$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$$

Cosmology RF

1. $ds^2 = \tilde{g}_{ij} dx^i dx^j = S(r)\, dr^2 + r^2 d\Omega_2$

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$\tilde{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

so,

$$\tilde{g}_{ij} = \begin{pmatrix} S(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad \tilde{g}^{ij} = \begin{pmatrix} \frac{1}{S} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

(diagonal)

$$\Rightarrow \tilde{\Gamma}^r_{rr} = \frac{1}{2} g^{rr} (\partial_r g_{rr}) = \frac{1}{2} \frac{1}{S} \frac{\partial}{\partial r}(S)$$

$$\therefore \tilde{\Gamma}^r_{rr} = \frac{1}{2S} \frac{dS}{dr}$$

$$\Gamma^r_{r\theta} = \frac{1}{2} g^{rr} (\partial_r \cancel{g_{\theta r}}^{0} + \cancel{\partial_\theta g_{rr}}^{0} - \partial_r \cancel{g_{r\theta}}^{0}) = 0$$

$\Gamma^r_{r\phi} = 0$ (same reason as $\Gamma^r_{r\theta}$)

$\Gamma^r_{\theta r} = \Gamma^r_{r\theta} = 0$

$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr} (-\partial_r g_{\theta\theta})$ (only non-cross terms)

$$= \frac{-1}{2S} \frac{\partial}{\partial r}(r^2) \Rightarrow \therefore \Gamma^r_{\theta\theta} = \frac{-r}{S}$$

$\Gamma^r_{\theta\phi} = 0$ (all cross terms)

$$\Gamma^r_{\phi\phi} = \frac{1}{2} g^{rr}\left(-\partial_r g_{\phi\phi}\right)$$

$$\therefore \; \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{S}$$

$$\Gamma^\theta_{rr} = \frac{1}{2} g^{\theta\theta}\left(-\cancel{\partial_\theta g_{rr}}^{\,0}\right) = 0$$

$$\Gamma^\theta_{r\theta} = \frac{1}{2} g^{\theta\theta}\left(\partial_r g_{\theta\theta}\right)$$

$$= \frac{1}{2r^2}\left(\frac{\partial}{\partial r}\left(r^2\right)\right) \Rightarrow \Gamma^\theta_{r\theta} = \frac{1}{r}$$

$\Gamma^\theta_{r\phi} = 0$ (all cross terms)

$$\cancel{\Gamma^\theta_{\theta r} = \Gamma^\theta}$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta}\left(\partial_\theta g_{\theta\theta}\right) = 0$$

$\Gamma^\theta_{\theta\phi} = 0$ (cross terms or ϕ derivative)

$$\Gamma^\theta_{\phi\phi} = -\frac{1}{2} g^{\theta\theta}\partial_\theta g_{\phi\phi}$$

$$= -\frac{1}{2\cancel{r^2}}\frac{\partial}{\partial\theta}\left(\cancel{r^2}\sin^2\theta\right)$$

$$= -\frac{1}{2} \times 2\sin\theta\cos\theta$$

$$\therefore \; \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$\Gamma^\phi_{rr} = 0$ ($g_{r\phi}$ or $\partial_\phi g_{rr}$)

$$\Gamma^\phi_{r\theta} = 0, \quad \Gamma^\phi_{r\phi} = \frac{1}{2} g^{\phi\phi}\left(\partial_r g_{\phi\phi}\right)$$

$$= \frac{1}{2r^2\cancel{\sin^2\theta}} \times 2r\cancel{\sin^2\theta} = \frac{1}{r}$$

$$\therefore \Gamma^{\phi}_{r\phi} = \frac{1}{r}$$

$$\Gamma^{\phi}_{\theta\theta} = 0 \quad (\partial_\phi g_{\theta\theta} \text{ or } g_{\phi\theta})$$

$$\Gamma^{\phi}_{\phi\phi} = 0 \quad (\cancel{\text{no}}\ \partial_\phi \text{ of anything} = 0)$$

$$\Gamma^{\phi}_{\theta\phi} = \frac{1}{2} g^{\phi\phi} (\partial_\theta g_{\phi\phi})$$

$$= \frac{1}{2\cancel{r^2}\sin^2\theta} \partial_\theta (\cancel{r^2}\sin^2\theta)$$

$$= \frac{1}{\cancel{2}\sin^2\theta} \times \cancel{2\sin\theta}\cos\theta$$

$$\therefore \Gamma^{\phi}_{\theta\phi} = \cot\theta$$

~~$R^{\rho}{}_{\sigma\mu\nu}$ =~~

$$R^{\rho}{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu}$$

$$\Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}$$

Possible ? non-zero Term? (independent)
only those needed for R!

