Cosmology and Particle Physics **Toby Wiseman**

Example sheet 3

Qu. 1 Consider the temperature $T \sim 10^{11} K$ where the light degrees of freedom are photons, e, $\bar{e}$ and the 3 generations of ν, $\bar{\nu}$ which we assume to be all in thermal equilibrium. Show that the effective number of degrees of freedom (counted as bosonic) is $g_{eff} = 43/4$, so that,

$$\rho = \frac{1}{2} g_{eff} a_B T^4 \tag{1}$$

Assume neutrinos instantaneously decouple at a temperature between $10^{10} K$ and $10^{11} K$, and approximate the electrons/positions as still being light in this temperature range.

For temperatures well below $10^{10} K$ the electrons/positons have annihilated. Show that then the effective number of degrees of freedom drops to,

$$g_{eff} = 2 + \frac{21}{4} \times \left(\frac{4}{11} \right)^{\frac{4}{3}} \simeq 3.36 \tag{2}$$

Show that the scale factor at temperature T is related to the current scale factor, a_0, and today's CMB photon temperature as,

$$\frac{T}{T_{CMB}} = \frac{a_0}{a(T)} \ , \qquad T \ll 10^{10} K \tag{3}$$

and

$$\frac{T}{T_{CMB}} \simeq \left(\frac{4}{11} \right)^{\frac{1}{3}} \frac{a_0}{a(T)} \ , \qquad 10^{10} K < T < 10^{11} K \tag{4}$$

Now consider very high temperatures above the weak scale $T \gg 10^{15} K$. Then the relativistic degrees of freedom are the photons, the gluons, and 3 generations of neutrinos, electrons and quarks, the W, Z and the Higgs. Compute the g_{eff} and the corresponding constant α in,

$$\frac{T}{T_{CMB}} \simeq \alpha \frac{a_0}{a(T)} \ , \qquad T \gg 10^{15} K \tag{5}$$

Qu. 2 Consider a WIMP dark matter model where a massive scalar particle X and its antiparticle $\bar{X}$ are initially in thermal equilibrium, and decay in a $2 \rightarrow 2$ process to light (effectively massless) scalar particles with rate,

$$\Gamma_X = \frac{\lambda^2}{m_X^2} n_X \tag{6}$$

with m_X the mass of the particle and its antiparticle, n_X the number density, and λ is the (dimensionless) coupling governing the 4 scalar particle interaction.

Assume that freeze out occurs at $T = T_{freeze}$ such that $kT_{freeze} \ll m_X$. Let us assume that $m_X = 1GeV$. What is the approximate temperature at which freeze out happens in this model in order to produce the correct relic density to act as today's dark matter? Find the value of the coupling λ (note that in this part of the analysis you can use the appropriate number of degrees of freedom in the radiation era from the previous question)?

Qu. 3 Consider the nuclear reaction forming a nucleus N_i with atomic number Z_i and atomic mass A_i with g_i spin states from neutrons and protons, so that;

$$(A_i - Z_i)\, n + Z_i p \rightarrow N_i \tag{7}$$

Take the binding energy liberated to be B_i. Deduce that in thermal equilibrium, at temperatures well below the nuclear scale (so all particles are non-relativistic) then the Saha equation for the reaction predicts;

$$X_i \simeq \frac{1}{2} g_i A_i^{\frac{3}{2}} e^{\frac{B_i}{kT}} X_p^{Z_i} X_n^{A_i - Z_i} \epsilon^{A_i - 1} \tag{8}$$

with the dimensionless,

$$\epsilon = \frac{1}{2} n_B \left(\frac{m_N k T}{2\pi\hbar^2} \right)^{-\frac{3}{2}} \tag{9}$$

where $m_N = m_p \simeq m_n$ is the nucleon mass, and X_a is the fraction of the nuclear i-species, $X_a \equiv n_a / n_B$, with n_B the total baryon number density (ie. the density number of protons plus neutrons in unbound or bound states, so in this case $n_B = n_n + n_p + A_i n_i$).

By using the current baryon density, compute the baryon density at the earlier time, and higher temperature T, and hence show that,

$$\epsilon \simeq 10^{-12} \Omega_B \left(\frac{T}{10^{10} K} \right)^{\frac{3}{2}} \tag{10}$$

and hence $\epsilon \ll 1$ around the time of nucleosynthesis. Hence show that the temperature T_i at which (in equilibrium) the nucleus becomes abundant is,

$$kT_i \simeq \frac{B_i}{(A_i - 1) |\ln \epsilon|} \tag{11}$$

Hence show that for deuterium, with $g_D = 3$ (ie. spin-1), with $B_D = 2.2 MeV$ then $T_D \simeq 0.8 \times 10^9 K$.

Compute the abundance temperatures for tritium and helium-3 (both with binding energies $\sim 8 MeV$) and helium-4 (binding energy $\sim 30 MeV$) to show the corresponding temperatures T_i are higher than for deuterium. Hence once deuterium has formed in the early universe, these other nuclei 'want' to be abundant in equilibrium (although the multi particle reactions above are too slow to maintain equilibrium, and hence these nuclei are produced instead from 2-body processes involving the deuterium once it is abundant).

Qu. 4 Consider Compton scattering of a photon with initial momentum $\vec{p}$, energy $E = |\vec{p}|$, on an electron mass m_e at rest. By considering 4-momentum conservation, show that for incident photon energies $E \ll m_e$ then the scattering is highly elastic, with the change in photon energy due to scattering, δE, going approximately as,

$$\frac{\delta E}{E} \sim \frac{E}{m} \tag{12}$$

where we ignore numerical $O(1)$ constants.

[Recall that perfectly elastic scattering cannot maintain equilibrium (it corresponds to a vanishing collision term C in the Boltzmann equation in FRW). One might think then that for $kT < m_e$ the photons would no longer be held in thermal equilibrium. In fact since there is some exchange of energy, given above, whilst small this is still sufficient to keep photon thermal equilibrium down to around $10^5 K$. After that thermal equilibrium is lost, but they still maintain their Bose distribution with an effective temperature that scales as $T \sim 1/a$. Note that they are still frequently interacting with charges at $10^5 K$, so they are not free streaming (travelling freely), but since the interactions are elastic they do not affect the photon phase space distribution.]

Qu. 5 Recall from the equilibrium Saha equation we found that the temperature of recombination for hydrogen (with binding energy $B_H = 13.6 eV$) is $T_H \simeq 3000 K$. Compute the equivalent temperature for helium (spin 0, binding energy for last electron $B_{He} = 25 eV$), showing it is higher than that for hydrogen. Photon last scattering is controlled by the last atom to recombine, and hence the hydrogen.