Example sheet 2

Answers

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Qu. 1 Recall that the kinetic relations

$$\rho = \int dp 4\pi p^2 n(p) E \ , \quad P = \int dp 4\pi p^2 n(p) \frac{p^2}{3E}$$

relate the density and pressure of a gas to its density distribution function $n(p)$, where $E = \sqrt{m^2 + p^2}$. Consider a bosonic(-) or fermion(+) field, with mass m, chemical potential μ, with g internal spin degrees of freedom. In thermal equilibrium the density distribution function is;

$$n(p) \ = \ \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{E-\mu}{kT}} \pm 1}$$

where $E = \sqrt{m^2 + p^2}$. In the ultra relativistic limit $kT \gg m, \mu$, then $E - \mu \sim E \sim p$ so this is well approximated by;

$$n(p) \ = \ \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{p}{kT}} \pm 1}$$

Thus compute the number density $n = \int dp 4\pi p^2 n(p)$ and energy density of an ultra relativistic gas (recall pressure $P = \rho/3$ from the above kinetic relations). You should find for a boson;

$$n_{boson} \ = \ \frac{15\zeta(3) a_B g}{k\pi^4} T^3 \ , \qquad \rho_{boson} = \frac{1}{2} g a_B T^4$$

and for a fermion;

$$n_{fermion} \ = \ \frac{3}{4} n_{boson} \ , \qquad \rho_{fermion} = \frac{7}{8} \rho_{boson}$$

where the radiation constant $a_B = \pi^2 k^4 / 15\hbar^3 c^3$ (although we are using units where $c = 1$). You may find the following integrals useful;

$$\int_0^\infty dx \frac{x^2}{e^x \pm 1} = \frac{7 \mp 1}{4} \zeta(3) \ , \quad \int_0^\infty dx \frac{x^3}{e^x \pm 1} = \frac{15 \mp 1}{240} \pi^4$$

where $\zeta(x)$ is the Riemann zeta function and in particular $\zeta(3) \simeq 1.202$.

Use the first law to show the equilibrium entropy density is $s = 4\rho/3T$. Also show that it implies $d\rho = T ds$, and check this is true for the expressions you have computed.

Qu. 1 answer Then,

$$\begin{aligned} n &= \int_0^\infty dp 4\pi p^2 n(p) = \frac{4\pi g}{(2\pi\hbar)^3}\int_0^\infty dp \frac{p^2}{e^{\frac{p}{kT}} \pm 1} \\ &= \frac{4\pi g(kT)^3}{(2\pi\hbar)^3}\int_0^\infty dx \frac{x^2}{e^x \pm 1} \end{aligned}$$

setting $x = p/kT$, and then using the integrals;

$$\begin{aligned} n &= \frac{4\pi g(kT)^3}{(2\pi\hbar)^3}\frac{7 \mp 1}{4}\zeta(3) = \frac{g(kT)^3}{8\pi^2\hbar^3}(7 \mp 1)\zeta(3) \\ &= \frac{15 a_B g}{8k\pi^4}(7 \mp 1)\zeta(3)T^3 \end{aligned}$$

So for bosons we obtain;

$$n_{boson} = \frac{15\zeta(3)a_B g}{k\pi^4}T^3$$

and for fermions we find;

$$n_{fermion} = \frac{6}{8}\frac{15\zeta(3)a_B g}{k\pi^4}T^3 = \frac{3}{4}n_{boson}$$

Then the energy density is (taking $E \simeq p$ in the ultra relativistic limit);

$$\begin{aligned} \rho &= \int_0^\infty dp 4\pi p^2 n(p) p = \frac{4\pi g}{(2\pi\hbar)^3}\int_0^\infty dp \frac{p^3}{e^{\frac{p}{kT}} \pm 1} \\ &= \frac{4\pi g(kT)^4}{(2\pi\hbar)^3}\int_0^\infty dx \frac{x^3}{e^x \pm 1} \\ &= \frac{g(kT)^4}{2\pi^2\hbar^3}\frac{15 \mp 1}{240}\pi^4 \\ &= \frac{1}{2}g a_B T^4 \frac{15 \mp 1}{16} \end{aligned}$$

Hence

$$\rho_{boson} = \frac{1}{2}g a_B T^4$$

and then,

$$\rho_{fermion} = \frac{1}{2}\frac{7}{8}g a_B T^4 = \frac{7}{8}\rho_{boson}$$

Assuming there is no chemical potential (it is negligible) for an ultra relativistic gas then, ρ and P and s only depend on T (and not on μ). Then the first law (ignoring chemical potential term);

$$dE = TdS - pdV$$

becomes, using $S = s(T)V$, $E = \rho(T)V$;

$$\rho dV + Vd\rho \quad = \quad TVds + TsdV - pdV$$

so that,

$$V\left(d\rho - Tds\right) \quad = \quad dV\left(Ts - \rho - p\right)$$

But there is only T dependence, under a variation of V there should be no change in s, ρ, P, and hence the right hand side must vanish implying;

$$s = \frac{\rho + P}{T} = \frac{4}{3}\frac{\rho}{T}$$

using $P = \rho/3$.

Note the left hand side then yields the first law; $d\rho = Tds$. To check this we note there is only T dependence, so we must check; $d\rho/dT = Tds/dT$.

Since $\rho = kT^4$ for a constant k, and $s = \frac{4\rho}{3T} = \frac{4k}{3}T^3$, then,

$$\frac{d\rho}{dT} = 4kT^3 \;, \qquad \frac{ds}{dT} = 3\frac{4k}{3}T^2 = 4kT^2 = \frac{1}{T}\frac{d\rho}{dT}$$

and hence indeed $d\rho/dT = Tds/dT$ is true.