$R^{r}{}_{\theta r\theta}$, $R^{r}{}_{\phi r\phi}$, $R^{\theta}{}_{\phi\theta\phi}$

$R^{r}{}_{\theta r\phi}$, $R^{\theta}{}_{r\theta\phi}$, $R^{\phi}{}_{r\phi\theta}$

So, only need 6 terms!

only $i=r \to \neq 0$

$R^{\theta}{}_{\phi\theta\phi}$

$= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \cancel{\partial_\phi \Gamma^{\theta}_{\theta\phi}}^{\,0} + \Gamma^{\theta}_{\theta i}\Gamma^{i}_{\phi\phi} - \Gamma^{\theta}_{\phi i}\Gamma^{i}_{\theta\phi}$

only $i=\phi$

$= \frac{\partial}{\partial\theta}(-\cos\theta\sin\theta)$

$+ \frac{1}{r}\left(\frac{-r}{s}\sin^2\theta\right) - (-\cos\theta\sin\theta)\cot\theta$

$= \sin^2\theta - \cancel{\cos^2\theta} - \frac{1}{s}\sin^2\theta$

$+ \cancel{\cos^2\theta}$

$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta\left(1 - \frac{1}{s}\right)$?

maybe...

$R^{r}{}_{\theta r\phi} = \partial_r \cancel{\Gamma^{r}_{\phi\theta}}^{\,0} - \partial_\phi \cancel{\Gamma^{r}_{r\theta}}^{\,0}$

$+ \Gamma^{r}_{ri}\Gamma^{i}_{\phi\theta} - \Gamma^{r}_{\phi i}\Gamma^{i}_{r\theta}$

must be r — must be ϕ

$\frac{s'}{2s}\cancel{\Gamma^{r}_{\phi\theta}}^{\,0} - \left(\frac{-r\sin^2\theta}{s}\right)\cancel{\Gamma^{\phi}_{r\theta}}^{\,0}$

$\Rightarrow R^{r}{}_{\theta r\phi} = 0$

$R^{\theta}{}_{r\theta\phi} = \partial_\theta \cancel{\Gamma^{\theta}_{\phi r}}^{\,0} - \partial_\phi \cancel{\Gamma^{\theta}_{\theta r}}^{\,0} \Rightarrow R^{\theta}{}_{r\theta\phi} = 0$

$+ \Gamma^{\theta}_{\theta i}\cancel{\Gamma^{i}_{\phi r}}^{\,0} - \Gamma^{\theta}_{\phi i}\Gamma^{i}_{\theta r}$

r — $\overset{0}{\text{if}}\ i=\theta$ — $i=\theta$

$$R^r{}_{\theta r\theta} = \partial_r\left(\Gamma^r_{\theta\theta}\right) - \partial_\theta\left(\cancel{\Gamma^r_{r\theta}}\right) \to 0$$

$$+ \Gamma^r_{ri}\Gamma^i_{\theta\theta} - \Gamma^r_{\theta i}\Gamma^i_{r\theta}$$

$$= -\frac{1}{S} + \Gamma^r_{rr}\Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta}$$

$$+\frac{rS'}{S^2}$$

only non-zero $\Gamma^i_{\theta\theta}$

$$= \frac{rS'}{S^2} - \frac{1}{S} + \frac{S'}{2S}\left(-\frac{r}{S}\right) - \left(-\frac{\cancel{r}}{S}\right)\frac{1}{\cancel{r}}$$

$$R^r{}_{\theta r\theta} = +\frac{S' r}{2S^2}$$

$$R^r{}_{\phi r\phi} = \partial_r \Gamma^r_{\phi\phi} - \partial_\phi \cancel{\Gamma^r_{r\phi}} \to 0$$

$$+ \Gamma^r_{ri}\Gamma^i_{\phi\phi} - \Gamma^r_{\phi i}\Gamma^i_{r\phi}$$

only $\Gamma^r_{rr} \neq 0$

only $\Gamma^r_{\phi\phi} \neq 0$

$$= \frac{\partial}{\partial r}\left(-\frac{r}{S}\sin^2\theta\right) + \Gamma^r_{rr}\Gamma^r_{\phi\phi} - \Gamma^r_{\phi\phi}\Gamma^\phi_{r\phi}$$

$$= -\frac{1}{S}\sin^2\theta + \frac{S'}{2S}\left(-\frac{r}{S}\sin^2\theta\right) + \frac{rS'\sin^2\theta}{S^2}$$

$$- \left(-\frac{r}{S}\sin^2\theta\right)\left(\frac{1}{r}\right)$$

$$R^r{}_{\phi r\phi} = +\frac{rS'}{2S^2}\sin^2\theta$$

$R^{\phi}{}_{r\phi\theta}$

$$= \partial_\phi \cancel{\Gamma^\phi_{\theta r}}^{\to 0} - \partial_\theta \cancel{\Gamma^\phi_{\phi r}}^{\to 0} + \Gamma^\phi_{\phi i}\Gamma^i_{\theta r} - \Gamma^\phi_{\theta i}\Gamma^i_{\phi r}$$

