# Example sheet 3

# Answers

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**Example sheet 3**

**Qu. 1 Note: Have slightly modified this question to hopefully make it more accessible.**
Consider the temperature $T \sim 10^{11} K$ where the light degrees of freedom are photons, $e$, $\bar{e}$ and the 3 generations of $\nu$, $\bar{\nu}$ which we assume to be all in thermal equilibrium. Show that the effective number of degrees of freedom (counted as bosonic) is $g_{eff} = 43/4$, so that,

$$\rho = \frac{1}{2} g_{eff} a_B T^4 \tag{1}$$

Assume neutrinos decouple between $10^{10} K$ and $10^{11} K$, and approximate the electrons/positions as still being light in this temperature range.

For temperatures well below $10^{10} K$ the electrons/positons have annihilated. Show that then the effective number of degrees of freedom drops to,

$$g_{eff} = 2 + \frac{21}{4} \times \left(\frac{4}{11}\right)^{\frac{4}{3}} \simeq 3.36 \tag{2}$$

Show that the scale factor at temperature $T$ is related to the current scale factor, $a_0$, and today's CMB photon temperature as,

$$\frac{T}{T_{CMB}} = \frac{a_0}{a(T)} , \qquad T \ll 10^{10} K \tag{3}$$

and

$$\frac{T}{T_{CMB}} \simeq \left(\frac{4}{11}\right)^{\frac{1}{3}} \frac{a_0}{a(T)} , \qquad 10^{10} K < T < 10^{11} K \tag{4}$$

**(note typo in previous version** $4/11$ **rather than** $11/4$**).**

Now consider very high temperatures above the weak scale $T \gg 10^{15} K$. Then the relativistic degrees of freedom are the photons, the gluons, and 3 generations of neutrinos, electrons and quarks, the $W, Z$ and the Higgs. Compute the $g_{eff}$ and the corresponding constant $\alpha$ in,

$$\frac{T}{T_{CMB}} \simeq \alpha \frac{a_0}{a(T)} , \qquad T \gg 10^{15} K \tag{5}$$

**Qu. 1 answer**

Recall from Qu 1 of sheet 2 for each relativistic boson degree of freedom; $\rho_{boson} = \frac{1}{2} a_B T^4$. For each relativistic fermonic $\rho_{fermion} = \frac{7}{8}\rho_{boson}$.

Then for photons, $e$, $\bar{e}$ and the 3 generations of $\nu$, $\bar{\nu}$ at around $10^{11}K$ when all are in equilibrium we have a total,

$$\rho = \frac{1}{2}\left(g_\gamma\right) a_B T^4 + \frac{1}{2}\frac{7}{8}\left(g_{e,\bar{e}} + g_{\nu,\bar{\nu}}\right) a_B T^4 \tag{6}$$

where the photons have 2 polarizations so $g_\gamma = 2(polarisation)$; electrons and anti electrons each have 2 helicities so $g_{e,\bar{e}} = 2(particle/antiparticle) \times 2(helicity) = 4$; and each generation of neutrinos and antineutrinos contributes a degree of freedom (remember neutrinos are left handed so the particle and antiparticle only have one helicity degree of freedom) so $g_{\nu,\bar{\nu}} = 3(generation) \times 2(particle/antiparticle) = 6$. So,

$$\begin{aligned}
\rho &= \frac{1}{2}\left(g_\gamma + \frac{7}{8}\left(g_{e,\bar{e}} + g_{\nu,\bar{\nu}}\right)\right) a_B T^4 \\
&= \frac{1}{2}\left(2 + \frac{7}{8}(4+6)\right) a_B T^4 \\
&= \frac{1}{2}\left(\frac{43}{4}\right) a_B T^4 = \frac{1}{2} g_{eff} a_B T^4
\end{aligned} \tag{7}$$

Between $10^{10}K - 10^{11}K$ the neutrinos decouple. They maintain a fermi distribution with temperature $T_\nu \sim 1/a$. Until the electrons/position mass becomes relevant we maintain $T = T_\nu$. Then for $T \ll 10^{10}K$ the electrons/positions annihilate so;

$$\begin{aligned}
\rho &= \frac{1}{2} g_\gamma a_B T^4 + \frac{1}{2}\frac{7}{8} g_{\nu,\bar{\nu}} a_B T_\nu^4 \\
&= \frac{1}{2}\left(g_\gamma + \frac{7}{8} g_{\nu,\bar{\nu}}\left(\frac{T_\nu}{T}\right)^4\right) a_B T^4
\end{aligned} \tag{8}$$

Entropy for the thermal components in equilibrium is conserved. Recall for a relativistic species $s = 4\rho/(3T)$. So at $T \sim 10^{10}K$ where electrons/positions are relativistic we have,

$$\begin{aligned}
s(T) &= \frac{4}{3T}\frac{1}{2}\left(g_\gamma + \frac{7}{8} g_{e,\bar{e}}\right) a_B T^4 \\
&= \frac{2}{3}\left(2 + \frac{7 \times 4}{8}\right) a_B T^3 = \frac{2}{3}\left(\frac{11}{2}\right) a_B T^3
\end{aligned} \tag{9}$$

And at this temperature $T \simeq T_\nu$ so,

$$s(T) = \frac{2}{3}\left(\frac{11}{2}\right) a_B T_\nu^3 \quad (10)$$

But for $T \ll 10^{10}K$ when the electrons/positions annihilate we have,

$$\begin{aligned} s(T) &= \frac{4}{3T}\frac{1}{2}\left(g_\gamma\right) a_B T^4 \\ &= \frac{2}{3}\left(2\right) a_B T^3 \end{aligned} \quad (11)$$

and $T \neq T_\nu$.

