}e^N \end{aligned}$$

where the number of e-foldings N between t_{inf} and the end of inflation t_{rad} is,

$$e^N = \frac{a_{rad}}{a_{inf}}$$

Hence requiring t_{inf} such that $R_{inf} > R_0$ we require,

$$\frac{1}{a_{eq}H_{eq}} \frac{Z_{eq}}{Z_{rad}} e^N > 2\sqrt{Z_{eq}} \frac{1}{a_{eq}H_{eq}}$$

so that,

$$e^N > \frac{2Z_{rad}}{\sqrt{Z_{eq}}}$$

Now the photon temperature $T \sim 1/a \sim Z$ in the matter and radiation era's (ignoring entropy injection from annihilation), and so,

$$Z_{rad} = \frac{T_{rad}}{T_0}$$

where $T_0 = T_{CMB} = 2.7K$. So,

$$e^N > 2\frac{T_{rad}}{T_{CMB}\sqrt{Z_{eq}}}$$

where we recall $Z_{eq} = 3600$.

Then for $T_{rad} = 10^{10}K$ we require $N > 19$.

Then for $T_{rad} = 10^{20}K$ we require $N > 42$.

Then for $T_{rad} = 10^{29}K$ we require $N > 62$.

Qu. 3 The inflaton is a scalar field ϕ with equation of motion,

$$\nabla^2\phi = V'(\phi)$$

and stress tensor,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)$$

Show that assuming homogeneity and isotropy in a flat FRW spacetime, $ds^2 = -dt^2 + a(t)^2 dx^i dx^j$, so that $\phi = \phi(t)$, then the scalar field equation yields,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0$$

and the Einstein equations yield,

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

and,

$$\dot{H} = -4\pi G\dot{\phi}^2$$

Qu. 3 answer So for a homogeneous $\phi = \phi(t)$ then,

$$\begin{aligned}\rho &= T_{tt} = \partial_t\phi\partial_t\phi - g_{tt}\left(\frac{1}{2}g^{tt}\partial_t\phi\partial_t\phi + V\right) = \dot{\phi}^2 + \left(-\frac{1}{2}\dot{\phi}^2 + V\right) \\ &= \frac{1}{2}\dot{\phi}^2 + V\end{aligned}$$

and,

$$P g_{ij} = a^2 P\delta_{ij} = T_{ij} = \partial_i\phi\partial_j\phi - g_{ij}\left(\frac{1}{2}g^{tt}\partial_t\phi\partial_t\phi + V\right) = -a^2\delta_{ij}\left(-\frac{1}{2}\dot{\phi}^2 + V\right)$$

so that,

$$P = \frac{1}{2}\dot{\phi}^2 - V$$

Hence the Friedmann equation is,

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$$

and also,

$$\dot{H} = -4\pi G\left(\rho + P\right) = -4\pi G\dot{\phi}^2$$

And for the scalar equation $\nabla^2\phi = V'(\phi)$, recall for flat FRW, the non-vanishing components of $\Gamma^{\mu}{}_{\alpha\beta}$ are,

$$\Gamma^{t}{}_{ij} = a\dot{a}\delta_{ij}\ , \qquad \Gamma^{i}{}_{tj} = \frac{\dot{a}}{a}\delta^{i}_{j}$$

Then let us consider the scalar equation,

$$\begin{aligned}
\nabla^2\phi &= g^{\mu\nu}\partial_\mu\partial_\nu\phi - g^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\beta}\partial_\mu\phi \\
&= g^{tt}\ddot{\phi} - g^{tt}\Gamma^{t}{}_{tt}\partial_t\phi - 2g^{ti}\Gamma^{t}{}_{ti}\partial_t\phi - g^{ij}\Gamma^{t}{}_{ij}\partial_t\phi \\
&= -\ddot{\phi} - \frac{1}{a^2}\delta^{ij}\Gamma^{t}{}_{ij}\partial_t\phi \\
&= -\ddot{\phi} - \frac{1}{a^2}\delta^{ij}a\dot{a}\delta_{ij}\dot{\phi} \\
&= -\ddot{\phi} - 3\frac{\dot{a}}{a}\dot{\phi}
\end{aligned}$$

So then the scalar equation $\nabla^2\phi = V'(\phi)$ implies,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0$$

Qu. 4 Define the slow roll function;

$$\epsilon(\phi) = \frac{1}{16\pi G} \left(\frac{V'(\phi)}{V(\phi)} \right)^2$$

Over a range of ϕ the potential supports slow roll inflation, so $\epsilon(\phi) \ll 1$. Suppose that V is monotonic over this range so we have $\phi = \phi(V)$ and so can think of the function $\epsilon(V)$, and further that $\epsilon(V)$ is well approximated by,

$$\epsilon(V) \simeq \epsilon(V_0) + \epsilon'(V_0)(V - V_0) + \frac{1}{2}\epsilon''(V_0)(V - V_0)^2$$

over this range, with ϕ_0 some value of the scalar within the range, and V_0 the potential at this point.

Assume that over this scalar range the potential varies so that $V/V_0 \sim O(1)$. Firstly show that the parameter characterizing second derivatives of the potential at ϕ_0,

$$\eta(\phi_0) = \frac{1}{8\pi G} \left| \frac{V''(\phi)}{V} \right|_{\phi=\phi_0} \ll 1$$

must be small. Then show that the parameter characterizing third derivatives is also small, so,

$$\gamma(\phi_0) = \frac{1}{(8\pi G)^{3/2}} \left| \frac{V'''(\phi)}{V} \right|_{\phi=\phi_0} \ll 1\,.$$

Qu. 4 answer

If $0 \leq \epsilon(V) \ll 1$ is small over the range of V where $V/V_0 \sim O(1)$, then if $\epsilon(V_0) \ll 1$ this also requires,

$$|(V - V_0)\epsilon'(V_0)| \sim |V\epsilon'(V_0)| \ll 1 \tag{1}$$

and

$$|(V - V_0)^2\epsilon''(V_0)| \sim |V^2\epsilon''(V_0)| \ll 1 \tag{2}$$

Consider the first condition;

$$\begin{aligned} V\frac{d\epsilon}{dV} &= \frac{V}{16\pi G}\frac{d}{dV}\left(\frac{V'(\phi)^2}{V(\phi)^2}\right) \\ &= -\frac{1}{8\pi G}\frac{V'(\phi)^2}{V(\phi)^2} + \frac{V'}{8\pi GV}\frac{dV'}{dV} \end{aligned}$$

Now,

$$\frac{dV'}{dV} = \frac{1}{V'}\frac{dV'}{d\phi} = \frac{V''}{V'}$$

so that,

$$V\frac{d\epsilon}{dV} = -2\epsilon + \frac{1}{8\pi G}\frac{V''}{V}$$

Then requiring $|V\frac{d\epsilon}{dV}|_{\phi=\phi_0} \ll 1$, implies,

$$\frac{1}{8\pi G}\left|\frac{V''}{V}\right|_{\phi=\phi_0} \ll 1$$

(since $\epsilon \ll 1$).