$\frac{1}{r}$ $\quad$ $i = \theta$ or r $\quad$ $i = \phi$

$$= \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\theta r} - \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\phi r}$$

$$= \frac{\cot\theta}{r} - \frac{\cot\theta}{r} = 0$$

$$\therefore \quad R^\phi{}_{r\phi\theta} = 0$$

only non-zeros are:

$$R^r{}_{\theta r\theta} = +\frac{rS'}{2S^2}$$

$$R^r{}_{\phi r\phi} = +\frac{rS'\sin^2\theta}{2S^2}$$

$$R^\theta{}_{\phi\theta\phi} = \sin^2\theta\left(1 - \frac{1}{S}\right)$$

so, $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$

· All off diagonals zero, as relevant Riem cmp's are zero.

$$\therefore R_{rr} = R^\theta{}_{r\theta r} + R^\phi{}_{r\phi r}$$

$$= g^{\theta\theta}g_{rr}R^r{}_{\theta r\theta} + g^{\phi\phi}g_{rr}R^r{}_{\phi r\phi}$$

$$\therefore R_{rr} = \frac{\cancel{S}}{r^2}\left(\frac{+S'\cancel{r}}{2S^{\cancel{2}}}\right) + \frac{\cancel{S}}{r^2\cancel{\sin^2\theta}}\left(\frac{+S'\cancel{r}\cancel{\sin^2\theta}}{2S^{\cancel{2}}}\right)$$

$$R_{rr} = +\frac{S'}{Sr}$$ sign?

$$R_{\theta\theta} = R^r{}_{\theta r\theta} + R^\phi{}_{\theta\phi\theta}$$

$$= R^r{}_{\theta r\theta} + g^{\phi\phi} g_{\theta\theta} R^\theta{}_{\phi\theta\phi}$$

$$= \frac{+rS'}{2S^2} + \frac{1}{\cancel{r^2\sin^2\theta}}\cancel{r^2\sin^2\theta}\left(1-\frac{1}{S}\right)$$

$$R_{\theta\theta} = \frac{rS'}{2S^2} + \left(1-\frac{1}{S}\right)$$

$$R_{\phi\phi} = R^r{}_{\phi r\phi} + R^\theta{}_{\phi\theta\phi}$$

$$= \frac{+rS'\sin^2\theta}{2S^2} + \cancel{r^2}\sin^2\theta\left(1-\frac{1}{S}\right)$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$$

so, $$R = g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}$$

$$= \frac{S'}{S^2 r} + \frac{S'}{2S^2 r} + \frac{1}{r^2}\left(1-\frac{1}{S}\right)$$

$$+\frac{S'}{2S^2 r} + \frac{1}{r^2}\left(1-\frac{1}{S}\right)$$

$$\Rightarrow R = \frac{2}{r^2}\left(1 - \frac{1}{s} + \frac{r s'}{s^2}\right)$$

Homogeneity $= \rho =$ const.

so, solve:

$$R = \frac{2}{r^2}\left(1 - \frac{1}{s} + \frac{r s'}{s^2}\right)$$

$$\therefore \frac{r^2}{2} R = 1 - \frac{1}{s} + \frac{r s'}{s^2}$$

$$s^2\left(\frac{r^2}{2} R - 1\right) + s = r s'$$

$$\therefore \cancel{ds} =$$

Solution?

note,
$$-\frac{1}{s} + \frac{r s'}{s^2} = \frac{d}{dr}\left(\frac{-r}{s}\right)$$

so,
$$R = \frac{2}{r^2}\left(1 + \frac{d}{dr}\left(\frac{-r}{s}\right)\right)$$

and if $y = +\frac{r}{s}$

$$\Rightarrow R = \frac{2}{r^2}\left(1 - \frac{dy}{dr}\right)$$

so, $\left(\frac{Rr^2}{2}-1\right) = -\frac{dy}{dr}$

$\therefore \frac{dy}{dr} = 1-\frac{Rr^2}{2}$

$\Rightarrow y = r - \frac{Rr^3}{6} + C$

$\Rightarrow \frac{r}{S} = r - \frac{Rr^3}{6} + C$

$\Rightarrow S(r) = \frac{1}{1-\frac{Rr^2}{6}+\frac{C}{r}}$

Smoothness at $r=0 \Rightarrow \underline{\underline{C=0}}$

so, $\rightarrow S(r) = \frac{1}{1-\frac{Rr^2}{6}} = \frac{1}{1-kr^2}$

where $\underline{k=\frac{R}{6}}$

note, $$R_{\mu\nu} = \begin{pmatrix} \frac{S'}{Sr} & 0 \quad \cancel{0} & 0 \\ 0 & \frac{rS'}{2S^2}+\left(1-\frac{1}{S}\right) & 0 \\ 0 & & \left[\frac{rS'}{2S^2}+1-\frac{1}{S}\right]\sin^2\theta \end{pmatrix}$$