Qu. 2 Repeat the calculations in Qu 1 in the non-relativistic limit $kT \ll E - \mu$ and $E \simeq m + \frac{p^2}{2m}$ so;

$$n(p) = \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu - m}{kT}} e^{-\frac{p^2}{2mkT}}$$

Firstly show that;

$$n = \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu - m}{kT}} (2\pi mkT)^{\frac{3}{2}}$$

and then show;

$$\rho = \left(m + \frac{3}{2}kT\right) n \ , \qquad P = kTn$$

You may find it useful to recall that for a Gaussian integral;

$$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

Use the 1st law in a closed system (so the total particle number cannot change) to show the equilibrium entropy density s in this case obeys;

$$n\, d\rho - nT ds = dn\, (\rho + p - Ts)$$

and hence integrate this to find;

$$s = k\, n \log\left(\frac{T^{3/2}}{c\, n}\right)$$

for a constant c.

Qu. 2 answer Now,

$$\begin{aligned} n &= \int_0^\infty dp 4\pi p^2 n(p) = \frac{4\pi g}{(2\pi\hbar)^3} e^{\frac{\mu-m}{kT}} \int_0^\infty dp p^2 e^{-\frac{p^2}{2mkT}} \\ &= \frac{g}{2\pi^2\hbar^3} e^{\frac{\mu-m}{kT}} (2mkT)^{\frac{3}{2}} \int_0^\infty dx x^2 e^{-x^2} \end{aligned}$$

setting $x = p/\sqrt{2mkT}$.

Using;

$$2\int_0^\infty e^{-ax^2} = \int_{-\infty}^\infty e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

then we find;

$$\int_0^\infty x^2 e^{-x^2} = -\frac{d}{da}\left(\int_0^\infty e^{-ax^2}\right)\Big|_{a=0} = -\frac{d}{da}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)\Big|_{a=0} = \frac{1}{4}\sqrt{\pi}$$

Thus,

$$\begin{aligned} n &= \frac{g\sqrt{\pi}}{8\pi^2\hbar^3} e^{\frac{\mu-m}{kT}} (2mkT)^{\frac{3}{2}} \\ &= \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu-m}{kT}} (2\pi mkT)^{\frac{3}{2}} \end{aligned}$$

Then,

$$\begin{aligned} \rho &= \int_0^\infty dp 4\pi p^2 n(p) E = \int_0^\infty dp 4\pi p^2 n(p)\left(m + \frac{p^2}{2m}\right) \\ &= \frac{4\pi g}{(2\pi\hbar)^3} e^{\frac{\mu-m}{kT}} \int_0^\infty dp p^2 e^{-\frac{p^2}{2mkT}} \left(m + \frac{p^2}{2m}\right) \\ &= mn + \frac{1}{2m}\frac{4\pi g}{(2\pi\hbar)^3} e^{\frac{\mu-m}{kT}} \int_0^\infty dp p^4 e^{-\frac{p^2}{2mkT}} \\ &= mn + \frac{1}{2m}\frac{4\pi g}{(2\pi\hbar)^3} e^{\frac{\mu-m}{kT}} (2mkT)^{\frac{5}{2}} \int_0^\infty dx x^4 e^{-x} \end{aligned}$$

setting $x = p/\sqrt{2mkT}$.

Now;

$$\int_0^\infty x^4 e^{-x^2} = \frac{d^2}{da^2}\left(\int_0^\infty e^{-ax^2}\right)\Big|_{a=0} = \frac{d^2}{da^2}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)\Big|_{a=0} = \frac{3}{8}\sqrt{\pi}$$

So,

$$
\begin{aligned}
\rho &= mn + \frac{3\sqrt{\pi}}{16m}\frac{4\pi g}{(2\pi\hbar)^3}e^{\frac{\mu-m}{kT}}(2mkT)^{\frac{5}{2}} \\
&= mn + \frac{3}{2}\frac{g}{(2\pi\hbar)^3}e^{\frac{\mu-m}{kT}}(2\pi mkT)^{\frac{3}{2}}(kT) \\
&= mn + \frac{3}{2}kTn
\end{aligned}
$$

And for pressure;

$$
\begin{aligned}
P &= \int_0^\infty dp 4\pi p^2 n(p)\frac{p^2}{3E} = \int_0^\infty dp 4\pi p^2 n(p)\frac{p^2}{3m} \\
&= \frac{1}{3m}\frac{4\pi g}{(2\pi\hbar)^3}e^{\frac{\mu-m}{kT}}\int_0^\infty dp\, p^4 e^{-\frac{p^2}{2mkT}} \\
&= \frac{1}{3m}\frac{4\pi g}{(2\pi\hbar)^3}e^{\frac{\mu-m}{kT}}(2mkT)^{\frac{5}{2}}\int_0^\infty dx x^4 e^{-x} \\
&= \frac{1}{3m}\frac{4\pi g}{(2\pi\hbar)^3}e^{\frac{\mu-m}{kT}}(2mkT)^{\frac{5}{2}}\frac{3}{8}\sqrt{\pi} \\
&= \frac{g}{(2\pi\hbar)^3}e^{\frac{\mu-m}{kT}}(2\pi mkT)^{\frac{3}{2}}(kT)
\end{aligned}
$$

again using $x = p/\sqrt{2mkT}$. So,

$$P = nkT$$

For fixed particle number the first law states;

$$dE = TdS - pdV$$

and we may write $S = sV$ and $E = \rho V$, and also $n \sim 1/V$ so that $dn/n = -dV/V$. Then,

$$\rho dV + V d\rho = TV ds + Ts dV - pdV$$

so that,

$$d\rho - Tds = \frac{dV}{V}(Ts - \rho - p)$$

and so,

$$nd\rho - nTds \;=\; dn\,(\rho + p - Ts)$$

Then,

$$sdn - nds \;=\; \frac{1}{T}dn\,(\rho + p) - \frac{1}{T}nd\rho$$

so,

$$n^2 d\left(\frac{s}{n}\right) \;=\; \frac{1}{T}nd\rho - \frac{1}{T}dn\,(\rho + p)$$

and,

$$d\left(\frac{s}{n}\right) \;=\; \frac{1}{nT}d\rho - \frac{1}{n^2T}dn\,(\rho + p)$$