Continuing to the second condition;

$$\begin{aligned} V^2\frac{d^2\epsilon}{dV^2} &= V\frac{d}{dV}\left(V\frac{d\epsilon}{dV}\right) \\ &= V\frac{d}{dV}\left(-2\epsilon + \frac{1}{8\pi G}\frac{V''}{V}\right) \\ &= -2V\frac{d\epsilon}{dV} + \frac{1}{8\pi G}V\frac{d}{dV}\left(\frac{V''}{V}\right) \\ &= -2V\frac{d\epsilon}{dV} - \frac{1}{8\pi G}\frac{V''}{V} + \frac{1}{8\pi G}\frac{dV''}{dV} \end{aligned}$$

Now,

$$\frac{dV''}{dV} = \frac{1}{V'}\frac{dV''}{d\phi} = \frac{V'''}{V'}$$

so,

$$V^2\frac{d^2\epsilon}{dV^2} \quad = \quad -2V\frac{d\epsilon}{dV} - \frac{1}{8\pi G}\frac{V''}{V} + \frac{1}{(8\pi G)^{3/2}}\frac{V'''}{V}\frac{(8\pi G)^{1/2}V}{V'}$$

Requiring $|V^2\frac{d^2\epsilon}{dV^2}|_{\phi=\phi_0} \ll 1$, and recalling $|V\frac{d\epsilon}{dV}|_{\phi=\phi_0} \ll 1$ and $\frac{1}{8\pi G}\left|\frac{V''}{V}\right|_{\phi=\phi_0} \ll 1$ we have the condition,

$$\left|\frac{1}{(8\pi G)^{3/2}}\frac{V'''}{V}\frac{(8\pi G)^{1/2}V}{V'}\right|_{\phi_0} \ll 1$$

so that,

$$\left|\frac{1}{(8\pi G)^{3/2}}\frac{V'''}{V}\right|_{\phi_0} \ll \left|\frac{V'}{(8\pi G)^{1/2}V}\right|_{\phi_0} = \sqrt{\epsilon(\phi_0)} \ll 1$$

Qu. 5 Linearize the inflaton equation of motion in a fixed flat FRW background,

$$\nabla^2 \phi = V'(\phi)$$

about a homogeneous isotropic classical solution $\phi_{cl}(t)$. Show that if we ignore back reaction on the metric, then perturbations to the inflaton which are not homogeneous or isotropic, so that $\phi(t, x) = \phi_{cl} + \delta\phi(t, x)$, obey,

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{a(t)^2}\delta^{ij}\partial_i\partial_j\delta\phi + V''(\phi_{cl})\delta\phi = 0$$

Show that we may write a solution to this as,

$$\delta\phi_{\vec{k}}(t, x) = \delta\phi_{\vec{k}}(t)e^{ik_i x^i}$$

where $k = |\vec{k}| = \sqrt{\delta^{ij}k_i k_j}$ is the comoving wavenumber and find the ordinary differential equation that the time dependence given by $\delta\phi_{\vec{k}}(t)$ obeys. Show that for fixed $\vec{k}$, then for $t \to -\infty$ such that $a \to 0$ and $k/a \gg H$, then provided the FRW background obeys the slow roll conditions, we may write a "WKB solution" as,

$$\delta\phi_{\vec{k}}(t) = \frac{c_{\vec{k}}}{a(t)}e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}}\left(1 + O\left(\frac{a}{k}\right)\right)$$

for some time t^* such that $k/a \gg H$ at that time, and $c_{\vec{k}}$ is an integration constant.

Confirm that for an exact de Sitter background, so $a = e^{Ht}$ with constant H and $V'' = 0$, that the full solution to the perturbation equation for $\delta\phi_k(t)$ is,

$$\delta\phi_{\vec{k}}(t) = \frac{c_{\vec{k}}}{a(t)}e^{-\frac{ik}{a(t)H}}\left(1 - \frac{ia(t)H}{k}\right)$$

Qu. 5 answer For the scalar equation $\nabla^2\phi = V'(\phi)$, recall for flat FRW, the non-vanishing components of $\Gamma^\mu{}_{\alpha\beta}$ are,

$$\Gamma^t{}_{ij} = a\dot{a}\delta_{ij}\ , \qquad \Gamma^i{}_{tj} = \frac{\dot{a}}{a}\delta^i_j$$

Then let us consider the scalar equation in general,

$$\begin{aligned}
\nabla^2\phi &= g^{\mu\nu}\partial_\mu\partial_\nu\phi - g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}\partial_\mu\phi \\
&= g^{tt}\ddot{\phi} + g^{ij}\partial_i\partial_j\phi - g^{tt}\Gamma^\mu{}_{tt}\partial_\mu\phi - 2g^{ti}\Gamma^\mu{}_{ti}\partial_\mu\phi - g^{ij}\Gamma^\mu{}_{ij}\partial_\mu\phi \\
&= -\ddot{\phi} + \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi - \frac{1}{a^2}\delta^{ij}\Gamma^t{}_{ij}\partial_t\phi \\
&= -\ddot{\phi} + \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi - \frac{1}{a^2}\delta^{ij}a\dot{a}\delta_{ij}\dot{\phi} \\
&= -\ddot{\phi} + \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi - 3\frac{\dot{a}}{a}\dot{\phi}
\end{aligned}$$

So then the scalar equation implies,

$$-\ddot{\phi} + \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi - 3\frac{\dot{a}}{a}\dot{\phi} = V'(\phi)$$

so,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi + V'(\phi) = 0$$

Now consider $\phi = \phi_{cl}(t) + \delta\phi(t,x)$ where $\phi_{cl}(t)$ is a homogeneous classical solution so it must obey,

$$\ddot{\phi}_{cl} + 3\frac{\dot{a}}{a}\dot{\phi}_{cl} + V'(\phi_{cl}) = 0$$

Then, the scalar equation gives,

$$\ddot{\phi}_{cl} + \ddot{\delta\phi} + 3\frac{\dot{a}}{a}\dot{\phi}_{cl} + 3\frac{\dot{a}}{a}\dot{\delta\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\delta\phi + V'(\phi_{cl} + \delta\phi) = 0$$

and we may expand,

$$V'(\phi_{cl} + \delta\phi) = V'(\phi_{cl}) + V''(\phi_{cl})\delta\phi + O(\delta\phi^2)$$

so the scalar equation gives,

$$\left(\ddot{\phi}_{cl} + 3\frac{\dot{a}}{a}\dot{\phi}_{cl} + V'(\phi_{cl})\right) + \left(\ddot{\delta\phi} + 3\frac{\dot{a}}{a}\dot{\delta\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\delta\phi + V''(\phi_{cl})\delta\phi\right) = O(\delta\phi^2)$$

and since $\phi_{cl}(t)$ is a classical solution the first bracket vanishes leaving,

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\delta\phi + V''(\phi_{cl})\delta\phi = 0$$

to linear order in $\delta\phi$.