$\frac{S'}{Sr} = ?$ $\quad \frac{S'}{S} = \frac{1}{r}\left(1+S\left(\frac{r^2}{2}R-1\right)\right)$

so, $\frac{S'}{S} = \frac{1}{r}\left(1+\frac{\left(\frac{r^2}{2}R-1\right)}{1-\frac{Rr^2}{6}}\right) = \frac{1-r^2\frac{R}{6}+\frac{3r^2R}{6}-1}{1-\frac{Rr^2}{6}}$

$$\frac{S'}{S} = \frac{\frac{r^2R}{3}}{1 - \frac{r^2R}{6}}$$

So, $$\frac{S'}{Sr} = \frac{\frac{R}{3}}{1 - \frac{r^2R}{6}} = \underline{\underline{2\kappa g_{rr}}}$$

and: $$\frac{rS'}{2S^2} + 1 - \frac{1}{S}$$

$$\cancel{= \frac{d}{dr}\left(\frac{-r}{S}\right) + 1}$$

$$= \frac{\frac{r^2R}{3}}{2\left(1 - \frac{r^2R}{6}\right)^2} + 1 - \left(1 - \frac{r^2R}{6}\right)$$

$$= \frac{\frac{r^2R}{6}}{\left(1 - \frac{r^2R}{6}\right)^2} + \frac{r^2R}{6}$$

$$= \frac{r^2R}{6}\left(\frac{1 - \left(1 - \frac{r^2R}{6}\right)^2}{\left(1 - \frac{r^2R}{6}\right)^2}\right)$$

$$= \frac{r^2R}{6}\left(\frac{1 - 1 + \frac{r^2R}{3} - \frac{r^4R^2}{36}}{\left(1 - \frac{r^2R}{6}\right)^2}\right)$$

$$= \left(\frac{r^2R}{6}\right)^2\left(\frac{2 - \frac{r^2R}{6}}{\left(1 - \frac{r^2R}{6}\right)^2}\right)$$

or, $\frac{rs'}{2s^2} + 1 - \frac{1}{s}$

$$= \frac{1}{2}\left(\frac{rs'}{\cancel{2}s^2} - \frac{1}{s}\right) + 1 - \frac{1}{2s}$$

$$= \frac{1}{2}\frac{d}{dr}\left(\frac{-r}{s}\right) + 1 - \frac{1}{2s}$$

But $\frac{d}{dr}\left(\frac{-r}{s}\right) = \frac{r^2 R}{2} - 1$

so, $\Rightarrow \frac{1}{4} r^2 R - \frac{1}{2} + 1 - \frac{1}{2s}$

$$= \frac{1}{4} r^2 R + \frac{1}{2}\left(1 - \frac{1}{s}\right)$$

$$= \frac{1}{4} r^2 R + \frac{1}{2}\left(1 - 1 + \frac{r^2 R}{6}\right)$$

$$= \frac{1}{4}\left(r^2 R + \frac{r^2 R}{3}\right)$$

$$= \frac{r^2 R}{3} = \underline{\underline{2 k r^2}}$$

so, ~~sim~~

$\Rightarrow R_{\mu\nu} = 2k g_{\mu\nu}$, $\underline{\underline{k = \frac{R}{6}}}$

$$R_{\mu\nu} = \frac{R}{3} g_{\mu\nu}$$

2. $ds^2 = -dt^2 + a^2 \check{g}_{ij}\, dx^i dx^j$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

so, $\cancel{\Gamma^t_{ij}}$ $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^2 \check{g}_{ij} \end{pmatrix}$

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{a^2} \check{g}^{ij} \end{pmatrix}$$

$$\Gamma^t_{tt} = 0 \quad (\dot{g}_{tt} = 0)$$

$$\Gamma^t_{ti} = \frac{1}{2} g^{tt} (\cancel{\partial_i g_{tt}}^{=0}) = 0$$

$$\Gamma^t_{ij} = \frac{1}{2} g^{tt} (-\partial_t g_{ij})$$

$$= -\frac{1}{2}\left(-\frac{\partial}{\partial t}(a^2 \check{g}_{ij})\right)$$

$$\Gamma^t_{ij} = a\dot{a}\, \check{g}_{ij}$$

$$\Gamma^k_{tt} = \frac{1}{2a^2} \check{g}^{kl} (\cancel{-\partial_l g_{tt}}^{0}) = 0$$

$$\cancel{\Gamma^k}\ \Gamma^i_{tj} = \frac{1}{2a^2} \check{g}^{ik} (\partial_t (a^2 \check{g}_{jk}) + 0)$$

$$\Gamma^i_{tj} = \cancel{a}\frac{\dot{a}}{a} \delta^i{}_j$$

$$\Gamma^k_{ij} = \frac{1}{2a^2}\check{g}^{kl}\left(\partial_i(a^2\check{g}_{jl}) + \partial_j(a^2\check{g}_{li}) - \partial_l(a^2\check{g}_{ij})\right)$$

$$= \frac{1}{2}\check{g}^{kl}\left(\partial_i\check{g}_{jl} + \partial_j\check{g}_{li} - \partial_l\check{g}_{ij}\right)$$

$$\Gamma^k_{ij} = \check{\Gamma}^k_{ij}$$

$$\hat{\check{R}}_{ij} = 2k\check{g}_{ij}$$

so, ~~whate~~ what Riemanns do we need?