Assuming adiabatic expansion we have $sa^3$ =constant. For all $T < 10^{11}K$ we have $a \sim 1/T_\nu$. And hence we have,

$$\frac{s(T)}{T_\nu^3} = C \quad (12)$$

for a constant $C$. For $T \sim 10^{10}K$ using the above $s(T) = \frac{2}{3}\left(\frac{11}{2}\right) a_B T_\nu^3$ then,

$$C = \frac{2}{3}\left(\frac{11}{2}\right) a_B \quad (13)$$

Then for $T \ll 10^{10}K$,

$$\begin{aligned} \frac{2}{3}\left(\frac{11}{2}\right) a_B &= \frac{s(T)}{T_\nu^3} \\ &= \frac{2}{3}\left(2\right) a_B \frac{T^3}{T_\nu^3} \end{aligned} \quad (14)$$

Hence we derive,

$$\frac{T^3}{T_\nu^3} = \frac{11}{4} \quad (15)$$

Then recall that from above for $T \ll 10^{10}K$ we have,

$$\begin{aligned} \rho &= \frac{1}{2}\left(g_\gamma + \frac{7}{8} g_{\nu,\bar{\nu}}\left(\frac{T_\nu}{T}\right)^4\right) a_B T^4 \\ &= \frac{1}{2}\left(g_\gamma + \frac{7}{8} g_{\nu,\bar{\nu}}\left(\frac{4}{11}\right)^{\frac{4}{3}}\right) a_B T^4 \\ &= \frac{1}{2} g_{eff} a_B T^4 \end{aligned} \quad (16)$$

so that,

$$\begin{aligned} g_{eff} &= g_\gamma + \frac{7}{8} g_{\nu,\bar{\nu}} \left(\frac{4}{11}\right)^{\frac{4}{3}} \\ &= 2 + \frac{7}{8} 3 \times 2 \left(\frac{4}{11}\right)^{\frac{4}{3}} \\ &\simeq 3.36 \end{aligned} \tag{17}$$

Now for temperatures $T \ll 10^{10}K$ we have $T \sim 1/a$ in the radiation era. Recall that thermal equilibrium of photons is maintained until they decouple around $T \sim 10^5 K$ in the radiation era (recall $T_{eq} \sim 10^4 K$). After that $T \sim 1/a$ as the photons free stream even in the matter era. Hence for all $T \ll 10^{10}K$ we have $T \sim 1/a$. Hence,

$$\frac{T}{T_{CMB}} = \frac{a_0}{a(T)} \tag{18}$$

where $T_{CMB}$ is today's CMB temperature and $a_0$ is today's scale factor.

However, looking to higher temperatures $10^{10}K < T < 10^{11}K$ then since during the period where electrons/positions annihilate they do not behave relativistically and hence we do not have exactly $T \sim 1/a$ during that period. However we know that $T_\nu \sim 1/a$ as the neutrinos have decoupled. Hence we have,

$$\frac{T_\nu(T)}{T_{\nu,0}} = \frac{a_0}{a(T)} \tag{19}$$

We also know that between $10^{10}K < T < 10^{11}K$ then neutrinos were in thermal equilibrium and electrons/positions were light and hence then $T = T_\nu$. From above we also know that for $T \ll 10^{10}K$ that $T/T_\nu = (11/4)^{1/3}$. Thus, $T_0/T_{\nu,0} = (11/4)^{1/3}$. Hence for $10^{10}K < T < 10^{11}K$, where $T_\nu = T$,

$$\frac{T}{\left(\frac{4}{11}\right)^{1/3} T_0} = \frac{a_0}{a(T)} \tag{20}$$

so, $(T_0 = T_{CMB})$,

$$\frac{T}{T_{CMB}} = \left(\frac{4}{11}\right)^{\frac{1}{3}} \frac{a_0}{a(T)} \tag{21}$$

Now going to much higher temperatures $T \gg 10^{15}K$ we include all the generations of quarks and gluons, Higgs W, Z. We must also remember 3 generations of electrons, so now,

$$g_{e,\bar{e}} = 2 \times 2 \times 3 \tag{22}$$

The gluons are gauge bosons with 2 polarizations (as for the photon) but there are eight of them for $SU(3)$. Then the 3 generations of quarks and antiquarks, each with two helicities and each with 3 colours. Hence the bosonic gluons have,

$$g_{gluon} = 2 \times 8 \tag{23}$$

and fermionic quarks,

$$g_{q,\bar{q}} = 3(generations) \times 2(particle/anti) \times 2(helicity) \times 3(colour) \tag{24}$$

And we can count the Higgs and W, Z degrees of freedom above the symmetry breaking scale as the gauge bosons of $SU(2)$ (helicity 2, and 3 generators). The Higgs is a complex scalar in the fundamental of $SU(2)$ so $2(complex) \times 2(SU(2))$ components. These are all bosons so,

$$g_{W,Z,Higgs} = 2 \times 3 + 2 \times 2 \tag{25}$$

So,

$$\begin{aligned} g_{eff} &= g_\gamma + g_{gluon} + g_{W,Z,Higgs} + \frac{7}{8}\left(g_{e,\bar{e}} + g_{\nu,\bar{\nu}} + g_{q,\bar{q}}\right) \\ &= 2 + 2 \times 8 + 2 \times 3 + 2 \times 2 + \frac{7}{8}\left(2 \times 2 \times 3 + 2 \times 3 + 3 \times 2 \times 2 \times 3\right) \\ &= 28 + \frac{7}{8}(54) = \frac{301}{4} \end{aligned} \tag{26}$$

Let us denote this $g_{UV}$.