We may write a solution as,

$$\delta\phi_{\vec{k}}(t, x) = \delta\phi_{\vec{k}}(t)e^{ik_ix^i}$$

and then substituting this in gives,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} + V''(\phi_{cl})\delta\phi_{\vec{k}} = 0$$

Now consider a slow roll background, so that, $H^2 \simeq 8\pi GV(\phi_{cl})/3 > 0$ and is roughly constant. Then a mode with fixed $\vec{k}$ as $t \to 0$ and $a \to 0$ has that $k/a \gg H$ at early times. However for slow roll (see earlier question) we have,

$$V'' \ll GV$$

and hence,

$$V'' \ll \frac{8\pi G}{3}V = H^2$$

so that,

$$V'' \ll H^2 \ll \frac{k^2}{a^2}$$

Hence we can safely ignore the last term, and the perturbation obeys at early times $t \to 0$,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} = 0$$

with $k/a \gg H$.

Now consider a WKB solution;

$$\delta\phi_{\vec{k}}(t) = f(t)e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}}$$

Then,

$$\dot{\delta\phi}_{\vec{k}}(t) = \left(f'(t) - \frac{ik}{a(t)}f(t)\right)e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}} = \left(\frac{f'}{f} - \frac{ik}{a}\right)\delta\phi_{\vec{k}}$$

and,

$$\begin{aligned}
\ddot{\delta\phi}_{\vec{k}}(t) &= \left(f'(t) - \frac{ik}{a(t)}f(t)\right)\frac{-ik}{a(t)}e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}} + \left(f''(t) - \frac{ik}{a(t)}f'(t) + ik\frac{\dot{a}}{a^2}f(t)\right)e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}} \\
&= \left(\frac{f''}{f} - 2ik\frac{f'}{af} + ik\frac{\dot{a}}{a^2} - \frac{k^2}{a^2}\right)\delta\phi_{\vec{k}} \\
&= \left(\frac{f''}{f} + i\frac{k}{a}\left(H - 2\frac{f'}{f}\right) - \left(\frac{k}{a}\right)^2\right)\delta\phi_{\vec{k}}
\end{aligned}$$

Then substituting into the scalar equation,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} = 0$$

gives,

$$\left(\frac{f''}{f} + i\frac{k}{a}\left(H - 2\frac{f'}{f}\right) - \left(\frac{k}{a}\right)^2\right)\delta\phi_{\vec{k}} + 3H\left(\frac{f'}{f} - \frac{ik}{a}\right)\delta\phi_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} = 0$$

so that,

$$\left(\frac{f''}{f} + 3H\frac{f'}{f}\right) - 2i\frac{k}{a}\left(H + \frac{f'}{f}\right) = 0$$

where we notice that by construction the k^2/a^2 terms cancel due to the WKB form. Since $k/a \gg H$, we see that the terms linear in k/a dominate the equation giving,

$$\frac{k}{a}\left(\frac{\dot{a}}{a} + \frac{f'}{f}\right) = 0$$

and hence

$$\frac{\dot{a}}{a} = -\frac{\dot{f}}{f}$$

so integrating,

$$f(t) = \frac{c_{\vec{k}}}{a(t)}$$

for an integratiob constant $c_{\vec{k}}$ to leading order, and we expect corrections in a/k so,

$$f(t) = \frac{c_{\vec{k}}}{a(t)}\left(1 + O\left(\frac{a}{k}\right)\right)$$

Then the WKB solution is,

$$\delta\phi_{\vec{k}}(t) = \frac{c_{\vec{k}}}{a(t)} e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}}\left(1 + O\left(\frac{a}{k}\right)\right)$$

For exact de Sitter, so that H =constant, and $V'' = 0$ the claim is that the full solution to,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} = 0$$

is,

$$\begin{aligned}
\delta\phi_{\vec{k}}(t) &= \frac{c_{\vec{k}}}{a(t)} e^{-\frac{ik}{a(t)\,H}}\left(1 - \frac{ia(t)H}{k}\right) \\
&= c_{\vec{k}} e^{-\frac{ik}{a(t)\,H}}\left(\frac{1}{a(t)} - \frac{iH}{k}\right)
\end{aligned}$$

We must check this. Firstly, using $\dot{a}/a = H$ and $\dot{H} = 0$,

$$\begin{aligned}
\dot{\delta\phi}_{\vec{k}}(t) &= c_{\vec{k}}\left(+\frac{ik\dot{a}}{a^2\,H}\right) e^{-\frac{ik}{a(t)\,H}}\left(\frac{1}{a(t)} - \frac{iH}{k}\right) + c_{\vec{k}} e^{-\frac{ik}{a(t)\,H}}\left(-\frac{\dot{a}}{a^2}\right) \\
&= c_{\vec{k}}\left(+\frac{ik}{a}\right) e^{-\frac{ik}{a(t)\,H}}\left(\frac{1}{a(t)} - \frac{iH}{k}\right) + c_{\vec{k}} e^{-\frac{ik}{a(t)\,H}}\left(-\frac{H}{a}\right) \\
&= c_{\vec{k}} e^{-\frac{ik}{a(t)\,H}}\left(\frac{ik}{a^2} + \frac{H}{a} - \frac{H}{a}\right) = c_{\vec{k}}\frac{ik}{a^2} e^{-\frac{ik}{a(t)\,H}}
\end{aligned}$$

Then,

$$\begin{aligned}
\ddot{\delta\phi}_{\vec{k}}(t) &= c_{\vec{k}}\frac{ik}{a^2}\left(+\frac{ik\dot{a}}{a^2 H}\right) e^{-\frac{ik}{a(t)\,H}} + c_{\vec{k}}\left(-2\frac{ik\dot{a}}{a^3}\right) e^{-\frac{ik}{a(t)\,H}} \\
&= c_{\vec{k}}\left(-\frac{k^2}{a^3} - 2\frac{ikH}{a^2}\right) e^{-\frac{ik}{a(t)\,H}}
\end{aligned}$$

