**Cosmology and Particle Physics** **Toby Wiseman**

**Example sheet 4**

**Qu. 1** Assume flat FRW, then in the radiation era $a = a_{eq} \left( t/t_{eq} \right)^{1/2}$ where $t_{eq}$ is the time of matter-radiation equality. Show that assuming an instantaneous transition from radiation to matter era at $t = t_{eq}$ that in the subsequent matter era,

$$a = a_{eq} \left( \frac{3t + t_{eq}}{4t_{eq}} \right)^{2/3}$$

Assuming the radiation era extends back to a big bang at $t = 0$, compute the particle horizon size at time $t$ in the matter era showing that the proper size at time $t > t_{eq}$ in the matter era is,

$$d_H(t) = \frac{2}{H} - \frac{a}{a_{eq} H_{eq}}$$

Ignore the presence of the dark energy today so that today the scale factor still evolves as for matter domination. Take $t_{eq}$ to be at redshift $Z = Z_{eq} \sim 3600$ and last scattering to be at $Z = Z_{ls} \sim 1100$. Then show that the particle horizon at last scattering subtends an angle $\theta$ on the sky which is,

$$\theta \simeq \frac{360^o}{2\pi} \frac{1}{\sqrt{Z_{ls}}}$$

Hence at last scattering the scales that today are less than $\sim 1^o$ on the sky were in causal contact in the early radiation era, but larger scales were not.

**Qu. 2** As in the previous question, neglect dark energy today and assume instantaneous transitions from matter to radiation, and from radiation to an inflationary era which you may approximate as de Sitter. Using this, compute the number of e-folds required to solve the horizon problem if the universe reheated just above nucleosynthesis temperatures, $\sim 10^{10} K$ (ie. $\sim 1 MeV$), or at an intermediate scale $\sim 10^{23}$ (ie. $\sim 10^{10} GeV$), or at the GUT scale $10^{29} K$ (ie. $\sim 10^{16} GeV$).

**Qu. 3** The inflaton is a scalar field $\phi$ with equation of motion,

$$\nabla^2 \phi = V'(\phi)$$

and stress tensor,

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left( \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right)$$

Show that assuming homogeneity and isotropy in a flat FRW spacetime, $ds^2 = -dt^2 + a(t)^2 dx^i dx^j$, so that $\phi = \phi(t)$, then the scalar field equation yields,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0$$

and the Einstein equations yield,

$$H^2 = \frac{8\pi G}{3} \left( \frac{1}{2}\dot{\phi}^2 + V(\phi) \right)$$

and,

$$\dot{H} = -4\pi G \dot{\phi}^2$$

**Qu. 4** Define the slow roll function;

$$\epsilon(\phi) = \frac{1}{16\pi G} \left( \frac{V'(\phi)}{V(\phi)} \right)^2$$

Over a range of $\phi$ the potential supports slow roll inflation, so $\epsilon(\phi) \ll 1$. Suppose that $V$ is monotonic over this range so we have $\phi = \phi(V)$ and so can think of the function $\epsilon(V)$, and further that $\epsilon(V)$ is well approximated by,

$$\epsilon(V) \simeq \epsilon(V_0) + \epsilon'(V_0)(V - V_0) + \frac{1}{2}\epsilon''(V_0)(V - V_0)^2$$

over this range, with $\phi_0$ some value of the scalar within the range, and $V_0$ the potential at this point.

Assume that over this scalar range the potential varies so that $V/V_0 \sim O(1)$. Firstly show that the parameter characterizing second derivatives of the potential at $\phi_0$,

$$\eta(\phi_0) = \frac{1}{8\pi G} \left| \frac{V''(\phi)}{V} \right|_{\phi=\phi_0} \ll 1$$

must be small. Then show that the parameter characterizing third derivatives is also small, so,

$$\gamma(\phi_0) = \frac{1}{(8\pi G)^{3/2}} \left| \frac{V'''(\phi)}{V} \right|_{\phi=\phi_0} \ll 1 \,.$$