At very high temperature,

$$s = \frac{2}{3} g_{UV} a_B T^3 \tag{27}$$

Then at temperatures $T \sim 10^{11}K$ we know that,

$$s = \frac{2}{3}\frac{43}{4} a_B T^3 \tag{28}$$

Now for the whole period above $T > 10^{11}K$ we can assume thermal equilibrium and adiabatic expansion. Hence $sa^3$ =constant. Hence,

$$\frac{2}{3}g_{UV}a_B T^3 a(T)^3 = \frac{2}{3}\frac{43}{4}a_B T_1^3 a(T_1)^3 \tag{29}$$

where $T_1 = 10^{11}K$. Hence,

$$\frac{T}{T_1} = \left(\frac{43}{4g_{UV}}\right)^{\frac{1}{3}} \frac{a(T_1)}{a(T)} \tag{30}$$

Now we know,

$$\frac{T_1}{T_{CMB}} = \left(\frac{4}{11}\right)^{\frac{1}{3}} \frac{a_0}{a(T_1)} \tag{31}$$

Hence, for very high $T$,

$$\begin{aligned}
\frac{T}{T_{CMB}} &= \frac{T}{T_1}\frac{T_1}{T_{CMB}} = \left(\frac{43}{4g_{UV}}\right)^{\frac{1}{3}} \frac{a(T_1)}{a(T)} \left(\frac{4}{11}\right)^{\frac{1}{3}} \frac{a_0}{a(T_1)} \\
&= \left(\frac{43}{11g_{UV}}\right)^{\frac{1}{3}} \frac{a_0}{a(T)} \\
&= \left(\frac{43 \times 4}{11 \times 301}\right)^{\frac{1}{3}} \frac{a_0}{a(T)} \\
&= \left(\frac{4}{77}\right)^{\frac{1}{3}} \frac{a_0}{a(T)}
\end{aligned} \tag{32}$$

**Qu. 2** Consider a WIMP dark matter model where a massive scalar particle $X$ and its antiparticle $\bar{X}$ are initially in thermal equilibrium, and decay in a $2 \rightarrow 2$ process to light (effectively massless) scalar particles with rate,

$$\Gamma_X = \frac{\lambda^2}{m_X^2} n_X \tag{33}$$

**(note typo in previous version - missing $n_X$ above)** with $m_X$ the mass of the particle and its antiparticle, $n_X$ the number density, and $\lambda$ is the (dimensionless) coupling governing the 4 scalar particle interaction.

Assume that freeze out occurs at $T = T_{freeze}$ such that $kT_{freeze} \ll m_X$. Let us assume that $m_X = 1GeV$. What is the approximate temperature at which freeze out happens in this model in order to produce the correct relic density to act as today's dark matter? Find the value of the coupling $\lambda$ (note that in this part of the analysis you can use the appropriate number of degrees of freedom in the radiation era from the previous question)?

**Qu. 2 answer**

Freeze out occurs at;

$$1 \sim \frac{\Gamma_X}{H} \tag{34}$$

Above $T > T_{freeze}$ $X$ will have a non-relativistic thermal distribution (assume no chemical potential), so its number density (from sheet 2) will be,

$$n = g e^{-\frac{m}{kT}} \left( \frac{mkT}{2\pi} \right)^{\frac{3}{2}} \tag{35}$$

For a scalar we have $g = 2$ since there is the particle and antiparticle.

Below freeze out the X particles simply dilute with the scale factor. Hence we approximate for $T < T_{freeze}$,

$$n(T) = n(T_{freeze}) \left( \frac{a_{freeze}}{a(T)} \right)^3 = n(T_{freeze}) \left( \frac{T}{T_{freeze}} \right)^3 \tag{36}$$

taking $T \sim 1/a$ for the photons. Hence today the relic density is $\rho_{DM} = m_X n(T_{CMB})$, so,

$$\begin{aligned} \rho_{DM} &= m_X n(T_{freeze}) \left( \frac{T_{CMB}}{T_{freeze}} \right)^3 \\ &= m_X \left( \frac{T_{CMB}}{T_{freeze}} \right)^3 g e^{-\frac{m_X}{kT_{freeze}}} \left( \frac{m_X k T_{freeze}}{2\pi} \right)^{\frac{3}{2}} \end{aligned} \tag{37}$$

Hence we have,

$$\Omega_{DM} \sim 0.3 \sim \frac{8\pi G \rho_{DM}}{3H_0^2} \tag{38}$$

So we want,

$$\begin{aligned} 0.3 &\sim \frac{8\pi G}{3H_0^2} m_X \left( \frac{T_{CMB}}{T_{freeze}} \right)^3 g e^{-\frac{m_X}{kT_{freeze}}} \left( \frac{m_X k T_{freeze}}{2\pi} \right)^{\frac{3}{2}} \\ &= \frac{8\pi G}{3H_0^2} m_X \left( \frac{kT_{CMB}}{kT_{freeze}} \right)^3 g e^{-\frac{m_X}{kT_{freeze}}} \left( \frac{m_X k T_{freeze}}{2\pi} \right)^{\frac{3}{2}} \\ &= \frac{8\pi G}{3H_0^2} m_X \left( kT_{CMB} \right)^3 g e^{-\frac{m_X}{kT_{freeze}}} \left( \frac{m_X}{2\pi k T_{freeze}} \right)^{\frac{3}{2}} \end{aligned} \tag{39}$$

Now we know from the previous example sheet that for the CMB photons today $\Omega_{CMB} \sim 5 \times 10^{-5}$. Hence we know,

$$\begin{aligned}
\Omega_{CMB} &= \frac{8\pi G \rho_{CMB}}{3H_0^2} = \frac{8\pi G a_B T_{CMB}^4}{3H_0^2} \\
&= \frac{8\pi G \pi^2 (kT_{CMB})^4}{15 \times 3H_0^2} \\
&= \frac{8\pi^3 G (kT_{CMB})^4}{45 H_0^2}
\end{aligned} \tag{40}$$

(recall $a_B = \pi^2 k^4/15$).