Using $P = nkT$ and $\rho = (m + 3kT/2)\,n$ we find;

$$\begin{aligned} d\left(\frac{s}{n}\right) &= \frac{1}{nT}d\left((m + 3kT/2)\,n\right) - \frac{1}{nT}dn\,(m + 5kT/2) \\ &= \frac{3k}{2nT}d\,(Tn) - \frac{5k}{2n}dn \\ &= k\left(\frac{3}{2T}dT - \frac{1}{n}dn\right) \\ &= d\left(k\log\frac{T^{3/2}}{n}\right) \end{aligned}$$

Hence we find integrating for a constant c;

$$s \;=\; k\,n\log\left(\frac{T^{3/2}}{c\,n}\right)$$

Qu. 3 Use the observed Hubble parameter today $H_0 \simeq 70kms^{-1}Mpc^{-1}$, and $\Omega_\Lambda \simeq 0.7$, $\Omega_{matter} \simeq 0.3$ and assuming a flat FRW geometry, compute the density of non-relativistic matter today. You should find a density of $\sim 2.7 \times 10^{-27} kg\, m^{-3}$.

The photon radiation today (CMB photons) while free streaming and not in equilibrium, has almost exactly a bose distribution with temperature $2.7K$. Hence show its (very small) contribution to the Hubble expansion today is;

$$\Omega_\gamma \sim 5 \times 10^{-5}$$

You will need the values of the constants;

$$\begin{aligned} 1pc &= 3.2\,\text{light years}\ , \quad c = 3.0 \times 10^8 ms^{-1} \\ \hbar &= 1.05 \times 10^{-34} m^2 kgs^{-1}\ , \quad k = 1.38 \times 10^{-23} m^2 kgs^{-2} K^{-1} \end{aligned}$$

The total radiation fraction today, $\Omega_R = 1.68\Omega_\gamma$ as we shall show later in the course due to the presence of neutrinos. Use the Friedmann equation to show that radiation came to dominate the Hubble expansion over matter at a redshift of $Z_{eq} \sim 3600$.

Qu. 3 answer

The critical energy density is;

$$\rho_c = \frac{3c^2}{8\pi G} H_0^2 = 8.3 \times 10^{-10} kgm^{-1}s^{-2}$$

using ($1yr = 3.2 \times 10^7 s$).

Thus the **density** (ie. not energy density) of matter today is;

$$\frac{1}{c^2}\rho_{matter} = \frac{1}{c^2}\Omega_{matter}\rho_c = 2.76 \times 10^{-27} kgm^{-3}$$

We have the radiation constant;

$$a_B = \frac{\pi^2 k^4}{15\hbar^3 c^3} = 7.6 \times 10^{-16} \frac{kg}{K^4 m s^2}$$

Hence the energy density today for photons is,

$$\rho_\gamma = a_B T^4 = a_B (2.7K)^4 = 4 \times 10^{-14} kgm^{-1}s^{-2}$$

Hence,

$$\Omega_\gamma = \frac{\rho_\gamma}{\rho_c} = 5 \times 10^{-5}$$

and hence the total radiation has,

$$\Omega_R = 1.68 \times 5 \times 10^{-5} = 8.2 \times 10^{-5}$$

Then the Friedmann equation is;

$$\rho(t) = \rho_{crit}\left(\Omega_\Lambda + \Omega_M(1+Z)^3 + \Omega_R(1+Z)^4\right)$$

In order for radiation to dominate the energy density we therefore require that $\Omega_R(1+Z)^4 > \Omega_M(1+Z)^3$ and $\Omega_R(1+Z)^4 > \Omega_\Lambda$. Consider the first, then matter-radiation equality happens at redshift Z_{eq} when;

$$\Omega_R(1+Z_{eq})^4 > \Omega_M(1+Z_{eq})^3$$

and hence,

$$1 + Z_{eq} = \frac{\Omega_M}{\Omega_R} = 3600$$

Note that certainly $\Omega_R(1+Z)^4 > \Omega_\Lambda$ is satisfied then.

Qu. 4 Consider the Boltzmann equation for the density distribution function $n(t,p)$ of a species with mass m and with chemical potential μ;

$$\frac{\partial n}{\partial t} - Hp\frac{\partial n}{\partial p} = C$$

where $H = \dot{a}/a$. Suppose at early times the interaction term C is very large and the species is in thermal equilibrium so;

$$n(t,p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{E-\mu(t)}{kT(t)}} \pm 1}$$

where $T(t)$ and $\mu(t)$ are the temperature and chemical potential of the heat bath the species is in equilibrium with at time t. However, suppose interactions rapidly turn off at time t_{freeze}, with temperature T_{freeze}, when the scale factor is a_{freeze}, and subsequently the species then free streams.

Firstly, show that if the interactions turn off in the ultra relativistic regime $kT \gg m, \mu$ then,

$$n(t,p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{p}{kT_{eff}(t)}} \pm 1}\ , \qquad T_{eff}(t) = \frac{a_{freeze}}{a(t)}T_{freeze}$$

Is it true in this case that the distribution is a thermal distribution simply with a redshifted temperature $T_{eff}(T)$? (imagine what happens when the temperature falls below the mass scale of the particle).

Secondly, show that if the interactions turn off in the non-relativistic regime $kT \ll E - \mu$ then,

$$n(t,p) = \frac{g}{(2\pi\hbar)^3}e^{\frac{\mu_{freeze}-m}{kT_{freeze}}}\, e^{-\frac{p^2}{2mkT_{eff}(t)}}\ , \qquad T_{eff}(t) = \left(\frac{a_{freeze}}{a(t)}\right)^2 T_{freeze}$$

Qu. 4 answer

The solution to the free Boltzmann equation;

$$\frac{\partial n}{\partial t} - Hp\frac{\partial n}{\partial p} = 0$$

is $n(t,p) = n(a(t)p)$. Then,

$$\frac{\partial n}{\partial t} = n'(a(t)p)\dot{a}p$$

and,

$$\frac{\partial n}{\partial p} = n'(a(t)p)a$$

so that,

$$\frac{\partial n}{\partial t} - Hp\frac{\partial n}{\partial p} = n'(a(t)p)\dot{a}p - \frac{\dot{a}}{a}pn'(a(t)p)a = 0$$

as required.