Then substituting into the scalar equation,

$$\begin{aligned}
&\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} \\
&\quad = c_{\vec{k}}\left(-\frac{k^2}{a^3} - 2\frac{ikH}{a^2}\right)e^{-\frac{ik}{a(t)\,H}} + 3Hc_{\vec{k}}\frac{ik}{a^2}e^{-\frac{ik}{a(t)\,H}} + \frac{k^2}{a^2}c_{\vec{k}}e^{-\frac{ik}{a(t)\,H}}\left(\frac{1}{a(t)} - \frac{iH}{k}\right) \\
&\quad = c_{\vec{k}}e^{-\frac{ik}{a(t)\,H}}\left(\left(-\frac{k^2}{a^3} - 2\frac{ikH}{a^2}\right) + 3H\frac{ik}{a^2} + \frac{k^2}{a^2}\left(\frac{1}{a(t)} - \frac{iH}{k}\right)\right) \\
&\quad = c_{\vec{k}}e^{-\frac{ik}{a(t)\,H}}\left(-\frac{k^2}{a^3} - 2\frac{ikH}{a^2} + 3\frac{ikH}{a^2} + \frac{k^2}{a^3} - \frac{ikH}{a^2}\right) = 0
\end{aligned}$$

and indeed we see that it does exactly solve the equation.

Qu. 6 As in lectures we quantise the (real) inflaton scalar field ϕ in flat FRW as,

$$\hat{\phi}(t,x) = \phi_{cl}(t) + \int d^3\vec{k}\delta\phi_{\vec{k}}(t)e^{ik_ix^i}a_{\vec{k}} + \delta\phi_{\vec{k}}(t)^\star e^{-ik_ix^i}a^\dagger_{\vec{k}}$$

where $\delta\phi_{\vec{k}}(t)$ obeys,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} + V''(\phi_{cl})\delta\phi_{\vec{k}} = 0$$

with appropriate boundary conditions. Show that if we choose the creation/annhilation operators so that,

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0 \qquad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k} - \vec{k}')$$

then the field obeys the equal time commutation relations,

$$[\hat{\phi}(t,x), \hat{\phi}(t,y)] = 0$$

and,

$$[\hat{\phi}(t,x), \hat{\pi}(t,y)] = a^3(t)\int d^3\vec{k}\left(\delta\phi_{\vec{k}}\dot{\delta\phi}_{\vec{k}}^{\ \star} - \delta\phi^\star_{\vec{k}}\dot{\delta\phi}_{\vec{k}}\right)e^{ik_i(x^i-y^i)}$$

where we recall that the conjugate momentum $\hat{\pi} = a^3(t)\dot{\hat{\phi}}$.

For $t \to -\infty$ and slow roll inflation (so that $a \to 0$ and $H \simeq$ constant) we may use the WKB approximation,

$$\delta\phi_{\vec{k}}(t) \simeq \frac{c_{\vec{k}}}{a(t)}e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}}$$

Then in order to obtain the conventional equal time flat space commutator as $t \to -\infty$,

$$[\hat{\phi}(t,x), \hat{\pi}(t,y)] = i\delta^{(3)}(\vec{x} - \vec{y})$$

show that the modes must be normalised so that,

$$|c_{\vec{k}}| = \frac{1}{(2\pi)^{3/2}\sqrt{2k}}$$

(and we conventionally choose $c_{\vec{k}}$ to be real).

Qu. 6 answer Now,

$$\begin{aligned}
[\hat{\phi}(t,x),\hat{\phi}(t,y)] &= \int d^3\vec{k}\int d^3\vec{k'}[\delta\phi_{\vec{k}}e^{ik_ix^i}a_{\vec{k}}+\delta\phi^\star_{\vec{k}}e^{-ik_ix^i}a^\dagger_{\vec{k}},\delta\phi_{\vec{k'}}e^{ik'_iy^i}a_{\vec{k'}}+\delta\phi^\star_{\vec{k'}}e^{-ik'_iy^i}a^\dagger_{\vec{k'}}] \\
&= \int d^3\vec{k}\int d^3\vec{k'}\delta\phi_{\vec{k}}(t)e^{ik_ix^i}\delta\phi_{\vec{k'}}(t)^\star e^{-ik'_iy^i}[a_{\vec{k}},a^\dagger_{\vec{k'}}] \\
&\quad +\int d^3\vec{k}\int d^3\vec{k'}\delta\phi_{\vec{k}}(t)^\star e^{-ik_ix^i}\delta\phi_{\vec{k'}}(t)e^{ik'_iy^i}[a^\dagger_{\vec{k}},a_{\vec{k'}}] \\
&= \int d^3\vec{k}\int d^3\vec{k'}\delta\phi_{\vec{k}}(t)e^{ik_ix^i}\delta\phi_{\vec{k'}}(t)^\star e^{-ik'_iy^i}\delta^{(3)}(\vec{k}-\vec{k}') \\
&\quad -\int d^3\vec{k}\int d^3\vec{k'}\delta\phi_{\vec{k}}(t)^\star e^{-ik_ix^i}\delta\phi_{\vec{k'}}(t)e^{ik'_iy^i}\delta^{(3)}(\vec{k}-\vec{k}') \\
&= \int d^3\vec{k}\left|\delta\phi_{\vec{k}}(t)\right|^2e^{ik_i(x^i-y^i)}-\int d^3\vec{k}\left|\delta\phi_{\vec{k}}(t)\right|^2e^{-ik_i(x^i-y^i)}
\end{aligned}$$

Now in the last term taking $\vec{k}\to-\vec{k}$, then $\int d^3\vec{k}\to\int d^3\vec{k}$ (remember we must switch the limits too!), so,

$$[\hat{\phi}(t,x),\hat{\phi}(t,y)] = \int d^3\vec{k}\left|\delta\phi_{\vec{k}}(t)\right|^2e^{ik_i(x^i-y^i)}-\int d^3\vec{k}\left|\delta\phi_{-\vec{k}}(t)\right|^2e^{ik_i(x^i-y^i)}=0$$

since $\delta\phi_{\vec{k}}(t)=\delta\phi_{-\vec{k}}(t)$ as is evident from the differential equation that it must satisfy (which only depends on $|\vec{k}|$).