$$R^t{}_{itj} \quad R^k{}_{tki}, \quad R^k{}_{ikj}$$

$$R^k{}_{tkt}$$

$$R^t{}_{itj} = \partial_t\Gamma^t_{ji} - \partial_j\cancel{\Gamma^t_{ti}}^{\,0}$$

$$+ \cancel{\Gamma^t_{t\lambda}}^{\,0}\Gamma^\lambda_{ji} - \Gamma^t_{j\lambda}\Gamma^\lambda_{ti}$$

$$= \partial_t(a\dot{a}\check{g}_{ij})$$

$$- \Gamma^t_{jl}\Gamma^l_{ti}$$

$$R^t{}_{itj} = (\dot{a}^2 + a\ddot{a})\check{g}_{ij} - \underbrace{a\dot{a}\check{g}_{jl}\frac{\dot{a}}{a}\delta^l{}_i}_{\dot{a}^2\check{g}_{jl}}$$

$$\Rightarrow R^t{}_{itj} = a\ddot{a}\hat{g}_{ij}$$

$$R^k{}_{tki} = \partial_k \Gamma^k_{it} - \partial_i \Gamma^k_{kt}$$
$$+ \Gamma^k_{k\lambda}\Gamma^\lambda_{it} - \Gamma^k_{i\lambda}\Gamma^\lambda_{kt}$$
$$= \cancel{\partial_k\left(\frac{\dot a}{a}\delta_i{}^k\right)} \to 0 - \cancel{\partial_i\left(\frac{\dot a}{a}\delta_k{}^k\right)} \to 0$$
$$+ \Gamma^k_{kl}\Gamma^l_{it} - \Gamma^k_{il}\Gamma^l_{kt}$$
$$= \check\Gamma^k_{kl}\,\frac{\dot a}{a}\,\delta_i{}^l - \check\Gamma^k_{il}\,\frac{\dot a}{a}\,\delta^l{}_k$$
$$= \frac{\dot a}{a}\left(\check\Gamma^k_{ki} - \check\Gamma^k_{ik}\right) = 0$$
$$R^k{}_{tki} = 0$$

$$R^k{}_{ikj} = \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki}$$
$$+ \Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}$$
$$+ \Gamma^k_{kt}\Gamma^t_{ji} - \Gamma^k_{jt}\Gamma^t_{ki}$$
$$= \check R^k{}_{ikj} + \frac{\dot a}{a}\,\delta_k{}^k\,a\dot a\,\check g_{ij}$$
$$- \frac{\dot a}{a}\,\delta_j{}^k\,a\dot a\,\check g_{ki}$$
$$= \check R_{ij} + \dot a^2 (N-1)\,\check g_{ij}$$

where $N = 3$ (# of spatial dims)

so, $R^k{}_{ikj} = \check R_{ij} + 2\dot a^2\,\check g_{ij}$

$$\cancel{R_{tt} = R^{k}{}_{tkt} =}$$

$$R^{k}{}_{tkt} = g^{kl}\,\cancel{g_{km}}\,g_{tt}\,R^{t}{}_{ltk}$$

$$= -\frac{1}{a^2}\,\breve{g}^{kl}\,a\ddot{a}\,\breve{g}_{lk}$$

$$= -N\frac{\ddot{a}}{a}\,\cancel{\Phi} = -\frac{3\ddot{a}}{a}$$

So, $$R_{tt} = R^{k}{}_{tkt} = \frac{-3\ddot{a}}{a}$$

$$R_{ij} = R^{t}{}_{itj} + R^{k}{}_{ikj}$$

$$= a\ddot{a}\,\breve{g}_{ij} + \breve{R}_{ij} + 2\dot{a}^2\,\breve{g}_{ij}$$

$$R_{ij} = \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\breve{g}_{ij}$$

3. $ds^2 = -dt^2 + a^2 \hat{g}_{ij} dx^i dx^j$

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & a^2 \rho \hat{g}_{ij} \end{pmatrix}, \quad R = g^{\mu\nu} R_{\mu\nu}$$

$$= g^{tt} R_{tt} + \frac{1}{a^2} \hat{g}^{ij} R_{ij}$$

$$= +\frac{3\ddot{a}}{a} + \frac{N}{a^2}(a\ddot{a} + 2\dot{a}^2 + 2k)$$

$$= 3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right)$$

$$R = \frac{6\ddot{a}}{a} + \frac{6}{a^2}(\dot{a}^2 + k)$$

So,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$= \begin{pmatrix} -\frac{3\ddot{a}}{a} & 0 \\ 0 & (a\ddot{a} + 2\dot{a}^2 + 2k)\hat{g}_{ij} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & a^2 \hat{g}_{ij} \end{pmatrix} R$$