For initial conditions at $T = T_{freeze}$ which are the thermal distribution for an ultra relativistic species, so,

$$n(t_{freeze}, p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{p}{kT_{freeze}}} \pm 1}$$

Hence,

$$n(t_{freeze}, p) = n(a_{freeze}p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{(a_{freeze}p)}{ka_{freeze}T_{freeze}}} \pm 1}$$

Then the solution at lower temperatures where the species freely streams, is then,

$$n(t,p) = n(a(t)p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{(a(t)p)}{ka_{freeze}T_{freeze}}} \pm 1} = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{p}{kT_{eff}}} \pm 1}$$

with,

$$T_{eff}(t) = \frac{a_{freeze}}{a(t)}T_{freeze}$$

For a massless particle so $E = p$ then this is simply a redshifted thermal distribution. However, if the particle has a small mass, then in the high temperature ultra-relativistic regime $kT \gg m$ then $E \simeq p$, but for $kT \ll m$ a thermal distribution would take a non-relativistic form. However the free streaming particles would maintain the above relativistic form even when $kT_{eff} \ll m$. Thus in this temperature range the free streaming behaviour would deviate strongly from a thermal behaviour at temperature T_{eff}.

For initial conditions at $T = T_{freeze}$ which are the thermal distribution for a non-relativistic species, so,

$$n(t_{freeze}, p) = \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu_{freeze} - m}{kT_{freeze}}} e^{-\frac{p^2}{2mkT_{freeze}}}$$

then,

$$n(t_{freeze}, p) = n(a_{freeze} p) = \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu_{freeze} - m}{kT_{freeze}}} e^{-\frac{(a_{freeze} p)^2}{2mka^2_{freeze} T_{freeze}}}$$

Then the solution at lower temperatures where the species freely streams, is then,

$$\begin{aligned} n(t, p) = n(a(t)p) &= \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu_{freeze} - m}{kT_{freeze}}} e^{-\frac{(a(t)p)^2}{2mka^2_{freeze} T_{freeze}}} \\ &= \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu_{freeze} - m}{kT_{freeze}}} e^{-\frac{p^2}{2mkT_{eff}}} \end{aligned}$$

with,

$$T_{eff}(t) = \left(\frac{a_{freeze}}{a(t)} \right)^2 T_{freeze}$$

Qu. 5 Consider the Boltzmann equation for a fermion species with mass m and vanishing chemical potential and number of spin degrees of freedom g interacting with the photons in the universe, which we approximate as a heat bath with temperature T related to the scale factor a as,

$$\frac{a}{a_0} = \frac{T_0}{T}$$

where a_0, T_0 are the scale factor and temperature today. Assume they do not interact with anything else. The Boltzmann equation is;

$$\frac{d \ln n^c(T)}{d \ln T} = \frac{\Gamma}{H}\left(1 - \frac{n^c_{eq}(T)}{n^c(T)}\right)$$

where the comoving number density and comoving equilibrium density are defined as;

$$n^c = \left(\frac{a}{a_0}\right)^3 n \ , \qquad n^c_{eq} = \left(\frac{a}{a_0}\right)^3 n_{eq}$$

for physical density n and equilibrium density n_{eq}. Assume that at early times then $n^c \simeq n^c_{eq}$ and $\Gamma/H \gg 1$ and when the temperature drops to $T = T_{freeze}$ then $\Gamma/H \sim 1$ and at later times (and lower temperatures) $\Gamma/H \ll 1$. Show that an approximate relic density today for this fermion species if the freeze out occurs when the species is non-relativistic is,

$$\rho_{relic}(T) \quad = \quad T^3 \frac{gm}{\hbar^3}\left(\frac{mk}{2\pi T_{freeze}}\right)^{\frac{3}{2}} e^{-\frac{m}{kT_{freeze}}}$$

Conversely if freeze out occurs in its ultra relativistic regime, temperature is $kT_{freeze} \gg m$, but the temperature today is low so that it is non-relativistic, show;

$$\rho_{relic}(T) \quad = \quad \frac{20\,\zeta(3) a_B g\, m}{k\pi^4} T^3$$

Show that these two answers are consistent with the results for the relic density distributions in Qu 4. by computing the relic densities from these density distributions.

Qu. 5 answer

For temperatures above T_{freeze} we have $n^c(T) = n^c_{eq}(T)$. Then at $T = T_{freeze}$ the species free streams so that for $T < T_{freeze}$ we have,

$$n^c(T)_{relic} = n^c_{eq}(T_{freeze})$$

is constant. Then the physical relic density is;

$$\begin{aligned}
n(T)_{relic} &= \left(\frac{a_0}{a}\right)^3 n^c = \left(\frac{a_0}{a}\right)^3 n^c_{eq}(T_{freeze}) = \left(\frac{a_0}{a}\right)^3 \left(\frac{a_0}{a_{freeze}}\right)^{-3} n_{eq}(T_{freeze}) \\
&= \left(\frac{T_0}{T}\right)^{-3} \left(\frac{T_0}{T_{freeze}}\right)^3 n_{eq}(T_{freeze}) \\
&= \left(\frac{T}{T_{freeze}}\right)^3 n_{eq}(T_{freeze})
\end{aligned}$$

At low temperatures where the species is non-relativistic, then the relic energy density will be,

$$\rho_{relic}(T) = m\, n(T)_{relic}$$

Suppose the freeze out happens when our fermion species is non-relativistic. Then (from Qu 2) with no chemical potential,

$$n_{eq} = \frac{g}{(2\pi\hbar)^3} e^{\frac{-m}{kT}} \left(2\pi mkT\right)^{\frac{3}{2}}$$