And also,

$$[\hat{\phi}(t,x),\hat{\pi}(t,y)] = a^3(t)[\hat{\phi}(t,x),\dot{\hat{\phi}}(t,y)]$$

So,

$$\begin{aligned}
[\hat{\phi}(t,x),\dot{\hat{\phi}}(t,y)] &= \int d^3\vec{k}\int d^3\vec{k'}[\delta\phi_{\vec{k}}e^{ik_ix^i}a_{\vec{k}}+\delta\phi^\star_{\vec{k}}e^{-ik_ix^i}a^\dagger_{\vec{k}},\delta\dot{\phi}_{\vec{k'}}e^{ik'_iy^i}a_{\vec{k'}}+\delta\dot{\phi}_{\vec{k'}}{}^\star e^{-ik'_iy^i}a^\dagger_{\vec{k'}}] \\
&= \int d^3\vec{k}\int d^3\vec{k'}\delta\phi_{\vec{k}}e^{ik_ix^i}\delta\dot{\phi}_{\vec{k'}}{}^\star e^{-ik'_iy^i}[a_{\vec{k}},a^\dagger_{\vec{k'}}] \\
&\quad +\int d^3\vec{k}\int d^3\vec{k'}\delta\phi^\star_{\vec{k}}e^{-ik_ix^i}\delta\dot{\phi}_{\vec{k'}}e^{ik'_iy^i}[a^\dagger_{\vec{k}},a_{\vec{k'}}] \\
&= \int d^3\vec{k}\int d^3\vec{k'}\delta\phi_{\vec{k}}e^{ik_ix^i}\delta\dot{\phi}_{\vec{k'}}{}^\star e^{-ik'_iy^i}\delta^{(3)}(\vec{k}-\vec{k}') \\
&\quad -\int d^3\vec{k}\int d^3\vec{k'}\delta\phi^\star_{\vec{k}}e^{-ik_ix^i}\delta\dot{\phi}_{\vec{k'}}e^{ik'_iy^i}\delta^{(3)}(\vec{k}-\vec{k}') \\
&= \int d^3\vec{k}\,\delta\phi_{\vec{k}}\delta\dot{\phi}_{\vec{k}}{}^\star e^{ik_i(x^i-y^i)}-\delta\phi^\star_{\vec{k}}\delta\dot{\phi}_{\vec{k}}e^{-ik_i(x^i-y^i)}
\end{aligned}$$

Then taking $\vec{k} \to -\vec{k}$ in the latter term,

$$[\hat{\phi}(t,x), \dot{\hat{\phi}}(t,y)] \;=\; \int d^3\vec{k}\, \delta\phi_{\vec{k}} \dot{\delta\phi_{\vec{k}}}^\star e^{ik_i(x^i-y^i)} - \delta\phi^\star_{-\vec{k}} \dot{\delta\phi}_{-\vec{k}} e^{ik_i(x^i-y^i)}$$

and recalling again that $\dot{\delta\phi}_{\vec{k}} = \dot{\delta\phi}_{-\vec{k}}$, so,

$$[\hat{\phi}(t,x), \dot{\hat{\phi}}(t,y)] \;=\; \int d^3\vec{k} \left(\delta\phi_{\vec{k}} \dot{\delta\phi_{\vec{k}}}^\star - \delta\phi^\star_{\vec{k}} \dot{\delta\phi}_{\vec{k}} \right) e^{ik_i(x^i-y^i)}$$

Hence we see,

$$[\hat{\phi}(t,x), \hat{\pi}(t,y)] \;=\; a^3(t) \int d^3\vec{k} \left(\delta\phi_{\vec{k}} \dot{\delta\phi_{\vec{k}}}^\star - \delta\phi^\star_{\vec{k}} \dot{\delta\phi}_{\vec{k}} \right) e^{ik_i(x^i-y^i)}$$

Now take the WKB approximation,

$$\delta\phi_{\vec{k}}(t) \simeq \frac{c_{\vec{k}}}{a(t)} e^{-ik\int_{t^*}^{t} \frac{dt'}{a(t')}}$$

so that,

$$\begin{aligned}
\delta\phi_{\vec{k}} \dot{\delta\phi_{\vec{k}}}^\star &= \frac{c_{\vec{k}}}{a(t)} e^{-ik\int_{t^*}^{t} \frac{dt'}{a(t')}} \frac{d}{dt} \left(\frac{c_{\vec{k}}}{a(t)} e^{-ik\int_{t^*}^{t} \frac{dt'}{a(t')}} \right)^\star \\
&= \frac{c_{\vec{k}} c^\star_{\vec{k}}}{a(t)} e^{-ik\int_{t^*}^{t} \frac{dt'}{a(t')}} \frac{d}{dt} \left(\frac{1}{a(t)} e^{+ik\int_{t^*}^{t} \frac{dt'}{a(t')}} \right) \\
&= \frac{|c_{\vec{k}}|^2}{a(t)} e^{-ik\int_{t^*}^{t} \frac{dt'}{a(t')}} \left(-\frac{\dot{a}}{a^2} e^{+ik\int_{t^*}^{t} \frac{dt'}{a(t')}} + \frac{ik}{a^2} e^{+ik\int_{t^*}^{t} \frac{dt'}{a(t')}} \right) \\
&= \frac{|c_{\vec{k}}|^2}{a^2} \left(-H + \frac{ik}{a} \right)
\end{aligned}$$

and,

$$\delta\phi^\star_{\vec{k}} \dot{\delta\phi}_{\vec{k}} = \left(\delta\phi_{\vec{k}} \dot{\delta\phi_{\vec{k}}}^\star \right)^\star = \frac{|c_{\vec{k}}|^2}{a^2} \left(-H - \frac{ik}{a} \right)$$

so that,

$$\delta\phi_{\vec{k}} \dot{\delta\phi_{\vec{k}}}^\star - \delta\phi^\star_{\vec{k}} \dot{\delta\phi}_{\vec{k}} \;=\; \frac{2ik|c_{\vec{k}}|^2}{a^3}$$

Then, for $t \to -\infty$,

$$\begin{aligned}
[\hat{\phi}(t,x), \hat{\pi}(t,y)] &= a^3(t) \int d^3\vec{k} \left(\delta\phi_{\vec{k}} \delta\dot{\phi}_{\vec{k}}^{\star} - \delta\phi_{\vec{k}}^{\star} \delta\dot{\phi}_{\vec{k}} \right) e^{ik_i(x^i - y^i)} \\
&= a^3(t) \int d^3\vec{k} \left(\frac{2ik|c_{\vec{k}}|^2}{a^3} \right) e^{ik_i(x^i - y^i)} \\
&= i \int d^3\vec{k} \left(2k|c_{\vec{k}}|^2 \right) e^{ik_i(x^i - y^i)}
\end{aligned}$$

and we wish this to have the flat space form,

$$[\hat{\phi}(t,x), \hat{\pi}(t,y)] = i\delta^{(3)}(\vec{x} - \vec{y})$$

Now recalling that,

$$(2\pi)^3 \delta^{(3)}(\vec{x} - \vec{y}) = \int d^3\vec{k} e^{ik_i(x^i - y^i)}$$

we see that we require,

$$2k|c_{\vec{k}}|^2 = \frac{1}{(2\pi)^3}$$

and hence,

$$|c_{\vec{k}}| = \frac{1}{(2\pi)^{3/2}\sqrt{2k}}$$

Choosing $c_{\vec{k}}$ to be real gives,

$$c_{\vec{k}} = \frac{1}{(2\pi)^{3/2}\sqrt{2k}}$$

Qu. 7 Assume inflation was nearly de Sitter, and that inflation ended instantaneously with the universe reheating at the GUT scale so the radiation era began at a temperature $\sim 10^{29}K$. Show that the number of e-folds, N, before the end of inflation when the comoving scale with wavenumber $\vec{k}$ left the inflationary 'de Sitter horizon' (so $|\vec{k}|/a = H$) that today corresponds to a physical scale $R_{phys} = a_0/k$ is,

$$e^N \simeq \frac{T_{rad}}{\sqrt{T_{eq}T_0}} H_0 R_{phys} \simeq \frac{R_{phys}}{0.2m}$$