**Qu. 5** Linearize the inflaton equation of motion in a fixed flat FRW background,

$$\nabla^2\phi = V'(\phi)$$

about a homogeneous isotropic classical solution $\phi_{cl}(t)$. Show that if we ignore back reaction on the metric, then perturbations to the inflaton which are not homogeneous or isotropic, so that $\phi(t,x) = \phi_{cl} + \delta\phi(t,x)$, obey,

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{a(t)^2}\delta^{ij}\partial_i\partial_j\delta\phi + V''(\phi_{cl})\delta\phi = 0$$

Show that we may write a solution to this as,

$$\delta\phi_{\vec{k}}(t,x) = \delta\phi_{\vec{k}}(t)e^{ik_i x^i}$$

where $k = |\vec{k}| = \sqrt{\delta^{ij}k_i k_j}$ is the comoving wavenumber and find the ordinary differential equation that the time dependence given by $\delta\phi_{\vec{k}}(t)$ obeys. Show that for fixed $\vec{k}$, then for $t \to -\infty$ such that $a \to 0$ and $k/a \gg H$, then provided the FRW background obeys the slow roll conditions, we may write a "WKB solution" as,

$$\delta\phi_{\vec{k}}(t) = \frac{c_{\vec{k}}}{a(t)}e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}}\left(1 + O\left(\frac{a}{k}\right)\right)$$

for some time $t^*$ such that $k/a \gg H$ at that time, and $c_{\vec{k}}$ is an integration constant.

Confirm that for an exact de Sitter background, so $a = e^{Ht}$ with constant $H$ and $V'' = 0$, that the full solution to the perturbation equation for $\delta\phi_k(t)$ is,

$$\delta\phi_{\vec{k}}(t) = \frac{c_{\vec{k}}}{a(t)}e^{-\frac{ik}{a(t)H}}\left(1 - \frac{ia(t)H}{k}\right)$$

**Qu. 6** As in lectures we quantise the (real) inflaton scalar field $\phi$ in flat FRW as,

$$\hat{\phi}(t,x) = \phi_{cl}(t) + \int d^3\vec{k}\delta\phi_{\vec{k}}(t)e^{ik_ix^i}a_{\vec{k}} + \delta\phi_{\vec{k}}(t)^{\star}e^{-ik_ix^i}a^{\dagger}_{\vec{k}}$$

where $\delta\phi_{\vec{k}}(t)$ obeys,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} + \frac{k^2}{a^2}\delta\phi_{\vec{k}} + V''(\phi_{cl})\delta\phi_{\vec{k}} = 0$$

with appropriate boundary conditions. Show that if we choose the creation/annhilation operators so that,

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0 \qquad [a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta^{(3)}(\vec{k} - \vec{k}')$$

then the field obeys the equal time commutation relations,

$$[\hat{\phi}(t,x), \hat{\phi}(t,y)] = 0$$

and,

$$[\hat{\phi}(t,x), \hat{\pi}(t,y)] = a^3(t)\int d^3\vec{k}\left(\delta\phi_{\vec{k}}\dot{\delta\phi}_{\vec{k}}^{\star} - \delta\phi^{\star}_{\vec{k}}\dot{\delta\phi}_{\vec{k}}\right)e^{ik_i(x^i-y^i)}$$

where we recall that the conjugate momentum $\hat{\pi} = a^3(t)\dot{\hat{\phi}}$.

For $t \to -\infty$ and slow roll inflation (so that $a \to 0$ and $H \simeq$ constant) we may use the WKB approximation,

$$\delta\phi_{\vec{k}}(t) \simeq \frac{c_{\vec{k}}}{a(t)}e^{-ik\int_{t^*}^{t}\frac{dt'}{a(t')}}$$

Then in order to obtain the conventional equal time flat space commutator as $t \to -\infty$,

$$[\hat{\phi}(t,x), \hat{\pi}(t,y)] = i\delta^{(3)}(\vec{x} - \vec{y})$$

show that the modes must be normalised so that,

$$|c_{\vec{k}}| = \frac{1}{(2\pi)^{3/2}\sqrt{2k}}$$

(and we conventionally choose $c_{\vec{k}}$ to be real).