Then we have a relic density;

$$\begin{aligned}
\rho_{relic}(T) &= m\left(\frac{T}{T_{freeze}}\right)^3 n_{eq}(T_{freeze}) = m\left(\frac{T}{T_{freeze}}\right)^3 \frac{g}{(2\pi\hbar)^3} e^{\frac{-m}{kT_{freeze}}} \left(2\pi mkT_{freeze}\right)^{\frac{3}{2}} \\
&= T^3 \frac{gm}{\hbar^3} \left(\frac{mk}{2\pi T_{freeze}}\right)^{\frac{3}{2}} e^{-\frac{m}{kT_{freeze}}}
\end{aligned}$$

Suppose the freeze out happens when our fermion species is relativistic (so $kT \gg m, \mu$). Then (from Qu 1),

$$n_{eq}(T) = \frac{3}{4} \frac{15\zeta(3) a_B g}{k\pi^4} T^3$$

Then we have a relic density at low temperatures (when the species is non-relativistic) of;

$$\begin{aligned}\rho_{relic}(T) &= m\left(\frac{T}{T_{freeze}}\right)^3 n_{eq}(T_{freeze}) = m\left(\frac{T}{T_{freeze}}\right)^3 \frac{3}{4}\frac{15\zeta(3)a_B g}{k\pi^4}T_{freeze}^3 \\ &= \frac{45\,\zeta(3)a_B g\, m}{4k\pi^4}T^3\end{aligned}$$

The relic density distribution for freeze out in the ultra-relativistic case is;

$$n_{relic}(t,p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{p}{kT_{eff}(t)}}+1}$$

where,

$$T_{eff} = \frac{a_{freeze}}{a(t)}T_{freeze} = \frac{a_{freeze}}{a_0}\frac{a_0}{a(t)}T_{freeze} = \frac{T_0}{T_{freeze}}\frac{T}{T_0}T_{freeze} = T$$

Hence we have,

$$n_{relic}(t,p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{p}{kT}}+1}$$

so the distribution is simple that of a massless equilibrium fermion at temperature T.

Then from Qu 1 the number density is,

$$n_{relic}(t) = \int_0^\infty dp 4\pi p^2 n_{relic}(t,p) = \frac{4}{3}\frac{15\zeta(3)a_B g}{k\pi^4}T^3$$

Recall from question 2 the energy density in the non-relativistic regime for any density distribution is $\rho_{relic} = \left(m + \frac{3}{2}kT\right)n_{relic} \simeq m\, n_{relic}$. Hence in the non-relativistic regime the relic density is,

$$\rho_{relic} = m n_{relic}(t) = m\frac{4}{3}\frac{15\zeta(3)a_B g}{k\pi^4}T^3 = \frac{20\zeta(3)a_B g\, m}{k\pi^4}T^3$$

agreeing with the above result in Qu 6.

From Qu 4, the relic density distribution for freeze out in the non-relativistic case (without chemical potential) is;

$$n_{relic}(t,p) = \frac{g}{(2\pi\hbar)^3}e^{-\frac{m}{kT_{freeze}}}e^{-\frac{p^2}{2mkT_{eff}(t)}}$$

where,

$$\begin{aligned}
T_{eff}(t) &= \left(\frac{a_{freeze}}{a(t)}\right)^2 T_{freeze} = \left(\frac{a_{freeze}}{a_0}\right)^2 \left(\frac{a_0}{a(t)}\right)^2 T_{freeze} = \left(\frac{T_0}{T_{freeze}}\right)^2 \left(\frac{T}{T_0}\right)^2 T_{freeze} \\
&= \frac{T^2}{T_{freeze}}
\end{aligned}$$

Hence we have,

$$n_{relic}(t,p) = \frac{g}{(2\pi\hbar)^3} e^{-\frac{m}{kT_{freeze}}} e^{-\frac{p^2 T_{freeze}}{2mkT^2}}$$

The number density of the relic is computed as;

$$\begin{aligned}
n_{relic} &= \int_0^\infty dp 4\pi p^2 n_{relic}(t,p) \\
&= \frac{g}{(2\pi\hbar)^3} e^{-\frac{m}{kT_{freeze}}} \int_0^\infty dp 4\pi p^2 e^{-\frac{p^2 T_{freeze}}{2mkT^2}} \\
&= \frac{g}{(2\pi\hbar)^3} e^{-\frac{m}{kT_{freeze}}} 4\pi \left(\frac{2mkT^2}{T_{freeze}}\right)^{\frac{3}{2}} \int_0^\infty dx x^2 e^{-x^2}
\end{aligned}$$

using the substitution $x = p\sqrt{\frac{T_{freeze}}{2mkT^2}}$. Then recalling from Qu 2 that,

$$\int_0^\infty dx x^2 e^{-x^2} = \frac{1}{4}\sqrt{\pi}$$

we have,

$$\begin{aligned}
n_{relic} &= \frac{1}{4}\sqrt{\pi} \frac{g}{(2\pi\hbar)^3} e^{-\frac{m}{kT_{freeze}}} 4\pi \left(\frac{2mkT^2}{T_{freeze}}\right)^{\frac{3}{2}} \\
&= T^3 \frac{g}{\hbar^3} e^{-\frac{m}{kT_{freeze}}} \left(\frac{mk}{2\pi T_{freeze}}\right)^{\frac{3}{2}}
\end{aligned}$$

and using $\rho_{relic} \simeq m\, n_{relic}$ we have,

$$\rho_{relic} = T^3 \frac{m\, g}{\hbar^3} e^{-\frac{m}{kT_{freeze}}} \left(\frac{mk}{2\pi T_{freeze}}\right)^{\frac{3}{2}}$$

which indeed agrees as it should.