So,

$$\begin{aligned}
0.3 &\sim 2 \times \frac{8\pi G}{3H_0^2} m_X \left(\frac{45\Omega_{CMB} H_0^2}{8\pi^3 G}\right)^{\frac{3}{4}} \left(\frac{m_X}{2\pi k T_{freeze}}\right)^{\frac{3}{2}} e^{-\frac{m_X}{kT_{freeze}}} \\
&= 2 \times \frac{8\pi}{3} \left(\frac{45}{8\pi^3}\right)^{\frac{3}{4}} \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \Omega_{CMB}^{\frac{3}{4}} \left(\frac{G m_X^4}{H_0^2}\right)^{\frac{1}{4}} \left(\frac{m_X}{kT_{freeze}}\right)^{\frac{3}{2}} e^{-\frac{m_X}{kT_{freeze}}} \\
&= 0.3\, \Omega_{CMB}^{\frac{3}{4}} \left(\frac{G m_X^4}{H_0^2}\right)^{\frac{1}{4}} \left(\frac{m_X}{kT_{freeze}}\right)^{\frac{3}{2}} e^{-\frac{m_X}{kT_{freeze}}}
\end{aligned} \tag{41}$$

So,

$$1 \sim \Omega_{CMB}^{\frac{3}{4}} \left(\frac{G m_X^4}{H_0^2}\right)^{\frac{1}{4}} \left(\frac{m_X}{kT_{freeze}}\right)^{\frac{3}{2}} e^{-\frac{m_X}{kT_{freeze}}} \tag{42}$$

Now we can put in some numbers; recall $G = 6.6 \times 10^{-11} m^3 kg^{-1} s^{-2}$, $H_0 = 2 \times 10^{-18} s^{-1}$, and we put in $m_X c^2 = 1GeV$; we also need $\hbar = 10^{-34} kg m^2 s^{-1}$ and $c = 3 \times 10^8 ms^{-1}$. Then;

$$\frac{G m_X^4}{H_0^2} = 3 \times 10^{45} \frac{\hbar^3}{c^3} \tag{43}$$

Let,

$$x \equiv \frac{m_X}{kT_{freeze}} \tag{44}$$

and recall that we want $x \gg 1$ (so that the freeze out temperature is below $m_X$ so it is indeed non-relativistic. So then,

$$1 \sim \left(5 \times 10^{-5}\right)^{\frac{3}{4}} \left(3 \times 10^{45}\right)^{\frac{1}{4}} x^{\frac{3}{2}} e^{-x} \tag{45}$$

So,

$$x^{\frac{3}{2}}e^{-x} \quad \sim \quad 10^{-8} \tag{46}$$

This is achieved (for $x < 1$) by,

$$x \sim 23 \tag{47}$$

Hence the freeze out temperature is,

$$T_{freeze} = \frac{1}{23}\frac{m_X c^2}{k} \sim 5 \times 10^{11} K \tag{48}$$

(recall $1 GeV \leftrightarrow 10^{13} K$).

Now we know the number density at freeze out, since,

$$\begin{aligned} n(T_{freeze}) &= g e^{-\frac{m_X}{kT_{freeze}}} \left(\frac{m_X kT_{freeze}}{2\pi}\right)^{\frac{3}{2}} \\ &= g m_X^3 \frac{1}{(2\pi)^{3/2}} e^{-\frac{m_X}{kT_{freeze}}} \left(\frac{kT_{freeze}}{m_X}\right)^{\frac{3}{2}} \\ &= g m_X^3 \frac{1}{(2\pi)^{3/2}} e^{-x} x^{-\frac{3}{2}} \\ &= 10^{-13} \times m_X^3 \end{aligned} \tag{49}$$

In radiation era we have,

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\frac{1}{2} g_{eff} a_B T^4 \tag{50}$$

and a little above $T \sim 10^{11} K$ from the previous question we have $g_{eff} \sim 43/4$, so,

$$H^2 \sim \frac{4\pi G}{3}\frac{43}{4}\frac{\pi^2}{15}(kT)^4 \tag{51}$$

(in units $\hbar = 1$, $c = 1$) and so,

$$H \sim 5.4\sqrt{G}(kT)^2 \tag{52}$$

Then freeze out occurs at;

$$
\begin{aligned}
1 &\sim \frac{\Gamma_X}{H} \sim \frac{\lambda^2}{m_X^2} n(T_{freeze}) \frac{1}{5.4\sqrt{G}(kT_{freeze})^2} \\
&= \frac{\lambda^2}{5.4\sqrt{G} m_X^4} n(T_{freeze}) \left( \frac{m_X}{kT_{freeze}} \right)^2 \\
&= \frac{\lambda^2}{5.4\sqrt{G} m_X^4} \left( 10^{-13} m_X^3 \right) x^2 \qquad (53)
\end{aligned}
$$

and hence,

$$10^{11} \sim \frac{\lambda^2}{\sqrt{G} m_X} \qquad (54)$$

Now,

$$G m_X^2 = 7 \times 10^{-39} \hbar c \qquad (55)$$

So then,

$$10^{11} \sim \frac{\lambda^2}{\sqrt{7 \times 10^{-39}}} \qquad (56)$$

so that,

$$\lambda^2 \sim 10^{-8} \qquad (57)$$

**Qu. 3** Consider the nuclear reaction forming a nucleus $N_i$ with atomic number $Z_i$ and atomic mass $A_i$ with $g_i$ spin states from neutrons and protons, so that;

$$(A_i - Z_i)\, n + Z_i p \to N_i \tag{58}$$

Take the binding energy liberated to be $B_i$. Deduce that in thermal equilibrium, at temperatures well below the nuclear scale (so all particles are non-relativistic) then the Saha equation for the reaction predicts;

$$X_i \simeq \frac{1}{2} g_i A_i^{\frac{3}{2}} e^{\frac{B_i}{kT}} X_p^{Z_i} X_n^{A_i - Z_i} \epsilon^{A_i - 1} \tag{59}$$

with the dimensionless,

$$\epsilon = \frac{1}{2} n_B \left( \frac{m_N k T}{2\pi\hbar^2} \right)^{-\frac{3}{2}} \tag{60}$$

where $m_N = m_p \simeq m_n$ is the nucleon mass, and $X_a$ is the fraction of the nuclear $i$-species, $X_a \equiv n_a / n_B$, with $n_B$ the total baryon number density (ie. the density number of protons plus neutrons in unbound or bound states, so in this case $n_B = n_n + n_p + A_i n_i$).