Compute the number of e-folds before the end of inflation that the comoving scales left the inflationary de Sitter horizon that today correspond to the following physical scales;

1. the largest scales observable today ($\sim 10\,Gpc$).
2. $\sim 1^o$ on the sky at last scattering, or ($\sim 100Mpc$)
3. galaxy cluster scales ($\sim 10Mpc$)
4. galaxy scales ($\sim kpc$)
5. solar system scales ($\sim 10^{12}m$)

Qu. 7 answer The scale factor at the time t_{rad} when inflation ends and the radiation era starts is a_{rad} and since the temperature of photons goes as $T \sim 1/a$, then,

$$\frac{a_{rad}}{a_0} = \frac{T_0}{T_{rad}}$$

Now during inflation the scale factor goes as,

$$a = a_{rad} e^{H_{rad}(t - t_{rad})}$$

so that both a and H are continuous at t_{rad}, and $H = H_{rad}$ during inflation.

Then,

$$a = a_{rad} e^{-N(t)}$$

where $N(t)$ is the number of e-folds at time t before the end of inflation.

A comoving mode with wavenumber $\vec{k}$ corresponds to a physical scale today,

$$R_{phys} \sim \frac{a_0}{k}$$

with $k = |\vec{k}|$.

The mode leaves the inflationary de Sitter horizon when,

$$\frac{k}{a} = H$$

so since $H = H_{rad}$ during inflation, the mode leaves when,

$$H_{rad} = \frac{k}{a_{rad} e^{-N}}$$

Hence for a comoving physical scale today R_{phys} the scale exits the horizon N e-folds before the end of inflation, where,

$$H_{rad} = \frac{a_0}{R_{phys} a_{rad} e^{-N}}$$

so that,

$$e^N = a_{rad} H_{rad} \frac{R_{phys}}{a_0} = \frac{a_{rad} H_{rad}}{a_0 H_0} H_0 R_{phys}$$

In the radiation era we have $a \sim t^{1/2}$ and $H \sim 1/t$ so that $H \sim 1/a^2$ and then,

$$\frac{H_{rad}}{H_{eq}} = \frac{a_{eq}^2}{a_{rad}^2} = \left(\frac{T_{rad}}{T_{eq}}\right)^2$$

In the matter era we have $a \sim t^{2/3}$ and $H \sim 1/t$ so that $H \sim 1/a^{3/2}$ and then,

$$\frac{H_{eq}}{H_0} = \frac{a_0^{3/2}}{a_{eq}^{3/2}} = \left(\frac{T_{eq}}{T_0}\right)^{3/2}$$

Hence,

$$\frac{H_{rad}}{H_0} = \frac{a_0^{3/2}}{a_{eq}^{3/2}} = \left(\frac{T_{eq}}{T_0}\right)^{3/2} \left(\frac{T_{rad}}{T_{eq}}\right)^2$$

and then,

$$\begin{aligned}\frac{a_{rad}H_{rad}}{a_0H_0} &= \frac{a_{rad}}{a_0}\left(\frac{T_{eq}}{T_0}\right)^{3/2}\left(\frac{T_{rad}}{T_{eq}}\right)^2 \\ &= \frac{T_0}{T_{rad}}\left(\frac{T_{eq}}{T_0}\right)^{3/2}\left(\frac{T_{rad}}{T_{eq}}\right)^2 = \frac{T_{rad}}{\sqrt{T_{eq}T_0}}\end{aligned}$$

Then,

$$e^N = \frac{T_{rad}}{\sqrt{T_{eq}T_0}}H_0R_{phys}$$

Then recall that $T_{rad} = 10^{29}K$, $T_0 = 2.7K$ and $T_{eq} = 10^4K$.

Now $H_0 = 70kms^{-1}MPc^{-1}$, and hence,

$$\frac{1}{c}H_0 = \frac{1}{3\times 10^8ms^{-1}}70\times 10^3ms^{-1}MPc^{-1} \sim \frac{1}{4300}MPc^{-1}$$

so in $c = 1$ units,

$$\frac{1}{H_0} \sim 4300MPc$$

Then,

$$e^N = \frac{T_{rad}}{\sqrt{T_{eq}T_0}}H_0R_{phys} = \frac{10^{29}}{\sqrt{2.7\times 10^4}}\frac{R_{phys}}{4300Mpc} = 10^{23}\frac{R_{phys}}{Mpc} \simeq \frac{R_{phys}}{0.2m}$$

Hence for the largest scales today $\sim 10000Mpc$,

$$N = \log\left(10^{23}\times 10000\right) \sim 63$$

For 1^o on the last scattering surface, $\sim 100Mpc$ today,

$$N = \log\left(10^{23}\times 100\right) \sim 58$$

For galaxy cluster scales $\sim 10Mpc$ scales today,

$$N = \log\left(10^{23}\times 10\right) \sim 56$$

For galaxy scales $\sim kpc$ scales today,

$$N = \log\left(10^{23} \times \frac{10kpc}{Mpc}\right) \sim 49$$

And for solar system scales $\sim 10^{12}m$,

$$N = \log\left(\frac{10^{12}m}{0.2m}\right) \sim 29$$

Qu. 8 Show that the 2-point function of the fluctuations in the inflaton about the classical trajectory obeys,

$$\langle 0|\hat{\delta\phi}(t,x)\hat{\delta\phi}(t,y)|0\rangle = \int d^3\vec{k}|\delta\phi_{\vec{k}}(t)|^2 e^{ik_i(x^i-y^i)}$$

Hence the inflaton 2-point function in comoving Fourier space is given by

$$\langle 0|\hat{\phi}(t)\hat{\phi}(t)|0\rangle(\vec{k}) = |\delta\phi_{\vec{k}}(t)|^2$$

Assume that at time t a mode with comoving wavenumber $\vec{k}$ goes from sub horizon to super horizon. During this period we approximate H as being constant, so that $a \propto e^{Ht}$. Let us denote the value of H when a wave mode $\vec{k}$ exits the horizon (so $k = aH$) as H_k. Then show for t after the time of horizon exit, we have,

$$\langle 0|\hat{\phi}(t)\hat{\phi}(t)|0\rangle(\vec{k}) \simeq \frac{H_k^2}{2(2\pi k)^3}$$

Provided $H^2 \gg |V''(\phi_{cl})|$ and a mode has exited the horizon so that $k/a \ll H$, then the time dependence of the mode is governed by,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} \simeq 0$$

Use this to argue that then even if H does vary in time after a mode exits the inflationary horizon, the formula above for the 2-point function Fourier transform remains true.