**Qu. 7** Assume inflation was nearly de Sitter, and that inflation ended instantaneously with the universe reheating at the GUT scale so the radiation era began at a temperature $\sim 10^{29}K$. Show that the number of e-folds, $N$, before the end of inflation when the comoving scale with wavenumber $\vec{k}$ left the inflationary 'de Sitter horizon' (so $|\vec{k}|/a = H$) that today corresponds to a physical scale $R_{phys} = a_0/k$ is,

$$e^N \simeq \frac{T_{rad}}{\sqrt{T_{eq}T_0}} H_0 R_{phys} \simeq \frac{R_{phys}}{0.2m}$$

Compute the number of e-folds before the end of inflation that the comoving scales left the inflationary de Sitter horizon that today correspond to the following physical scales;

1. the largest scales observable today ($\sim 10\,Gpc$).
2. $\sim 1^o$ on the sky at last scattering, or ($\sim 100Mpc$)
3. galaxy cluster scales ($\sim 10Mpc$)
4. galaxy scales ($\sim kpc$)
5. solar system scales ($\sim 10^{12}m$)

**Qu. 8** Show that the 2-point function of the fluctuations in the inflaton about the classical trajectory obeys,

$$\langle 0|\hat{\delta\phi}(t,x)\hat{\delta\phi}(t,y)|0\rangle = \int d^3\vec{k}|\delta\phi_{\vec{k}}(t)|^2 e^{ik_i(x^i-y^i)}$$

Hence the inflaton 2-point function in comoving Fourier space is given by

$$\langle 0|\hat{\phi}(t)\hat{\phi}(t)|0\rangle(\vec{k}) = |\delta\phi_{\vec{k}}(t)|^2$$

Assume that at time $t$ a mode with comoving wavenumber $\vec{k}$ goes from sub horizon to super horizon. During this period we approximate $H$ as being constant, so that $a \propto e^{Ht}$. Let us denote the value of $H$ when a wave mode $\vec{k}$ exits the horizon (so $k = aH$) as $H_k$. Then show for $t$ after the time of horizon exit, we have,

$$\langle 0|\hat{\phi}(t)\hat{\phi}(t)|0\rangle(\vec{k}) \simeq \frac{H_k^2}{2(2\pi k)^3}$$

Provided $H^2 \gg |V''(\phi_{cl})|$ and a mode has exited the horizon so that $k/a \ll H$, then the time dependence of the mode is governed by,

$$\ddot{\delta\phi}_{\vec{k}} + 3H\dot{\delta\phi}_{\vec{k}} \simeq 0$$

Use this to argue that then even if $H$ does vary in time after a mode exits the inflationary horizon, the formula above for the 2-point function Fourier transform remains true.

**Qu. 9** Consider the quadratic potential,

$$V(\phi) = m^2 \phi^2$$

Show the slow roll conditions imply that slow roll may occur when the field is sufficiently far from the minimum $\phi = 0$ so that,

$$1 \ll \sqrt{G} |\phi| \tag{1}$$

Assume that the inflaton starts far from the minimum at $\phi_0$ and slow roll inflation occurs and ends when the above slow roll condition is violated, ie. when $1 \sim \sqrt{G}\phi$. Compute the number of e-folds of inflation as a function of $\phi_0$.

For this model consider an inflaton fluctuation with comoving wavenumber $\vec{k}$ that exits the inflationary 'horizon' at time $t = t_k$ when the scalar is at $\phi(t_k) = \phi_k$ and the Hubble parameter is $H = H_k$. Recall from lectures that the temperature fluctuation on the comoving scale $\vec{k}$ is estimated by,

$$\frac{\delta T}{T} \simeq \frac{H_k^2}{|\dot{\phi}_k|} \tag{2}$$

Assume the universe reheats at the GUT scale (with temperature $T = 10^{29} K$). Recall (from Qu 7) that the modes on the largest scales today left the inflationary horizon around $\sim 60$ e-folds before the end of inflation. If we wish to have $\delta T / T \sim 10^{-5}$ for these longest modes in order to account for the temperature fluctuations in the CMB on the largest scales, then give an estimate for the mass $m$. You should find that the mass is constrained to be orders of magnitude below the Planck scale (ie. $\sqrt{G} m \ll 1$).