By using the current baryon density, compute the baryon density at the earlier time, and higher temperature $T$, and hence show that,

$$\epsilon \simeq 10^{-12} \Omega_B \left( \frac{T}{10^{10} K} \right)^{\frac{3}{2}} \tag{61}$$

and hence $\epsilon \ll 1$ around the time of nucleosynthesis. Hence show that the temperature $T_i$ at which (in equilibrium) the nucleus becomes abundant is,

$$kT_i \simeq \frac{B_i}{(A_i - 1) |\ln \epsilon|} \tag{62}$$

Hence show that for deuterium, with $g_D = 3$ (ie. spin-1), with $B_D = 2.2 MeV$ then $T_D \simeq 0.8 \times 10^9 K$.

Compute the abundance temperatures for tritium and helium-3 (both with binding energies $\sim 8 MeV$) and helium-4 (binding energy $\sim 30 MeV$) to show the corresponding temperatures $T_i$ are higher than for deuterium. Hence once deuterium has formed in the early universe, these other nuclei 'want' to be abundant in equilibrium (although the multi particle reactions above are too slow to maintain equilibrium, and hence these nuclei are produced instead from 2-body processes involving the deuterium once it is abundant).

**Qu. 3 answer**

As each species is non-relativistic then the number density of species $i$, $n_i$, is;

$$n_i = g_i e^{\frac{\mu_i - m_i}{kT}} \left(\frac{m_i kT}{2\pi}\right)^{\frac{3}{2}} \tag{63}$$

Hence for the reaction,

$$(A_i - Z_i)\, n + Z_i p \to N_i \tag{64}$$

where we have,

$$(A_i - Z_i)\, \mu_n + Z_i \mu_p = \mu_{N_i} \tag{65}$$

then the chemical potentials will cancel if we consider the ratio,

$$\frac{n_{N_i}}{n_n^{A_i - Z_i} n_p^{Z_i}} = \frac{g_{N_i}}{g_n^{A_i - Z_i} g_p^{Z_i}} e^{\frac{(A_i - Z_i)m_n + Z_i m_p - m_{N_i}}{kT}} \left(\frac{m_n kT}{2\pi}\right)^{-(A_i - Z_i)\frac{3}{2}} \left(\frac{m_p kT}{2\pi}\right)^{-Z_i \frac{3}{2}} \left(\frac{m_{N_i} kT}{2\pi}\right)^{\frac{3}{2}} \tag{66}$$

Now the binding energy liberated by the forward reaction is,

$$B_i = (A_i - Z_i)m_n + Z_i m_p - m_{N_i} \tag{67}$$

and hence,

$$\begin{aligned}
\frac{n_{N_i}}{n_n^{A_i - Z_i} n_p^{Z_i}} &= \frac{g_{N_i}}{g_n^{A_i - Z_i} g_p^{Z_i}} e^{\frac{B_i}{kT}} \left(\frac{kT}{2\pi}\right)^{\frac{3}{2}(-(A_i - Z_i) - Z_i + 1)} \left(\frac{m_{N_i}}{m_n^{A_i - Z_i} m_p^{Z_i}}\right)^{\frac{3}{2}} \\
&= \frac{g_{N_i}}{g_n^{A_i - Z_i} g_p^{Z_i}} e^{\frac{B_i}{kT}} \left(\frac{kT}{2\pi}\right)^{\frac{3}{2}(1 - A_i)} \left(\frac{m_{N_i}}{m_n^{A_i - Z_i} m_p^{Z_i}}\right)^{\frac{3}{2}}
\end{aligned} \tag{68}$$

Now in the last term we approximate, $m_n \sim m_p = m_N$ the nucleon mass (not to be confused with $m_{N_i}$ the mass of the nucleus $N_i$), and $m_{N_i} \sim A_i m_N$, so that,

$$\frac{m_{N_i}}{m_n^{A_i - Z_i} m_p^{Z_i}} \simeq \frac{m_{N_i}}{m_N^{A_i}} \simeq \frac{A_i m_N}{m_N^{A_i}} \simeq A_i m_N^{1 - A_i} \tag{69}$$

Then,

$$\begin{aligned}
\frac{n_{N_i}}{n_n^{A_i - Z_i} n_p^{Z_i}} &= \frac{g_{N_i}}{g_n^{A_i - Z_i} g_p^{Z_i}} e^{\frac{B_i}{kT}} \left(\frac{kT}{2\pi}\right)^{\frac{3}{2}(1 - A_i)} \left(A_i m_N^{1 - A_i}\right)^{\frac{3}{2}} \\
&= A_i^{\frac{3}{2}} \frac{g_{N_i}}{g_n^{A_i - Z_i} g_p^{Z_i}} \left(\frac{m_N kT}{2\pi}\right)^{\frac{3}{2}(1 - A_i)} e^{\frac{B_i}{kT}}
\end{aligned} \tag{70}$$

Now the spin of the neutron and protons are spin-1/2, hence $g_n = g_p = 2$, so,

$$\frac{n_{N_i}}{n_n^{A_i - Z_i} n_p^{Z_i}} = \frac{A_i^{\frac{3}{2}}}{2^{A_i}} g_{N_i} \left( \frac{m_N kT}{2\pi} \right)^{\frac{3}{2}(1-A_i)} e^{\frac{B_i}{kT}} \tag{71}$$

