**Cosmology and Particle Physics** **Toby Wiseman**

**Example sheet 2**

**Qu. 1** Recall that the kinetic relations

$$\rho = \int dp 4\pi p^2 n(p) E \ , \quad P = \int dp 4\pi p^2 n(p) \frac{p^2}{3E}$$

relate the density and pressure of a gas to its density distribution function $n(p)$, where $E = \sqrt{m^2 + p^2}$. Consider a bosonic(-) or fermion(+) field, with mass $m$, chemical potential $\mu$, with $g$ internal spin degrees of freedom. In thermal equilibrium the density distribution function is;

$$n(p) \quad = \quad \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{E-\mu}{kT}} \pm 1}$$

where $E = \sqrt{m^2 + p^2}$. In the ultra relativistic limit $kT \gg m, \mu$, then $E - \mu \sim E \sim p$ so this is well approximated by;

$$n(p) \quad = \quad \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{p}{kT}} \pm 1}$$

Thus compute the number density $n = \int dp 4\pi p^2 n(p)$ and energy density of an ultra relativistic gas (recall pressure $P = \rho/3$ from the above kinetic relations). You should find for a boson;

$$n_{boson} \quad = \quad \frac{15\zeta(3) a_B g}{k\pi^4} T^3 \ , \qquad \rho_{boson} = \frac{1}{2} g a_B T^4$$

and for a fermion;

$$n_{fermion} \quad = \quad \frac{3}{4} n_{boson} \ , \qquad \rho_{fermion} = \frac{7}{8} \rho_{boson}$$

where the radiation constant $a_B = \pi^2 k^4 / 15\hbar^3 c^3$ (although we are using units where $c = 1$). You may find the following integrals useful;

$$\int_0^\infty dx \frac{x^2}{e^x \pm 1} = \frac{7 \mp 1}{4} \zeta(3) \ , \quad \int_0^\infty dx \frac{x^3}{e^x \pm 1} = \frac{15 \mp 1}{240} \pi^4$$

where $\zeta(x)$ is the Riemann zeta function and in particular $\zeta(3) \simeq 1.202$.

Use the first law to show the equilibrium entropy density is $s = 4\rho/3T$. Also show that it implies $d\rho = T ds$, and check this is true for the expressions you have computed.

**Qu. 2** Repeat the calculations in Qu 1 in the non-relativistic limit $kT \ll E - \mu$ and $E \simeq m + \frac{p^2}{2m}$ so;

$$n(p) \;=\; \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu - m}{kT}} e^{-\frac{p^2}{2mkT}}$$

Firstly show that;

$$n \;=\; \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu - m}{kT}} (2\pi m kT)^{\frac{3}{2}}$$

and then show;

$$\rho = \left( m + \frac{3}{2} kT \right) n \,, \qquad P = kTn$$

You may find it useful to recall that for a Gaussian integral;

$$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

Use the 1st law in a closed system (so the total particle number cannot change) to show the equilibrium entropy density $s$ in this case obeys;

$$n\, d\rho - nT ds \;=\; dn\, (\rho + p - Ts)$$

and hence integrate this to find;

$$s \;=\; k\, n \log \left( \frac{T^{3/2}}{c\, n} \right)$$

for a constant $c$.

**Qu. 3** Use the observed Hubble parameter today $H_0 \simeq 70 kms^{-1}Mpc^{-1}$, and $\Omega_\Lambda \simeq 0.7$, $\Omega_{matter} \simeq 0.3$ and assuming a flat FRW geometry, compute the density of non-relativistic matter today. You should find a density of $\sim 2.7 \times 10^{-27} kg\, m^{-3}$.

The photon radiation today (CMB photons) while free streaming and not in equilibrium, has almost exactly a bose distribution with temperature $2.7K$. Hence show its (very small) contribution to the Hubble expansion today is;

$$\Omega_\gamma \sim 5 \times 10^{-5}$$

You will need the values of the constants;

$$\begin{aligned} 1pc &= 3.2\,\text{light years}\,, \quad c = 3.0 \times 10^8 ms^{-1} \\ \hbar &= 1.05 \times 10^{-34} m^2 kg s^{-1}\,, \quad k = 1.38 \times 10^{-23} m^2 kg s^{-2} K^{-1} \end{aligned}$$

The total radiation fraction today, $\Omega_R = 1.68\Omega_\gamma$ as we shall show later in the course due to the presence of neutrinos. Use the Friedmann equation to show that radiation came to dominate the Hubble expansion over matter at a redshift of $Z_{eq} \sim 3600$.