So i

$$G_{\mu\nu} = \begin{pmatrix} -\frac{3\ddot{a}}{a} & 0 \\ 0 & (a\ddot{a} + 2\dot{a}^2 + 2k)\hat{g}_{ij} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{3\ddot{a}}{a} + \frac{3}{a^2}(\dot{a}^2 + k) & 0 \\ 0 & \left[-3a\ddot{a} - 3(\dot{a}^2 + k)\right]\hat{g}_{ij} \end{pmatrix}$$

$$G_{\mu\nu} = \begin{pmatrix} \frac{3}{a^2}(\dot{a}^2 + k) & 0 \\ 0 & \left[-2a\ddot{a} - \dot{a}^2 - k\right]\tilde{g}_{ij} \end{pmatrix}$$

so,

$$8\pi G_N T_{\mu\nu} = \begin{pmatrix} 8\pi G_N \rho & 0 \\ 0 & 8\pi G_N a^2 p\, \tilde{g}_{ij} \end{pmatrix}$$

$$\Rightarrow \frac{3}{a^2}(\dot{a}^2 + k) = 8\pi G_N \rho$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G_N}{3}\rho$$ Friedmann's Eq.!

spatial:

$$-2a\ddot{a} - \dot{a}^2 - k = 8\pi G_N a^2 p$$

$$\dot{a}^2 + k = a^2 \frac{8\pi G_N}{3}\rho$$

$$\text{so,} \quad -2a\ddot{a} = a^2 8\pi G_N \left(p + \frac{\rho}{3}\right)$$

$$\Rightarrow \ddot{a} = -\frac{4\pi G_N}{3} a (3p + \rho)$$ acceleration Eqs.

$\nabla^\mu T_{\mu\nu} = 0$

so,

$\cancel{\partial^\mu T_{\mu\nu} = \Gamma^\mu_{\mu\lambda}}$

$= g^{\mu\rho}(\nabla_\rho T_{\mu\nu})$

$= g^{\mu\rho}(\partial_\rho T_{\mu\nu} - \Gamma^\lambda_{\rho\mu} T_{\lambda\nu}$
$\quad - \Gamma^\lambda_{\rho\nu} T_{\mu\lambda})$

Use t comp:

$g^{\mu\rho}(\partial_\rho T_{\mu t} - \Gamma^\lambda_{\rho\mu} T_{\lambda t} - \Gamma^\lambda_{\rho t} T_{\mu\lambda}) = 0$

so, $g^{tt}\partial_t T_{tt} - g^{\rho\mu}\Gamma^t_{\rho\mu} T_{tt} - g^{\mu\rho}\Gamma^\lambda_{\rho t} T_{\mu\lambda} = 0$

note, $\Gamma^t_{\rho\mu} \Rightarrow$ only $rm \neq 0$.

so, $g^{\rho\mu}\Gamma^t_{\rho\mu} = g^{rm}\Gamma^t_{rm}$

$= \frac{1}{a^2}\hat{g}^{rm} a\dot{a}\hat{g}_{rm} = \frac{N\dot{a}}{a} = \underline{\underline{\frac{3\dot{a}}{a}}}$

while, $\Gamma^\lambda_{\rho t} g^{\mu\rho} T_{\mu\lambda} = \Gamma^l_{rt} g^{mr} T_{ml}$

$= \frac{\dot{a}}{a}\delta^l{}_r \frac{1}{a^2}\hat{g}^{mr} a^2 \hat{g}_{ml} P$

$= \frac{N\dot{a}}{a}P = \frac{3\dot{a}}{a}P$

$$T_{tt} = \rho$$

$$\Rightarrow -\partial_t \rho - \frac{3\dot{a}}{a}(\rho + P) = 0$$

$$\therefore \dot{\rho} + \frac{3\dot{a}}{a}(\rho + P) = 0$$

So,

$$\frac{2\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - \frac{2k}{a^3}\dot{a} = \frac{8\pi G_N}{3}\dot{\rho}$$

$$= -\frac{8\pi G_N}{3}\frac{3\dot{a}}{a}(\rho + P)$$

$$\frac{2\dot{a}}{a}\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2}\right]$$

$$= \frac{2\dot{a}}{a}\left[\frac{\ddot{a}}{a} - \frac{8\pi G_N}{3}\rho\right]$$

$$\Rightarrow \frac{2\cancel{\dot{a}}}{\cancel{a}}\left[\frac{\ddot{a}}{a} - \frac{8\pi G_N}{3}\rho\right] = -\frac{8\pi G_N}{3}\frac{3\cancel{\dot{a}}}{\cancel{a}}(\rho + P)$$

$$\Rightarrow \frac{\ddot{a}}{a} - \frac{8\pi G_N}{3}\rho = -4\pi G_N(\rho + P)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(3\rho + 3P - 2\rho) = -\frac{4\pi G_N}{3}(\rho + 3P)$$

4. $P = w\rho$, $w = \text{const}$.