Define the total baryon number density $n_B = n_n + n_p + A_i n_{N_i}$, and the fractions $X_a = n_a / n_B$. Then,

$$\begin{aligned} \frac{n_{N_i}}{n_n^{A_i - Z_i} n_p^{Z_i}} &= \frac{X_{N_i}}{X_n^{A_i - Z_i} X_p^{Z_i}} \left( \frac{n_B}{n_B^{A_i - Z_i} n_B^{Z_i}} \right) \\ &= \frac{X_{N_i}}{X_n^{A_i - Z_i} X_p^{Z_i}} n_B^{1 - A_i} \end{aligned} \tag{72}$$

So then,

$$\begin{aligned} \frac{X_{N_i}}{X_n^{A_i - Z_i} X_p^{Z_i}} &= n_B^{A_i - 1} \frac{n_{N_i}}{n_n^{A_i - Z_i} n_p^{Z_i}} \\ &= n_B^{A_i - 1} \frac{A_i^{\frac{3}{2}}}{2^{A_i}} g_{N_i} \left( \frac{m_N kT}{2\pi} \right)^{\frac{3}{2}(1-A_i)} e^{\frac{B_i}{kT}} \\ &= \frac{1}{2} \left( \frac{n_B}{2} \right)^{A_i - 1} A_i^{\frac{3}{2}} g_{N_i} \left( \frac{m_N kT}{2\pi} \right)^{\frac{3}{2}(1-A_i)} e^{\frac{B_i}{kT}} \\ &= \frac{1}{2} A_i^{\frac{3}{2}} g_{N_i} e^{\frac{B_i}{kT}} \left( \frac{n_B}{2} \left( \frac{m_N kT}{2\pi} \right)^{-\frac{3}{2}} \right)^{A_i - 1} \\ &= \frac{1}{2} A_i^{\frac{3}{2}} g_{N_i} e^{\frac{B_i}{kT}} \epsilon^{A_i - 1} \end{aligned} \tag{73}$$

where,

$$\epsilon = \frac{n_B}{2} \left( \frac{m_N kT}{2\pi} \right)^{-\frac{3}{2}} \tag{74}$$

where $\hbar = 1$.

The current baryon density $\rho_{B,0} = m_N n_{B,0} = \Omega_B \rho_{crit}$ (with $\Omega_B \sim 0.03$), so,

$$m_N n_{B,0} = \Omega_B \frac{3 H_0^2}{8\pi G} \tag{75}$$

Now using $m_N c^2 \sim 1GeV$, $G = 6.6\times10^{-11}m^3kg^{-1}s^{-2}$ and $H_0 \sim 70kms^{-1}Mpc^{-1}$ we find,

$$n_{B,0} \simeq 5\Omega_B m^{-3} \tag{76}$$

Conservation of baryon number implies $n_B a^3$ =constant so that at another time,

$$\frac{n_B}{n_{B,0}} = \frac{a_0^3}{a^3} \tag{77}$$

Now for photons we have that their temperature (or effective temperature when they are decoupled) goes as $T \sim 1/a$. Hence, in terms of photon temperature $T$,

$$\frac{n_B}{n_{B,0}} = \frac{T^3}{T_{CMB}^3} \tag{78}$$

where $T_{CMB}$ is the photon temperature today. Thus,

$$n_B(T) = \frac{T^3}{T_{CMB}^3} n_{B,0} = \frac{T^3}{(2.7K)^3} 5\Omega_B m^{-3} = 0.3\Omega_B T^3 m^{-3} K^{-3} \tag{79}$$

Then,

$$\begin{aligned} \epsilon &= \frac{n_B}{2}\left(\frac{m_N kT}{2\pi}\right)^{-\frac{3}{2}} \\ &= 0.5 \times 0.3\Omega_B T^3 m^{-3} K^{-3} \left(\frac{m_N kT}{2\pi}\right)^{-\frac{3}{2}} \end{aligned} \tag{80}$$

Now,

$$m_N k = 2 \times 10^{18} \frac{\hbar^2}{Km^2} \tag{81}$$

so then,

$$\begin{aligned} \epsilon &= 0.5 \times 0.3\Omega_B T^3 m^{-3} K^{-3} \left(\frac{2 \times 10^{18} T}{2\pi} \frac{1}{Km^2}\right)^{-\frac{3}{2}} \\ &= \Omega_B \times 8 \times 10^{-28} \times \left(\frac{T}{K}\right)^{\frac{3}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \Omega_B \times 8 \times 10^{-13} \times \left(\frac{T}{10^{10}K}\right)^{\frac{3}{2}} \\
&\simeq \Omega_B \times 10^{-12} \times \left(\frac{T}{10^{10}K}\right)^{\frac{3}{2}} \qquad (82)
\end{aligned}
$$

Hence around nucleosyntheisis $\epsilon \ll 1$. For the nucleus $N_i$ to become abundant we require $X_{N_i} \sim 1$, and from,

$$
X_i \simeq \frac{1}{2} g_i A_i^{\frac{3}{2}} e^{\frac{B_i}{kT}} X_p^{Z_i} X_n^{A_i - Z_i} \epsilon^{A_i - 1} \qquad (83)
$$

and assuming that neutrons and protons are abundant so $X_n \sim X_p \sim 1$, then this implies,

$$
1 \sim g_i A_i^{\frac{3}{2}} e^{\frac{B_i}{kT}} \epsilon^{A_i - 1} \qquad (84)
$$

Taking a log, then,

$$
\ln g_i + \frac{3}{2} \ln A_i + \frac{B_i}{kT} + (A_i - 1) \ln \epsilon = 0 \qquad (85)
$$