Qu. 8 answers

$$\hat{\phi}(t,x) = \phi_{cl}(t) + \int d^3\vec{k}\,\delta\phi_{\vec{k}}(t)e^{ik_ix^i}a_{\vec{k}} + \delta\phi_{\vec{k}}(t)^\star e^{-ik_ix^i}a^\dagger_{\vec{k}}$$

so the fluctuation obeys,

$$\hat{\delta\phi}(t,x) = \int d^3\vec{k}\,\delta\phi_{\vec{k}}(t)e^{ik_ix^i}a_{\vec{k}} + \delta\phi_{\vec{k}}(t)^\star e^{-ik_ix^i}a^\dagger_{\vec{k}}$$

Then,

$$\begin{aligned}
\langle 0|\hat{\delta\phi}(t,x)\hat{\delta\phi}(t,y)|0\rangle &= \langle 0|\int d^3\vec{k}\,\delta\phi_{\vec{k}}(t)e^{ik_ix^i}a_{\vec{k}} + \delta\phi_{\vec{k}}(t)^\star e^{-ik_ix^i}a^\dagger_{\vec{k}} \\
&\qquad \int d^3\vec{k'}\,\delta\phi_{\vec{k'}}(t)e^{ik'_iy^i}a_{\vec{k'}} + \delta\phi_{\vec{k'}}(t)^\star e^{-ik'_iy^i}a^\dagger_{\vec{k'}}|0\rangle \\
&= \int d^3\vec{k}\int d^3\vec{k'}\,\delta\phi_{\vec{k}}(t)e^{ik_ix^i}\delta\phi_{\vec{k'}}(t)^\star e^{-ik'_iy^i}\langle 0|a_{\vec{k}}a^\dagger_{\vec{k'}}|0\rangle \\
&= \int d^3\vec{k}\int d^3\vec{k'}\,\delta\phi_{\vec{k}}(t)e^{ik_ix^i}\delta\phi_{\vec{k'}}(t)^\star e^{-ik'_iy^i}\langle 0|a^\dagger_{\vec{k'}}a_{\vec{k}} + [a_{\vec{k}},a^\dagger_{\vec{k'}}]|0\rangle \\
&= \int d^3\vec{k}\int d^3\vec{k'}\,\delta\phi_{\vec{k}}(t)e^{ik_ix^i}\delta\phi_{\vec{k'}}(t)^\star e^{-ik'_iy^i}\langle 0|[a_{\vec{k}},a^\dagger_{\vec{k'}}]|0\rangle \\
&= \int d^3\vec{k}\int d^3\vec{k'}\,\delta\phi_{\vec{k}}(t)e^{ik_ix^i}\delta\phi_{\vec{k'}}(t)^\star e^{-ik'_iy^i}\delta^{(3)}(\vec{k}-\vec{k'})\langle 0|0\rangle \\
&= \int d^3\vec{k}\,\delta\phi_{\vec{k}}(t)e^{ik_ix^i}\delta\phi_{\vec{k}}(t)^\star e^{-ik_iy^i} \\
&= \int d^3\vec{k}\,\left|\delta\phi_{\vec{k}}(t)\right|^2 e^{ik_i(x^i-y^i)}
\end{aligned}$$

Assume de Sitter behaviour with Hubble parameter H_k corresponding to the Hubble constant when the mode with wavenumber $\vec{k}$ crosses the horizon, so $a = e^{H_kt}$.

Then from the earlier questions 5 and 6 we have,

$$\delta\phi_{\vec{k}}(t) = \frac{c_{\vec{k}}}{a(t)}e^{-\frac{ik}{a(t)H_k}}\left(1 - \frac{ia(t)H_k}{k}\right)$$

with $c_{\vec{k}} = 1/((2\pi)^{3/2}\sqrt{2k})$ so,

$$\delta\phi_{\vec{k}}(t) = \frac{1}{(2\pi)^{3/2}\sqrt{2k}\,a(t)}e^{-\frac{ik}{a(t)H_k}}\left(1 - \frac{ia(t)H_k}{k}\right)$$

so at times after the mode leaves the horizon so $k/a \ll H_k$, then,

$$\begin{aligned}\delta\phi_{\vec{k}}(t) &\simeq \frac{1}{(2\pi)^{3/2}\sqrt{2k}\,a(t)}\left(1-\frac{iH_k}{k/a}\right)\\ &\simeq -\frac{1}{(2\pi)^{3/2}\sqrt{2k}\,a}\frac{iH_k}{k/a}\\ &\simeq -\frac{iH_k}{(2\pi k)^{3/2}\sqrt{2}}\end{aligned}$$

Then,

$$\left|\delta\phi_{\vec{k}}\right|^2 \simeq \frac{H_k^2}{2(2\pi k)^3}$$

Now once a mode has left the horizon, provided we can ignore the $V''(\phi_{cl})$ term and k^2/a^2 term in,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} + V''(\phi_{cl})\delta\phi_{\vec{k}} = 0$$

as the question tells us, so that,

$$0 = \ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} = \ddot{\delta\phi}_{\vec{k}} + 3\frac{\dot{a}}{a}\dot{\delta\phi}_{\vec{k}}$$

then the solutions are

$$\delta\phi_{\vec{k}} = b_{\vec{k}} + c_{\vec{k}}\frac{1}{a^3(t)}$$

for constants $b_{\vec{k}}$ and $c_{\vec{k}}$, for any time dependence of H. Now just after exit when $H \sim H_{\vec{k}}$ then,

$$\delta\phi_{\vec{k}}(t) = -\frac{iH_k}{(2\pi k)^{3/2}\sqrt{2}}$$

and so is approximately constant in time, and hence,

$$b_{\vec{k}} \simeq -\frac{iH_k}{(2\pi k)^{3/2}\sqrt{2}}\ , \qquad c_{\vec{k}} \simeq 0$$

Thus afterwards during inflation even if H varies we still have,

$$\delta\phi_{\vec{k}}(t) \simeq -\frac{iH_k}{(2\pi k)^{3/2}\sqrt{2}}$$

as even if $c_{\vec{k}}$ is not exactly zero, as $a(t)$ is rapidly increasing that term quickly becomes irrelevant anyway.