so,

$$\dot{\rho} + \frac{3\dot{a}}{a}(1+w)\rho = 0$$

$$d\rho = -\frac{3da}{a}(1+w)\rho = 0$$

$$\frac{d\rho}{\rho} = -\frac{3\,da}{a}(1+w)$$

$$\log\rho = -3(1+w)\log a$$

$$\Rightarrow \rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}$$

Friedmann:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G_N}{3}\rho$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)} - \frac{k}{a^2}$$

$$\frac{da}{a\left(\frac{8\pi G_N}{3}\rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)} - \frac{k}{a^2}\right)^{\frac{1}{2}}} = dt$$

$$\frac{da}{\left(\frac{8\pi G_N}{3}\rho_0\left(\frac{1}{a_0}\right)^{-3(1+w)} a^{-(1+3w)} - k\right)^{\frac{1}{2}}} = dt$$

$$\int \frac{dx}{\sqrt{Ax^p - B}} = ?$$

hard to do in general (hypergeometric!)

$$\int \frac{dx}{\sqrt{Ax^p - B}} = \frac{x\sqrt{\frac{B - Ax^p}{B}}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{p}, 1+\frac{1}{p}, \frac{Ax^p}{B}\right)}{\sqrt{-B + Ax^p}}$$

so, set $k = 0$?

$$dt = \frac{da \quad a^{\left(\frac{1+3w}{2}\right)}}{\sqrt{\frac{8\pi G_N}{3}\rho_0 \; a_0^{\frac{3(1+w)}{2}} \cancel{a^{\frac{(1+3w)}{2}}}}}$$

$$t = \frac{\frac{2}{3(1+w)}\left(\frac{a}{a_0}\right)^{\frac{3}{2}(1+w)} + C}{\sqrt{\frac{8\pi G_N \rho_0}{3}}}$$

if $H_0 = \sqrt{\frac{8\pi G_N \rho_0}{3}}$ is initial H

as $\left.\frac{\dot{a}}{a}\right|_{t_0} = \sqrt{\frac{8\pi G_N \rho_0}{3}}$

$$\Rightarrow a = a_0\left(\frac{3(1+w)}{2} H_0 (t - \bar{t})\right)^{\frac{2}{3(1+w)}}$$

$\bar{t} = 0$ by convention!

So,

$a = a_0$

$$\Rightarrow \left(H_0 \frac{3}{2}(1+w) t_0 \right)^{\frac{2}{3(1+w)}} = 1$$

$$\Rightarrow \quad t_0 = \frac{2}{3(1+w) H_0}$$ Age of the universe

For $w < -\frac{1}{3}$?

$\cancel{\frac{2}{3(1+w)}}$ $\quad 1+w < \frac{2}{3}$

$$\frac{1}{1+w} > \frac{3}{2}$$

$$\frac{2}{3} \frac{1}{(1+w)} > 1$$

So, ~~a = a_0 (t)~~

$$a = C t^p \text{ where } p > 1$$

$$\Rightarrow \ddot{a} = C p(p-1) t^{p-2}$$

$$p > 1 \Rightarrow \ddot{a} > 0$$

accelerated expansion!

5. $ds^2 = -dt^2 + a^2\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right)$

E.L.

$\downarrow$

$$L = \int d\lambda\, \mathcal{L}$$

$$= \int d\lambda \left[-\left(\frac{dT}{d\lambda}\right)^2 + \frac{a^2(T)}{1-kR^2}\left(\frac{dR}{d\lambda}\right)^2 + a^2R^2\left(\frac{d\theta}{d\lambda}\right)^2 + a^2R^2\sin^2\theta\left(\frac{d\Phi}{d\lambda}\right)^2 \right]$$

$$\frac{\partial \mathcal{L}}{\partial T} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{T}}\right) = 0$$

so, $2a\dot{a}\left[\frac{1}{1-kR^2}R'^2 + R^2\theta'^2 + R^2\sin^2\theta\,\Phi'^2\right]$

$$-\frac{d}{d\lambda}\left(-2T'\right) = 0 \quad \text{(Horrible!)}$$

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \Phi'}\right) = 0$$

$0 \Rightarrow 2a^2R^2\sin^2\theta\,\Phi' = \text{const}$

$$\frac{\partial \mathcal{L}}{\partial R} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial R'}\right) = 0 \; ?$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right) = 0$$

so,
$$2a^2 R^2 \sin\theta \cos\theta \, (\Phi')^2 - \frac{d}{d\lambda}\left(2a^2 R^2 \theta'\right) = 0$$

consider ray at $r = 0$

$\Rightarrow 2a^2 R^2 \sin^2\theta \, \Phi' = \text{const}$

$R \to 0$

$\Rightarrow \text{const} = 0$

$\therefore 2a^2 R^2 \sin^2\theta \, \Phi' = 0$

$\Rightarrow \Phi = \text{const}$

and $\Phi' = 0$

$\Rightarrow \frac{d}{d\lambda}\left(2a^2 R^2 \theta'\right) = 0$

$\therefore 2a^2 R^2 \theta' = \text{const}$

const $= 0$ (cos passes through $R = 0$)