Now assuming that $A_i \sim O(1)$ and $g_i \sim O(1)$, but $\epsilon \ll 1$ then this gives,

$$
\frac{B_i}{kT} + (A_i - 1) \ln \epsilon \simeq 0 \qquad (86)
$$

and hence,

$$
kT = -\frac{B_i}{(A_i - 1) \ln \epsilon} = \frac{B_i}{(A_i - 1) \, |\ln \epsilon|} \qquad (87)
$$

For Deuterium $B_D = 2.2 MeV$ and $A_D = 2$. Using $\Omega_B \sim 0.03$ we have,

$$
\begin{aligned}
\epsilon &= \Omega_B \times 10^{-12} \times \left(\frac{T}{10^{10}K}\right)^{\frac{3}{2}} \\
&= 3 \times 10^{-14} \times \left(\frac{T}{10^{10}K}\right)^{\frac{3}{2}} \qquad (88)
\end{aligned}
$$

Assume the temperature isn't far from $10^{10}K$, then,

$$
\ln \epsilon = -31 + \frac{3}{2} \log \frac{T}{10^{10}K} \simeq -31 \qquad (89)
$$

and so,

$$kT \simeq \frac{B_D}{(A_D - 1)\,|\ln\epsilon|} \simeq \frac{2.2MeV}{1 \times 30} \tag{90}$$

so that $kT \sim 0.07MeV$ and,

$$T \simeq 0.8 \times 10^9 K \tag{91}$$

For tritium (A = 3) and helium-3 (A = 3) both have binding energies $B = 8MeV$ and hence both have approximately the same abundance temperature,

$$kT \simeq \frac{8MeV}{(3-1) \times 30} \simeq 0.13MeV \tag{92}$$

so $T \simeq 1.5 \times 10^9 K$.

For helium-4 (A = 4) and $B = 30MeV$ giving,

$$kT \simeq \frac{30MeV}{(4-1) \times 30} \simeq 0.33MeV \tag{93}$$

so $T \simeq 4 \times 10^9 K$.

These are certainly higher than that of Deuterium.

**Qu. 4** Consider Compton scattering of a photon with initial momentum $\vec{p}$, energy $E = |\vec{p}|$, on an electron mass $m_e$ at rest. By considering 4-momentum conservation, show that for incident photon energies $E \ll m_e$ then the scattering is highly elastic, with the change in photon energy due to scattering, $\delta E$, going approximately as,

$$\frac{\delta E}{E} \sim \frac{E}{m} \tag{94}$$

where we ignore numerical $O(1)$ constants.

**Qu. 5** Recall from the equilibrium Saha equation we found that the temperature of recombination for hydrogen (with binding energy $B_H = 13.6eV$) is $T_H \simeq 3000K$. Compute the equivalent temperature for helium (spin 0, binding energy for last electron $B_{He} = 25eV$), showing it is higher than that for hydrogen. Photon last scattering is controlled by the last atom to recombine, and hence the hydrogen.

**Qu. 4 answer**

Take initial rest frame for electron - 4-momentum $k = (m, \vec{0})$. Then incoming photon has 4-momentum $p = (E, \vec{p})$ with $E = |\vec{p}|$. Outgoing photon has 4-momentum $p' = (E', \vec{p'})$ with $E' = |\vec{p'}|$ and electron recoils with momentum $k'$.

Conservation gives $p + k = p' + k'$. Energy conservation,

$$m_e + E = E' + k'^0 \tag{95}$$

and momentum,

$$\vec{p} = \vec{p'} + \vec{k'} \tag{96}$$

Now $k'^0 = \sqrt{m_e^2 + |\vec{k'}|^2} > m_e$. Hence from $m_e + E = E' + k'^0$ since each term is positive we must have $E' < E$.

We now consider the situation where $E \ll m_e$. Hence this also implies $E' \ll m_e$. Hence $|E' - E| \ll m_e$ and the scattering is elastic. It also implies that $k'^0 - m_e \ll m_e$ and hence that the outgoing electron is moving slowly with velocity $\vec{v} \ll 1$. Hence we may expand,

$$k'^0 = \sqrt{m_e^2 + |\vec{k'}|^2} \simeq m_e + \frac{1}{2m_e}|\vec{k'}|^2 + \ldots \tag{97}$$

Then the energy conservation gives,

$$E - E' = k'^0 - m_e \simeq \frac{1}{2m_e}|\vec{k'}|^2 = \frac{1}{2m_e}|\vec{p'} - \vec{p}|^2 \tag{98}$$

Now $|\vec{p'} - \vec{p}|^2 < |\vec{p'}|^2 + |\vec{p}|^2 = E^2 + E'^2$. Hence we see,

$$\delta E = E - E' < \frac{1}{2m_e}\left(E^2 + E'^2\right) \tag{99}$$

and so,

$$\frac{\delta E}{E} < \frac{1}{2}\frac{E}{m_e}\left(1 + \frac{E'^2}{E^2}\right) \tag{100}$$

Recall that $E' < E$, and hence,

$$\frac{\delta E}{E} < \frac{E}{m_e} \tag{101}$$

For generic low energy collisions we would expect,

$$\frac{\delta E}{E} \sim \frac{E}{m_e} \tag{102}$$

although bounded as above.