Qu. 6 Consider the Boltzmann equation as in the previous question,

$$\frac{d \ln n^c(T)}{d \ln T} = \frac{\Gamma}{H} \left(1 - \frac{n^c_{eq}(T)}{n^c(T)} \right)$$

Suppose we treat Γ/H as being constant, so that $\alpha = \Gamma/H$. Let us consider starting the system at a temperature T_i with the in thermal equilibrium so that $n^c(T_i) = n^c_{eq}(T_i)$. Then consider evolving to lower temperatures $T < T_i$.

Confirm that the solution of the Boltzmann equation for $T \leq T_i$ (assuming α is constant) is then,

$$n^c(T) = \left(\frac{T}{T_i} \right)^{\alpha} n^c_{eq}(T_i) + \alpha\, T^{\alpha} \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T')$$

and check that it obeys the boundary condition at $T = T_i$.

For sufficiently large α, and $T < T_i$, and assuming $n^c_{eq}(T)$ is smooth, we may approximate;

$$\int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T') \simeq \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T)$$

since the function $1/T'^{1+\alpha}$ is very strongly peaked at the lower limit T of the integral. Use this to show that for sufficiently large α, then,

$$n^c(T) \simeq n^c_{eq}(T)$$

for $T < T_i$. Further, by Taylor expanding about the lower limit, show that (the constant) α must be large on a scale determined by the form of n^c_{eq}, so,

$$\alpha \gg \frac{d \log n^c_{eq}}{d \log T}$$

Consider a universe with thermal bath temperature $T \sim 1/a$. Show that for a relativistic species the above bound is always true provided $\alpha \gg 1$, and then equilibrium is maintained for all low temperatures $T < T_i$. However, for a non-relativistic species with mass m and with no chemical potential, for equilibrium to persist to a temperature $T < T_i$ then the constant α must be bounded as,

$$\alpha \gg \frac{m}{kT} > 1$$

and hence at sufficiently low temperatures equilibrium cannot be maintained even for large values of α. Basically the decay rate is not sufficiently high to reduce the density particles to their very rapidly decreasing equilibrium value.

Qu. 6 answer The Boltzmann equation;

$$\frac{d\ln n^c(T)}{d\ln T} = \alpha\left(1 - \frac{n^c_{eq}(T)}{n^c(T)}\right)$$

so that,

$$\frac{T}{n^c}\frac{dn^c}{dT} = \alpha\left(1 - \frac{n^c_{eq}(T)}{n^c(T)}\right)$$

so,

$$\frac{dn^c}{dT} = \frac{\alpha}{T}\left(n^c - n^c_{eq}\right)$$

Start with the solution in the question,

$$n^c(T) = \left(\frac{T}{T_i}\right)^\alpha n^c_{eq}(T_i) + \alpha\, T^\alpha \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T')$$

Clearly this satisfies the boundary conditions;

$$n^c(T_i) = n^c_{eq}(T_i)$$

We may write this solution as,

$$\begin{aligned} n^c(T) &= T^\alpha\left(k + \alpha\int_T^{T_i}\frac{dT'}{T'^{1+\alpha}}n^c_{eq}(T')\right) \\ &= T^\alpha\left(k - \alpha\int_{T_i}^{T}\frac{dT'}{T'^{1+\alpha}}n^c_{eq}(T')\right) \end{aligned}$$

where $k = \left(\frac{1}{T_i}\right)^\alpha n^c_{eq}(T_i)$.

Then,

$$\begin{aligned} \frac{d}{dT}n^c &= \frac{d}{dT}\left(T^\alpha\left(k - \alpha\int^T\frac{dT'}{T'^{1+\alpha}}n^c_{eq}(T')\right)\right) \\ &= \alpha T^{\alpha-1}\left(k - \alpha\int^T\frac{dT'}{T'^{1+\alpha}}n^c_{eq}(T')\right) - \alpha T^\alpha\frac{1}{T^{1+\alpha}}n^c_{eq}(T) \\ &= \alpha\frac{1}{T}n^c(T) - \alpha\frac{1}{T}n^c_{eq}(T) = \frac{\alpha}{T}\left(n^c - n^c_{eq}\right) \end{aligned}$$

so we see that indeed it solves the Boltzmann equation.

Using the approximation;

$$\int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T') \simeq \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T)$$

then,

$$\begin{aligned} n^c(T) &= \left(\frac{T}{T_i}\right)^\alpha n^c_{eq}(T_i) + \alpha\, T^\alpha \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T') \\ &\simeq \left(\frac{T}{T_i}\right)^\alpha n^c_{eq}(T_i) + \alpha\, T^\alpha n^c_{eq}(T) \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} \\ &= \left(\frac{T}{T_i}\right)^\alpha n^c_{eq}(T_i) + \alpha\, T^\alpha n^c_{eq}(T) \left[-\frac{1}{\alpha}\frac{1}{T'^\alpha}\right]_T^{T_i} \\ &= \left(\frac{T}{T_i}\right)^\alpha n^c_{eq}(T_i) + \alpha\, T^\alpha n^c_{eq}(T) \left(\frac{1}{\alpha}\frac{1}{T^\alpha} - \frac{1}{\alpha}\frac{1}{T_i^\alpha}\right) \\ &= n^c_{eq}(T) + \left(\frac{T}{T_i}\right)^\alpha \left(n^c_{eq}(T_i) - n^c_{eq}(T)\right) \end{aligned}$$

and for $T < T_i$ and $\alpha \gg 1$ then $\left(\frac{T}{T_i}\right)^\alpha \simeq 0$ so,

$$n^c(T) \simeq n^c_{eq}(T)$$

If we Taylor expand about the lower limit, we find,

$$\alpha\, T^\alpha \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T') \simeq \alpha\, T^\alpha \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} \left(n^c_{eq}(T) + (T'-T)\frac{d}{dT} n^c_{eq}(T) + \dots\right)$$