**Qu. 4** Consider the Boltzmann equation for the density distribution function $n(t,p)$ of a species with mass $m$ and with chemical potential $\mu$;

$$\frac{\partial n}{\partial t} - Hp\frac{\partial n}{\partial p} = C$$

where $H = \dot{a}/a$. Suppose at early times the interaction term $C$ is very large and the species is in thermal equilibrium so;

$$n(t,p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{E-\mu(t)}{kT(t)}} \pm 1}$$

where $T(t)$ and $\mu(t)$ are the temperature and chemical potential of the heat bath the species is in equilibrium with at time $t$. However, suppose interactions rapidly turn off at time $t_{freeze}$, with temperature $T_{freeze}$, when the scale factor is $a_{freeze}$, and subsequently the species then free streams.

Firstly, show that if the interactions turn off in the ultra relativistic regime $kT \gg m, \mu$ then,

$$n(t,p) = \frac{g}{(2\pi\hbar)^3}\frac{1}{e^{\frac{p}{kT_{eff}(t)}} \pm 1} , \qquad T_{eff}(t) = \frac{a_{freeze}}{a(t)}T_{freeze}$$

Is it true in this case that the distribution is a thermal distribution simply with a redshifted temperature $T_{eff}(T)$? (imagine what happens when the temperature falls below the mass scale of the particle).

Secondly, show that if the interactions turn off in the non-relativistic regime $kT \ll E - \mu$ then,

$$n(t,p) = \frac{g}{(2\pi\hbar)^3}e^{\frac{\mu_{freeze}-m}{kT_{freeze}}} e^{-\frac{p^2}{2mkT_{eff}(t)}} , \qquad T_{eff}(t) = \left(\frac{a_{freeze}}{a(t)}\right)^2 T_{freeze}$$

**Qu. 5** Consider the Boltzmann equation for a fermion species with mass $m$ and vanishing chemical potential and number of spin degrees of freedom $g$ interacting with the photons in the universe, which we approximate as a heat bath with temperature $T$ related to the scale factor $a$ as,

$$\frac{a}{a_0} = \frac{T_0}{T}$$

where $a_0, T_0$ are the scale factor and temperature today. Assume they do not interact with anything else. The Boltzmann equation is;

$$\frac{d \ln n^c(T)}{d \ln T} = \frac{\Gamma}{H}\left(1 - \frac{n^c_{eq}(T)}{n^c(T)}\right)$$

where the comoving number density and comoving equilibrium density are defined as;

$$n^c = \left(\frac{a}{a_0}\right)^3 n \ , \qquad n^c_{eq} = \left(\frac{a}{a_0}\right)^3 n_{eq}$$

for physical density $n$ and equilibrium density $n_{eq}$. Assume that at early times then $n^c \simeq n^c_{eq}$ and $\Gamma/H \gg 1$ and when the temperature drops to $T = T_{freeze}$ then $\Gamma/H \sim 1$ and at later times (and lower temperatures) $\Gamma/H \ll 1$. Show that an approximate relic density today for this fermion species if the freeze out occurs when the species is non-relativistic is,

$$\rho_{relic}(T) \quad = \quad T^3 \frac{gm}{\hbar^3}\left(\frac{mk}{2\pi T_{freeze}}\right)^{\frac{3}{2}} e^{-\frac{m}{kT_{freeze}}}$$

Conversely if freeze out occurs in its ultra relativistic regime, temperature is $kT_{freeze} \gg m$, but the temperature today is low so that it is non-relativistic, show;

$$\rho_{relic}(T) \quad = \quad \frac{20\,\zeta(3) a_B g\, m}{k\pi^4} T^3$$

Show that these two answers are consistent with the results for the relic density distributions in Qu 4. by computing the relic densities from these density distributions.