Qu. 9 Consider the quadratic potential,

$$V(\phi) = m^2\phi^2$$

Show the slow roll conditions imply that slow roll may occur when the field is sufficiently far from the minimum $\phi = 0$ so that,

$$1 \ll \sqrt{G}|\phi| \tag{3}$$

Assume that the inflaton starts far from the minimum at ϕ_0 and slow roll inflation occurs and ends when the above slow roll condition is violated, ie. when $1 \sim \sqrt{G}\phi$. Compute the number of e-folds of inflation as a function of ϕ_0.

For this model consider an inflaton fluctuation with comoving wavenumber $\vec{k}$ that exits the inflationary 'horizon' at time $t = t_k$ when the scalar is at $\phi(t_k) = \phi_k$ and the Hubble parameter is $H = H_k$. Recall from lectures that the temperature fluctuation on the comoving scale $\vec{k}$ is estimated by,

$$\frac{\delta T}{T} \simeq \frac{H_k^2}{|\dot{\phi}_k|} \tag{4}$$

Assume the universe reheats at the GUT scale (with temperature $T = 10^{29}K$). Recall (from Qu 7) that the modes on the largest scales today left the inflationary horizon around ~ 60 e-folds before the end of inflation. If we wish to have $\delta T/T \sim 10^{-5}$ for these longest modes in order to account for the temperature fluctuations in the CMB on the largest scales, then give an estimate for the mass m. You should find that the mass is constrained to be orders of magnitude below the Planck scale (ie. $\sqrt{G}m \ll 1$).

Qu. 9 answers

The first slow roll condition is,

$$\frac{|V'|}{\sqrt{G}V} \ll 1 \tag{5}$$

and the second is,

$$\frac{|V''|}{GV} \ll 1 \tag{6}$$

and really there are an infinite number,

$$\left| \frac{d^{p+1}}{d\phi^{p+1}} V \right| \ll \left| \sqrt{G} \frac{d^p}{d\phi^p} \right| \tag{7}$$

For a quadratic potential with $V' = 2m^2\phi$ and $V'' = 2m^2$, these simply yield,

$$\frac{V'}{\sqrt{G}V} = \frac{2m^2\phi}{\sqrt{G}m^2\phi^2} = \frac{2}{\sqrt{G}\phi} \, , \qquad \frac{V''}{GV} = \frac{2m^2}{Gm^2\phi^2} = \frac{2}{G\phi^2} \tag{8}$$

and hence these both conditions imply the same condition, that,

$$1 \ll \sqrt{G}|\phi| \tag{9}$$

and hence the scalar must be sufficiently far from the minimum of the potential for slow roll to occur. Near the bottom of the potential the field will start to move too quickly.

Recall from lectures that the number of e-folds of inflation for ϕ moving from its starting point ϕ_0 to the point where it ends slow roll ϕ_1 so that,

$$1 = \sqrt{G}\phi_1 \tag{10}$$

is then,

$$\begin{aligned} N &= -\int_{\phi_0}^{\phi_1} \frac{8\pi G V}{V'} = -\int_{\phi_0}^{\phi_1} \frac{8\pi G \times m^2\phi^2}{2m^2\phi} = -4\pi G \int_{\phi_0}^{\phi_1} \phi \\ &= -4\pi G \left[\frac{1}{2}\phi^2 \right]_{\phi_0}^{\phi_1} = 2\pi G \left({\phi_0}^2 - {\phi_1}^2 \right) \simeq 2\pi G \left({\phi_0}^2 - \frac{1}{G} \right) \\ &\simeq 2\pi \left(G{\phi_0}^2 - 1 \right) \end{aligned} \tag{11}$$

and we want,

$$N = 2\pi \left(G{\phi_0}^2 - 1 \right) \gg 1 \tag{12}$$

and hence $\sqrt{G}|\phi_0| \gg 1$ which is compatible with the slow roll condition.

During slow roll inflation we have,

$$H^2 \simeq \frac{8\pi G}{3} V \ , \qquad \dot{\phi} = -\frac{1}{3H} V'(\phi) \tag{13}$$

and let us call ϕ_k the value of ϕ when a mode $\vec{k}$ leaves the horizon, so we have,

$$H_k^2 \simeq \frac{8\pi G}{3} V(\phi_k) \ , \qquad \dot{\phi}_k = -\frac{1}{3H_k} V'(\phi_k) \tag{14}$$

and then,

$$\begin{aligned} \frac{\delta T}{T} \simeq \frac{H_k^2}{|\dot{\phi}_k|} &= \frac{3H_k^3}{|V'(\phi_k)|} = \frac{3}{|V'(\phi_k)|} \left(\frac{8\pi G}{3} V(\phi_k) \right)^{3/2} \\ &= \frac{3}{2m^2|\phi_k|} \left(\frac{8\pi G}{3} m^2 \phi_k^2 \right)^{3/2} \\ &= \frac{3}{2} \left(\frac{8\pi G}{3} \right)^{3/2} m\phi_k^2 \\ &\sim \sqrt{G} m (\sqrt{G}\phi_k)^2 \end{aligned} \tag{15}$$

dropping the $O(1)$ constants and only keeping parametric dependence.

The modes corresponding to the largest scales today left the inflationary horizon around 60 e-folds before the end of inflation. The number of e-folds $N(\phi)$ after the inflaton reaches ϕ in the potential is, from the above,

$$N(\phi) = 2\pi \left(G\phi^2 - 1 \right) \tag{16}$$

and so the scalar was as,

$$60 = 2\pi \left(G\phi^2 - 1 \right) \tag{17}$$

so,

$$\sqrt{G}\phi \simeq 3.2 \tag{18}$$

Then, for the modes that leave 60 e-folds and re-enter today on the largest scales, we have ϕ_k so that $\sqrt{G}\phi_k \simeq 3.2$ and then for these modes,

$$\begin{aligned} \frac{\delta T}{T} &\simeq \sqrt{G}m(\sqrt{G}\phi_k)^2 \simeq \sqrt{G}m(3.2)^2 \\ &\simeq \sqrt{G}m \end{aligned} \tag{19}$$

again dropping order one numbers. We require $\frac{\delta T}{T} \sim 10^{-5}$ and hence we need m to be constrained to parametrically be of order,

$$\sqrt{G}m \sim 10^{-5} \tag{20}$$