**Qu. 5 answer**

The reaction for the ionisation of He (ignoring the potential loss of a second electron to form $He^{++}$) is,

$$He \to He^+ + e \tag{103}$$

and hence the chemical potentials obey,

$$\mu_{He} = \mu_{He^+} + \mu_e \tag{104}$$

We treat the particles as non-relativistic, then for each particle type $i$ we have,

$$n_i = g_i e^{\frac{\mu_i - m_i}{kT}} \left( \frac{m_i kT}{2\pi} \right)^{\frac{3}{2}} \tag{105}$$

Now the $He$ atom has spin-0. Since the electron has spin-1/2, we must have that the $He^+$ produced has spin-1/2 too. Hence $g_{He} = 1$, but $g_{He^+} = g_e = 2$. We obtain the Saha relation by considering the ratio where the chemical potentials cancel;

$$\begin{aligned} \frac{n_{He}}{n_{He^+} n_e} &= \frac{g_{He}}{g_{He^+} g_e} e^{\frac{m_{He^+} + m_e - m_{He}}{kT}} \left( \frac{m_{He} kT}{2\pi} \right)^{\frac{3}{2}} \left( \frac{m_{He^+} kT}{2\pi} \right)^{-\frac{3}{2}} \left( \frac{m_e kT}{2\pi} \right)^{-\frac{3}{2}} \\ &= \frac{1}{4} e^{\frac{B_{He}}{kT}} \left( \frac{m_{He}}{m_{He^+}} \right)^{\frac{3}{2}} \left( \frac{m_e kT}{2\pi} \right)^{-\frac{3}{2}} \end{aligned} \tag{106}$$

where,

$$B_{He} = m_{He^+} + m_e - m_{He} = 25eV \tag{107}$$

is the binding energy of Helium atoms for the last electron. Using the fact that $m_{He} \simeq m_{He^+}$ then gives,

$$\frac{n_{He}}{n_{He^+} n_e} = \frac{1}{4} e^{\frac{B_{He}}{kT}} \left( \frac{m_e kT}{2\pi} \right)^{-\frac{3}{2}} \tag{108}$$

Define the ionised fraction $X = n_{He^+}/(n_{He} + n_{He^+})$. Then,

$$\frac{n_{He}}{n_{He^+}} = \frac{1-X}{X} \tag{109}$$

and hence we have the Saha equation,

$$\frac{X}{1-X} = \frac{4}{n_e} e^{-\frac{B_{He}}{kT}} \left( \frac{m_e kT}{2\pi} \right)^{\frac{3}{2}} \tag{110}$$

Now the number density of electrons is determined by charge neutrality to be equal to the number of protons. Now recall that nucleosynthesis predicts the total number of protons is predicted to be $\sim 0.76 n_B$ where $n_B$ is the total density of protons and neutrons, so to leading approximation we may take $n_e \sim n_B$.

Exactly as in the previous question 3; The current baryon density $\rho_{B,0} = m_N n_{B,0} = \Omega_B \rho_{crit}$ (with $\Omega_B \sim 0.03$), so,

$$m_N n_{B,0} = \Omega_B \frac{3H_0^2}{8\pi G} \tag{111}$$

Now using $m_N c^2 \sim 1GeV$, $G = 6.6 \times 10^{-11} m^3 kg^{-1} s^{-2}$ and $H_0 \sim 70 kms^{-1} Mpc^{-1}$ we find,

$$n_{B,0} \simeq 5\Omega_B m^{-3} \tag{112}$$

Conservation of baryon number implies $n_B a^3$ =constant so that at another time,

$$\frac{n_B}{n_{B,0}} = \frac{a_0^3}{a^3} \tag{113}$$

Now for photons we have that their temperature (or effective temperature when they are decoupled) goes as $T \sim 1/a$. Hence, in terms of photon temperature $T$,

$$\frac{n_B}{n_{B,0}} = \frac{T^3}{T_{CMB}^3} \tag{114}$$

where $T_{CMB}$ is the photon temperature today. Thus,

$$n_e(T) \simeq n_B(T) = \frac{T^3}{T_{CMB}^3} n_{B,0} = \frac{T^3}{(2.7K)^3} 5\Omega_B m^{-3} \tag{115}$$

Recalling that $m_e = 511KeV$, and that, $B_{He}/k = 2.9 \times 10^5 K$, then,

then we see that the ionisation occurs so that $X \sim 0.5$ when,

$$
\begin{aligned}
1 &\sim \frac{4}{n_e} e^{-\frac{B_{He}}{kT}} \left( \frac{m_e kT}{2\pi} \right)^{\frac{3}{2}} \\
&= 4 \left( \frac{T^3}{(2.7K)^3} 5\Omega_B m^{-3} \right)^{-1} e^{-\frac{B_{He}}{kT}} \left( \frac{m_e kT}{2\pi} \right)^{\frac{3}{2}} \\
&\sim \frac{1}{\Omega_B} \left( \frac{2.7K}{T} \right)^3 m^3 e^{-\frac{2.9\times 10^5 K}{T}} \left( 1.1 \times 10^{15} \hbar^2 \frac{T}{Km^2} \right)^{\frac{3}{2}} \\
&= 2 \times 10^{25} \left( \frac{K}{T} \right)^3 e^{-\frac{2.9\times 10^5 K}{T}} \left( \frac{T}{K} \right)^{\frac{3}{2}} \\
&= 2 \times 10^{25} \left( \frac{K}{T} \right)^{3/2} e^{-\frac{2.9\times 10^5 K}{T}} \\
&= 1.4 \times 10^{17} \left( \frac{2.9 \times 10^5 K}{T} \right)^{3/2} e^{-\frac{2.9\times 10^5 K}{T}} \qquad (116)
\end{aligned}
$$

Hence, setting $x = (290000K)/T$, then,

$$x^{3/2} e^{-x} \sim 7 \times 10^{-18} \qquad (117)$$

and hence $x \sim 45$.

Thus one finds a temperature,

$$T \sim \frac{290000K}{45} = 6400K \qquad (118)$$

which is around double that for the ionisation of $H$. Hence below $T \sim 6400K$ the He is largely neutral. Note that this only removes a small number of electrons from the total since the density of $He$ is much less than that of $H$. Hence we ignore this in the usual Saha calculation of $H$ recombination.