**Qu. 6** Consider the Boltzmann equation as in the previous question,

$$\frac{d \ln n^c(T)}{d \ln T} = \frac{\Gamma}{H} \left( 1 - \frac{n^c_{eq}(T)}{n^c(T)} \right)$$

Suppose we treat $\Gamma/H$ as being constant, so that $\alpha = \Gamma/H$. Let us consider starting the system at a temperature $T_i$ with the in thermal equilibrium so that $n^c(T_i) = n^c_{eq}(T_i)$. Then consider evolving to lower temperatures $T < T_i$.

Confirm that the solution of the Boltzmann equation for $T \leq T_i$ (assuming $\alpha$ is constant) is then,

$$n^c(T) = \left( \frac{T}{T_i} \right)^{\alpha} n^c_{eq}(T_i) + \alpha \, T^{\alpha} \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T')$$

and check that it obeys the boundary condition at $T = T_i$.

For sufficiently large $\alpha$, and $T < T_i$, and assuming $n^c_{eq}(T)$ is smooth, we may approximate;

$$\int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T') \simeq \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n^c_{eq}(T)$$

since the function $1/T'^{1+\alpha}$ is very strongly peaked at the lower limit $T$ of the integral. Use this to show that for sufficiently large $\alpha$, then,

$$n^c(T) \simeq n^c_{eq}(T)$$

for $T < T_i$. Further, by Taylor expanding about the lower limit, show that (the constant) $\alpha$ must be large on a scale determined by the form of $n^c_{eq}$, so,

$$\alpha \gg \frac{d \log n^c_{eq}}{d \log T}$$

Consider a universe with thermal bath temperature $T \sim 1/a$. Show that for a relativistic species the above bound is always true provided $\alpha \gg 1$, and then equilibrium is maintained for all low temperatures $T < T_i$. However, for a non-relativistic species with mass $m$ and with no chemical potential, for equilibrium to persist to a temperature $T < T_i$ then the constant $\alpha$ must be bounded as,

$$\alpha \gg \frac{m}{kT} > 1$$

and hence at sufficiently low temperatures equilibrium cannot be maintained even for large values of $\alpha$. Basically the decay rate is not sufficiently high to reduce the density particles to their very rapidly decreasing equilibrium value.

**Qu. 7** Consider a metric with coordinates $x^\mu = (t, x^i)$;

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + g_{ij}(t,x) dx^i dx^j$$

The Liouville condition for the phase space distribution $n(t, x^i, p_j)$ of free particles is;

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial n}{\partial p_i}\frac{dp_i}{dt}$$

Show that for a particular free particle with affine parameter $\lambda$ and 4-momentum $p^\mu = dx^\mu / d\lambda$ then;

$$\frac{dx^i}{dt} = \frac{p^i}{p^0}$$

Consider the Christoffel connection for the metric $\tilde{g}_{\mu\nu}$ and compute the following components, $\tilde{\Gamma}^t{}_{\mu\nu}$, to show;

$$\begin{aligned}
\tilde{\Gamma}^i{}_{tt} &= 0 \\
\tilde{\Gamma}^i{}_{tk} &= \frac{1}{2} g^{ij} \partial_t g_{jk} \\
\tilde{\Gamma}^i{}_{kl} &= \Gamma^i{}_{kl}
\end{aligned}$$

where $\Gamma^i{}_{jk}$ are the Christoffel components of the spatial metric $g_{ij}(t,x)$. Now show that (note the index is 'down'),

$$\frac{dp_i}{dt} = p^j \frac{\partial g_{ij}}{\partial t} + \frac{1}{p^0} p^j p^k \frac{\partial g_{ij}}{\partial x^k} + g_{ij} \frac{dp^j}{dt}$$

and show that (note the index is 'up'),

$$\frac{dp^i}{dt} = \frac{1}{p^0}\frac{d^2 x^i}{d\lambda^2}$$

and hence use the geodesic equation for the free particle to show that,

$$\dot{p}_i = \frac{1}{2}\frac{1}{p^0} p^j p^k \partial_i g_{jk}$$

Hence we derive the Boltzmann equation for free particles;

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{p^i}{p^0}\frac{\partial n}{\partial x^i} + \frac{1}{2}\frac{1}{p^0} p^j p^k \partial_i g_{jk} \frac{\partial n}{\partial p_i} = 0$$