Using our approximations we therefore find;

$$\begin{aligned} \alpha\, T^\alpha \int_T^{T_i} & \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T') \simeq n^c_{eq}(T)\alpha\, T^\alpha \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} + \frac{d}{dT} n^c_{eq}(T)\alpha\, T^\alpha \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}}(T'-T) + \dots \\ &= \alpha\, T^\alpha n^c_{eq}(T) \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} + \alpha\, T^\alpha \frac{d}{dT} n^c_{eq}(T) \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}}(T'-T) + \dots \\ &= \alpha\, T^\alpha n^c_{eq}(T) \left[-\frac{1}{\alpha}\frac{1}{T'^\alpha}\right]_T^{T_i} + \alpha\, T^\alpha \frac{d}{dT} n^c_{eq}(T) \left[-\frac{1}{\alpha-1}\frac{1}{T'^{\alpha-1}} + \frac{1}{\alpha}\frac{T}{T'^\alpha}\right]_T^{T_i} + \dots \\ &= \alpha\, T^\alpha n^c_{eq}(T) \left(\frac{1}{\alpha}\frac{1}{T^\alpha} - \frac{1}{\alpha}\frac{1}{T_i^\alpha}\right) + \alpha\, T^\alpha \frac{d}{dT} n^c_{eq}(T) \Big(\frac{1}{\alpha-1}\frac{1}{T^{\alpha-1}} - \frac{1}{\alpha-1}\frac{1}{T_i^{\alpha-1}} \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{\alpha}\frac{T}{T_i^\alpha} - \frac{1}{\alpha}\frac{T}{T^\alpha}\Big) + \dots \\
= \quad & n_{eq}^c(T) - n_{eq}^c(T)\left(\frac{T}{T_i}\right)^\alpha + T\frac{d}{dT}n_{eq}^c(T)\Big(\frac{\alpha}{\alpha-1} - \frac{\alpha}{\alpha-1}\left(\frac{T}{T_i}\right)^{\alpha-1} \\
& + \left(\frac{T}{T_i}\right)^\alpha - 1\Big) + \dots
\end{aligned}
$$

Now, for $T < T_i$ then, $(T/T_i)^\alpha \gg 1$, so then,

$$
\begin{aligned}
\alpha\, T^\alpha \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n_{eq}^c(T') \quad & \simeq \quad n_{eq}^c(T) + T\frac{d}{dT}n_{eq}^c(T)\Big(\frac{\alpha}{\alpha-1} - 1\Big) \\
& \simeq \quad n_{eq}^c(T) + T\frac{d}{dT}n_{eq}^c(T)\Big(\frac{1}{\alpha-1}\Big) \simeq n_{eq}^c(T) + \frac{1}{\alpha}T\frac{d}{dT}n_{eq}^c(T)
\end{aligned}
$$

Finally we conclude that the correction is small provided;

$$
n_{eq}^c \gg \frac{1}{\alpha}T\frac{dn_{eq}^c}{dT}
$$

which implies,

$$
\alpha \gg \frac{T}{n_{eq}^c}\frac{dn_{eq}^c}{dT}
$$

and hence gives the bound,

$$
\alpha \gg \frac{d\log n_{eq}^c}{d\log T}
$$

For a relativistic species we have,

$$
n_{eq} \propto T^3\ , \qquad n_{eq}^c = a^3 n_{eq} \sim \text{const}
$$

Hence,

$$
\frac{d\log n_{eq}^c}{d\log T} = 0
$$

and hence for any $\alpha \gg 1$ will ensure equilibrium for all temperatures $T < T_i$.

However, for a non-relativistic species we have,

$$
n_{eq} \propto T^{3/2} e^{-\frac{m}{kT}}\ , \qquad n_{eq}^c = a^3 n_{eq} \sim bT^{-3/2} e^{-\frac{m}{kT}}
$$

for a constant b. Then,

$$\begin{aligned}\frac{d\log n_{eq}^c}{d\log T} &= \frac{T}{n_{eq}^c}\frac{dn_{eq}^c}{dT} = \frac{T}{bT^{-3/2}e^{-\frac{m}{kT}}}\frac{d}{dT}\left(bT^{-3/2}e^{-\frac{m}{kT}}\right)\\ &= -\frac{3}{2}+\frac{m}{kT}\end{aligned}$$

Hence for sufficiently low temperature so that $m/kT \sim \alpha$ then equilibrium will not be preserved.

Qu. 7 Consider a metric with coordinates $x^\mu = (t, x^i)$;

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + g_{ij}(t,x) dx^i dx^j$$

The Liouville condition for the phase space distribution $n(t, x^i, p_j)$ of free particles is;

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial n}{\partial p_i}\frac{dp_i}{dt}$$

Show that for a particular free particle with affine parameter λ and 4-momentum $p^\mu = dx^\mu / d\lambda$ then;

$$\frac{dx^i}{dt} = \frac{p^i}{p^0}$$

Consider the Christoffel connection for the metric $\tilde{g}_{\mu\nu}$ and compute the following components, $\tilde{\Gamma}^t{}_{\mu\nu}$, to show;

$$\begin{aligned} \tilde{\Gamma}^i{}_{tt} &= 0 \\ \tilde{\Gamma}^i{}_{tk} &= \frac{1}{2} g^{ij} \partial_t g_{jk} \\ \tilde{\Gamma}^i{}_{kl} &= \Gamma^i{}_{kl} \end{aligned}$$

where $\Gamma^i{}_{jk}$ are the Christoffel components of the spatial metric $g_{ij}(t,x)$. Now show that (note the index is 'down'),

$$\frac{dp_i}{dt} = p^j \frac{\partial g_{ij}}{\partial t} + \frac{1}{p^0} p^j p^k \frac{\partial g_{ij}}{\partial x^k} + g_{ij} \frac{dp^j}{dt}$$

and show that (note the index is 'up'),

$$\frac{dp^i}{dt} = \frac{1}{p^0} \frac{d^2 x^i}{d\lambda^2}$$

and hence use the geodesic equation for the free particle to show that,

$$\dot{p}_i = \frac{1}{2} \frac{1}{p^0} p^j p^k \partial_i g_{jk}$$

Hence we derive the Boltzmann equation for free particles;

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{p^i}{p^0} \frac{\partial n}{\partial x^i} + \frac{1}{2} \frac{1}{p^0} p^j p^k \partial_i g_{jk} \frac{\partial n}{\partial p_i} = 0$$

Qu. 7 answer Consider a particle with curve $x^\mu(\lambda)$, affine parameter λ and 4-momentum $p^\mu = dx^\mu/d\lambda$. Then,

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{dx^i}{d\lambda}\left(\frac{dt}{d\lambda}\right)^{-1} = p^i(p^0)^{-1}$$

as required.

