Example sheet 2

Qu. 1 Recall that the kinetic relations
\[ \rho = \int dp 4\pi p^2 n(p) E, \quad P = \int dp 4\pi p^2 n(p) \frac{p^2}{3E} \]
relate the density and pressure of a gas to its density distribution function \(n(p)\), where \(E = \sqrt{m^2 + p^2}\). Consider a bosonic(-) or fermion(+) field, with mass \(m\), chemical potential \(\mu\), with \(g\) internal spin degrees of freedom. In thermal equilibrium the density distribution function is;
\[ n(p) = \frac{g}{(2\pi \hbar)^3} \frac{1}{e^{E-\mu/kT} \pm 1} \]
where \(E = \sqrt{m^2 + p^2}\). In the ultra relativistic limit \(kT \gg m, \mu\), then \(E-\mu \sim E \sim p\) so this is well approximated by;
\[ n(p) = \frac{g}{(2\pi \hbar)^3} \frac{1}{e^{E/kT} \pm 1} \]
Thus compute the number density \(n = \int dp 4\pi p^2 n(p)\) and energy density of an ultra relativistic gas (recall pressure \(P = \rho/3\) from the above kinetic relations). You should find for a boson;
\[ n_{\text{boson}} = \frac{15\zeta(3) a_B g}{k\pi^4} T^3, \quad \rho_{\text{boson}} = \frac{1}{2} g a_B T^4 \]
and for a fermion;
\[ n_{\text{fermion}} = \frac{4}{3} n_{\text{boson}}, \quad \rho_{\text{fermion}} = \frac{7}{8} \rho_{\text{boson}} \]
where the radiation constant \(a_B = \pi^2 k^4/15\hbar^3 c^3\) (although we are using units where \(c = 1\)). You may find the following integrals useful;
\[ \int_0^\infty dx \frac{x^2}{e^x \pm 1} = \frac{7 + 1}{4} \zeta(3), \quad \int_0^\infty dx \frac{x^2}{e^x \pm 1} = \frac{15 + 1}{240} \pi^4 \]
where \(\zeta(x)\) is the Riemann zeta function and in particular \(\zeta(3) \approx 1.202\).
Use the first law to show the equilibrium entropy density is \(s = 4\rho/3T\).
Also show that it implies \(d\rho = Tds\), and check this is true for the expressions you have computed.
Qu. 2 Repeat the calculations in Qu 1 in the non-relativistic limit \(kT \ll E - \mu\) and \(E \simeq m + \frac{p^2}{2m}\) so;

\[
n(p) = \frac{g}{(2\pi \hbar)^3} e^{\frac{\mu-m}{kT}} e^{-\frac{p^2}{2mkT}}
\]

Firstly show that;

\[
n = \frac{g}{(2\pi \hbar)^3} e^{\frac{\mu-m}{kT}} (2\pi mkT)^{\frac{3}{2}}
\]

and then show;

\[
\rho = \left( m + \frac{3}{2} kT \right) n, \quad P = kT n
\]

You may find it useful to recall that for a Gaussian integral;

\[
\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}
\]

Use the 1st law in a closed system (so the total particle number cannot change) to show the equilibrium entropy density \(s\) in this case obeys;

\[
n d\rho - nT ds = dn (\rho + p - Ts)
\]

and hence integrate this to find;

\[
s = kn \log \left( \frac{T^{3/2}}{c n} \right)
\]

for a constant \(c\).
Qu. 3 Use the observed Hubble parameter today $H_0 \simeq 70\text{km}s^{-1}\text{Mpc}^{-1}$, and $\Omega_\Lambda \simeq 0.7$, $\Omega_{\text{matter}} \simeq 0.3$ and assuming a flat FRW geometry, compute the density of non-relativistic matter today. You should find a density of $\sim 2.7 \times 10^{-27}\text{kgm}^{-3}$.

The photon radiation today (CMB photons) while free streaming and not in equilibrium, has almost exactly a bose distribution with temperature $2.7K$. Hence show its (very small) contribution to the Hubble expansion today is;

$$\Omega_\gamma \sim 5 \times 10^{-5}$$

You will need the values of the constants;

$$1\text{pc} = 3.2 \text{light years},\quad c = 3.0 \times 10^8 \text{ms}^{-1}$$

$$\hbar = 1.05 \times 10^{-34}\text{m}^2\text{kgs}^{-1},\quad k = 1.38 \times 10^{-23}\text{m}^2\text{kgs}^{-2}\text{K}^{-1}$$

The total radiation fraction today, $\Omega_R = 1.68\Omega_\gamma$, as we shall show later in the course due to the presence of neutrinos. Use the Friedmann equation to show that radiation came to dominate the Hubble expansion over matter at a redshift of $Z_{eq} \sim 3600$. 
Qu. 4 Consider the Boltzmann equation for the density distribution function $n(t,p)$ of a species with mass $m$ and with chemical potential $\mu$;

$$\frac{\partial n}{\partial t} - H_p \frac{\partial n}{\partial p} = C$$

where $H = \dot{a}/a$. Suppose at early times the interaction term $C$ is very large and the species is in thermal equilibrium so;

$$n(t,p) = \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{E-\mu(t)}{kT(t)}}} \pm 1$$

where $T(t)$ and $\mu(t)$ are the temperature and chemical potential of the heat bath the species is in equilibrium with at time $t$. However, suppose interactions rapidly turn off at time $t_{\text{freeze}}$, with temperature $T_{\text{freeze}}$, when the scale factor is $a_{\text{freeze}}$, and subsequently the species then free streams.

Firstly, show that if the interactions turn off in the ultra relativistic regime $kT \gg m, \mu$ then,

$$n(t,p) = \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{p}{\sqrt{2}mT_{\text{eff}}(t)}}} \pm 1, \quad T_{\text{eff}}(t) = \frac{a_{\text{freeze}}}{a(t)} T_{\text{freeze}}$$

Is it true in this case that the distribution is a thermal distribution simply with a redshifted temperature $T_{\text{eff}}(T)$? (imagine what happens when the temperature falls below the mass scale of the particle).

Secondly, show that if the interactions turn off in the non-relativistic regime $kT \ll E - \mu$ then,

$$n(t,p) = \frac{g}{(2\pi\hbar)^3} e^{\frac{\mu_{\text{freeze}}-m}{\sqrt{2}mk^2_{\text{eff}(t)}}}, \quad T_{\text{eff}}(t) = \left(\frac{a_{\text{freeze}}}{a(t)}\right)^2 T_{\text{freeze}}$$
Consider the Boltzmann equation for a fermion species with mass $m$ and vanishing chemical potential and number of spin degrees of freedom $g$ interacting with the photons in the universe, which we approximate as a heat bath with temperature $T$ related to the scale factor $a$ as,

$$\frac{a}{a_0} = \frac{T_0}{T}$$

where $a_0, T_0$ are the scale factor and temperature today. Assume they do not interact with anything else. The Boltzmann equation is;

$$\frac{d \ln n^c(T)}{d \ln T} = \frac{\Gamma}{H} \left( 1 - \frac{n^c_{eq}(T)}{n^c(T)} \right)$$

where the comoving number density and comoving equilibrium density are defined as;

$$n^c = \left( \frac{a}{a_0} \right)^3 n, \quad n^c_{eq} = \left( \frac{a}{a_0} \right)^3 n_{eq}$$

for physical density $n$ and equilibrium density $n_{eq}$. Assume that at early times then $n^c \simeq n^c_{eq}$ and $\Gamma/H \gg 1$ and when the temperature drops to $T = T_{freeze}$ then $\Gamma/H \sim 1$ and at later times (and lower temperatures) $\Gamma/H \ll 1$. Show that an approximate relic density today for this fermion species if the freeze out occurs when the species is non-relativistic is,

$$\rho_{\text{relic}}(T) = T^3 \frac{g m}{\hbar^3} \left( \frac{mk}{2\pi T_{\text{freeze}}} \right)^{\frac{3}{2}} e^{-\frac{m}{kT_{\text{freeze}}}}$$

Conversely if freeze out occurs in its ultra relativistic regime, temperature is $kT_{\text{freeze}} \gg m$, but the temperature today is low so that it is non-relativistic, show;

$$\rho_{\text{relic}}(T) = \frac{20 \zeta(3) a_B g m}{k \pi^4} T^3$$

Show that these two answers are consistent with the results for the relic density distributions in Qu 4. by computing the relic densities from these density distributions.
Qu. 6 Consider the Boltzmann equation as in the previous question,
\[
\frac{d \ln n^c(T)}{d \ln T} = \frac{\Gamma}{H} \left( 1 - \frac{n_{eq}^c(T)}{n^c(T)} \right)
\]
Suppose we treat \( \Gamma/H \) as being constant, so that \( \alpha = \Gamma/H \). Let us consider starting the system at a temperature \( T_i \) with the in thermal equilibrium so that \( n^c(T_i) = n_{eq}^c(T_i) \). Then consider evolving to lower temperatures \( T < T_i \).