The geodesic equation for the metric;

$$ds^2 = \tilde{g}_{\mu\nu}(t,x)dx^\mu dx^\mu = -dt^2 + g_{ij}(t,x)dx^i dx^j$$

is,

$$\frac{d^2x^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

Now;

$$\begin{aligned}
\tilde{\Gamma}^i{}_{\alpha\beta} &= \frac{1}{2}\tilde{g}^{i\rho}\left(\partial_\alpha\tilde{g}_{\rho\beta} + \partial_\beta\tilde{g}_{\rho\alpha} - \partial_\rho\tilde{g}_{\alpha\beta}\right) \\
&= \frac{1}{2}\tilde{g}^{ij}\left(\partial_\alpha\tilde{g}_{j\beta} + \partial_\beta\tilde{g}_{j\alpha} - \partial_j\tilde{g}_{\alpha\beta}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\tilde{\Gamma}^i{}_{tt} &= \frac{1}{2}\tilde{g}^{ij}\left(\partial_t\tilde{g}_{jt} + \partial_t\tilde{g}_{jt} - \partial_j\tilde{g}_{tt}\right) = 0 \\
\tilde{\Gamma}^i{}_{tk} &= \frac{1}{2}\tilde{g}^{ij}\left(\partial_t\tilde{g}_{jk} + \partial_k\tilde{g}_{jt} - \partial_j\tilde{g}_{tk}\right) = \frac{1}{2}g^{ij}\partial_t g_{jk} \\
\tilde{\Gamma}^i{}_{kl} &= \frac{1}{2}\tilde{g}^{ij}\left(\partial_k\tilde{g}_{jl} + \partial_l\tilde{g}_{jk} - \partial_j\tilde{g}_{kl}\right) = \Gamma^i{}_{kl}
\end{aligned}$$

So;

$$\begin{aligned}
0 &= \frac{d^2x^i}{d\lambda^2} + \tilde{\Gamma}^i{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \\
&= \frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} + g^{ij}\partial_t g_{jk}\frac{dx^t}{d\lambda}\frac{dx^k}{d\lambda} \\
&= \frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{jk}p^j p^k + g^{ij}\partial_t g_{jk}p^0 p^k
\end{aligned}$$

Now;

$$\begin{aligned}
\frac{dp_i}{dt} &= \frac{d(g_{ij}p^j)}{dt} = p^j\frac{dg_{ij}}{dt} + g_{ij}\frac{dp^j}{dt} \\
&= p^j\frac{\partial g_{ij}}{\partial t} + p^j\frac{dx^k}{dt}\frac{\partial g_{ij}}{\partial x^k} + g_{ij}\frac{dp^j}{dt} \\
&= p^j\frac{\partial g_{ij}}{\partial t} + \frac{1}{p^0}p^jp^k\frac{\partial g_{ij}}{\partial x^k} + g_{ij}\frac{dp^j}{dt}
\end{aligned}$$

Now;

$$\frac{dp^i}{dt} = \frac{d\lambda}{dt}\frac{dp^i}{d\lambda} = \frac{d\lambda}{dt}\frac{dp^i}{d\lambda} = \frac{1}{p^0}\frac{dp^i}{d\lambda} = \frac{1}{p^0}\frac{d^2x^i}{d\lambda^2}$$

So,

$$\begin{aligned}
\frac{dp_i}{dt} &= p^j\frac{\partial g_{ij}}{\partial t} + \frac{1}{p^0}p^jp^k\frac{\partial g_{ij}}{\partial x^k} + g_{ij}\frac{1}{p^0}\frac{d^2x^j}{d\lambda^2} \\
&= p^j\partial_t g_{ij} + \frac{1}{p^0}p^jp^k\partial_k g_{ij} - g_{ij}\frac{1}{p^0}\left(\Gamma^j{}_{mn}p^mp^n + g^{jm}\partial_t g_{mn}p^0p^n\right) \\
&= p^j\partial_t g_{ij} + \frac{1}{p^0}p^jp^k\partial_k g_{ij} - g_{ij}\frac{1}{p^0}\Gamma^j{}_{mn}p^mp^n - p^n\delta^m_i\partial_t g_{mn} \\
&= \frac{1}{p^0}p^jp^k\partial_k g_{ij} - \frac{1}{p^0}g_{ij}\Gamma^j{}_{mn}p^mp^n \\
&= \frac{1}{p^0}p^jp^k\partial_k g_{ij} - \frac{1}{2}\frac{1}{p^0}g_{ij}p^mp^ng^{jk}\left(\partial_m g_{kn} + \partial_n g_{km} - \partial_k g_{mn}\right) \\
&= \frac{1}{p^0}p^jp^k\partial_k g_{ij} - \frac{1}{2}\frac{1}{p^0}p^mp^n\left(\partial_m g_{in} + \partial_n g_{im} - \partial_i g_{mn}\right) \\
&= \frac{1}{p^0}p^jp^k\partial_k g_{ij} - \frac{1}{2}\frac{1}{p^0}p^jp^k\left(2\partial_k g_{ij} - \partial_i g_{jk}\right) \\
&= \frac{1}{2}\frac{1}{p^0}p^jp^k\partial_i g_{jk}
\end{aligned}$$