Confirm that the solution of the Boltzmann equation for \( T \leq T_i \) (assuming \( \alpha \) is constant) is then,
\[
n^c(T) = \left( \frac{T}{T_i} \right)^\alpha n_{eq}^c(T_i) + \alpha T^\alpha \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n_{eq}^c(T')
\]
and check that it obeys the boundary condition at \( T = T_i \).

For sufficiently large \( \alpha \), and \( T < T_i \), and assuming \( n_{eq}^c(T) \) is smooth, we may approximate;
\[
\int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n_{eq}^c(T') \simeq \int_T^{T_i} \frac{dT'}{T'^{1+\alpha}} n_{eq}^c(T)
\]
since the function \( 1/T'^{1+\alpha} \) is very strongly peaked at the lower limit \( T \) of the integral. Use this to show that for sufficiently large \( \alpha \), then,
\[
n^c(T) \simeq n_{eq}^c(T)
\]
for \( T < T_i \). Further, by Taylor expanding about the lower limit, show that (the constant) \( \alpha \) must be large on a scale determined by the form of \( n_{eq}^c \), so,
\[
\alpha \gg \frac{d \log n_{eq}^c}{d \log T}
\]

Consider a universe with thermal bath temperature \( T \sim 1/a \). Show that for a relativistic species the above bound is always true provided \( \alpha \gg 1 \), and then equilibrium is maintained for all low temperatures \( T < T_i \). However, for a non-relativistic species with mass \( m \) and with no chemical potential, for equilibrium to persist to a temperature \( T < T_i \) then the constant \( \alpha \) must be bounded as,
\[
\alpha \gg \frac{m}{kT} > 1
\]
and hence at sufficiently low temperatures equilibrium cannot be maintained even for large values of \( \alpha \). Basically the decay rate is not sufficiently high to reduce the density particles to their very rapidly decreasing equilibrium value.
**Qu. 7** Consider a metric with coordinates $x^\mu = (t, x^i)$;

$$ds^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + g_{ij}(t, x)dx^i dx^j$$

The Liouville condition for the phase space distribution $n(t, x^i, p_j)$ of free particles is;

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial n}{\partial p_i} \frac{dp_i}{dt}$$

Show that for a particular free particle with affine parameter $\lambda$ and 4-momentum $p^\mu = dx^\mu / d\lambda$ then;

$$\frac{dx^i}{dt} = \frac{p^i}{p^0}$$

Consider the Christoffel connection for the metric $\tilde{g}_{\mu\nu}$ and compute the following components, $\tilde{\Gamma}^i_{\mu\nu}$, to show;

$$\tilde{\Gamma}^i_{tt} = 0$$
$$\tilde{\Gamma}^i_{tk} = \frac{1}{2} g^{ij} \partial_t g_{jk}$$
$$\tilde{\Gamma}^i_{kl} = \Gamma^i_{kl}$$

where $\Gamma^i_{jk}$ are the Christoffel components of the spatial metric $g_{ij}(t, x)$. Now show that (note the index is ‘down’),

$$\frac{dp_i}{dt} = p^j \frac{\partial g_{ij}}{\partial t} + \frac{1}{p^0} p^j p^k \frac{\partial g_{ij}}{\partial x^k} + g_{ij} \frac{dp^j}{dt}$$

and show that (note the index is ‘up’),

$$\frac{dp^i}{dt} = \frac{1}{p^0} \frac{d^2 x^i}{d\lambda^2}$$

and hence use the geodesic equation for the free particle to show that,

$$\dot{p}_i = \frac{1}{2} \frac{1}{p^0} p^j p^k \partial_i g_{jk}$$

Hence we derive the Boltzmann equation for free particles;

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{p^i}{p^0} \frac{\partial n}{\partial x^i} + \frac{1}{2} \frac{1}{p^0} p^j p^k \partial_i g_{jk} \frac{\partial n}{\partial p_i} = 0$$