$\Rightarrow \theta' = 0$; $\underline{\underline{\theta = \text{const}}}$

$$\frac{\partial \mathcal{L}}{\partial R} - \frac{d}{dR}\left(\frac{\partial \mathcal{L}}{\partial R'}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial R} = + \frac{a^2 2kR}{(1-kR^2)^2}(R')^2 \cancel{+ 2a^2 R (\theta')^2}$$

$$+ 2R\left(a^2(\theta')^2 + a^2 \sin^2\theta (\phi')^2\right)$$

$\theta' = \phi' = 0$

so, $$\frac{\partial \mathcal{L}}{\partial R} = \frac{2a^2 kR}{(1-kR^2)^2}(R')^2$$

$$\frac{\partial \mathcal{L}}{\partial R'} = \frac{2a^2}{(1-kR^2)} R'$$

so, $$\frac{2a^2 kR}{(1-kR^2)^2}(R')^2 - \frac{d}{d\lambda}\left(\frac{2a^2 R'}{1-kR^2}\right) = 0$$

note,

$$\frac{2a^2 kR}{(1-kR^2)^2}(R')^2 - \frac{d}{d\lambda}(2a^2 R')\frac{1}{(1-kR^2)} - 2a^2 R' \frac{d}{d\lambda}\left(\frac{1}{1-kR^2}\right) = 0$$

now, use:

$$\frac{d}{d\lambda}\left(\frac{1}{\sqrt{1-kR^2}}\right) = -\frac{1}{2}(1-kR^2)^{-\frac{3}{2}} *$$

$$\frac{d}{d\lambda}\left(\frac{1}{1-kR^2}\right) = \frac{d}{d\lambda}\left(\left(\frac{1}{\sqrt{1-kR^2}}\right)^2\right)$$

$$= \frac{2}{\sqrt{1-kR^2}}\frac{d}{d\lambda}\left(\frac{1}{\sqrt{1-kR^2}}\right)$$

so:

$$\frac{2a^2kR^{\cancel{4}}(R')^2}{(1-kR^2)^2} - \frac{d}{d\lambda}\left(2a^2R'\right)\frac{1}{1-kR^2}$$

$$-\frac{4a^2R'}{\sqrt{1-kR^2}}\frac{d}{d\lambda}\left(\frac{1}{\sqrt{1-kR^2}}\right) = 0$$

so,

$$\frac{2a^2kR^{\cancel{k}}(R')^2}{(1-kR^2)^{\frac{3}{2}}} - \frac{2a^2R'}{\cancel{\quad}}\frac{d}{d\lambda}\left(\frac{1}{\sqrt{1-kR^2}}\right)$$

$$-\frac{d}{d\lambda}\left(\frac{2a^2R'}{\sqrt{1-kR^2}}\right) = 0$$

$$\frac{d}{d\lambda}\left(\frac{2a^2R'}{\sqrt{1-kR^2}}\right) = \frac{2a^2kR(R')^2}{(1-kR^2)^{\frac{3}{2}}}$$

$$- \frac{2a^2R' \times \cancel{+}\frac{1}{\cancel{2}}(\cancel{-2}kRR')}{(1-kR^2)^{\frac{3}{2}}}$$

$$= \frac{2a^2kR(R')^2 - 2a^2kR'^2}{(1-kR^2)^{\frac{3}{2}}} = 0$$

$$\Rightarrow \frac{2a^2R'}{\sqrt{1-kR^2}} = \text{const.}$$

integration constant

$$\frac{dR}{d\lambda} = \frac{\nu}{a^2}\sqrt{1-kR^2}$$

so:

$$2T'' + \frac{2a\dot{a}\,R'^2}{1-kR^2} = 0$$

or, use:

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \mu$$

$$\Rightarrow -\left(\frac{dT}{d\lambda}\right)^2 + \frac{a^2}{1-kR^2}\left(\frac{dR}{d\lambda}\right)^2 + a^2R^2\cancel{\dot{\theta}^2}^{\,0} + a^2R^2\sin^2\theta\,\cancel{\dot{\phi}^2}^{\,0} = \mu$$

so:

$$\frac{d\tau}{d\lambda} = \sqrt{\frac{a^2}{1-kr^2}(r')^2 - \mu}$$

$$= \sqrt{\cancel{\frac{a^2}{1-kr^2}}\frac{\nu^2}{a^4 2}\cancel{(1-kr^2)} - \mu}$$

$$\frac{d\tau}{d\lambda} = \sqrt{\frac{\nu^2}{a^2} - \mu}$$

$$\frac{d\tau}{d\lambda} = \frac{\nu}{a}\sqrt{1 - \frac{\mu a^2}{\nu^2}}$$

if $\mu = -1$

Energy measured by an observer?

$E = -V^\mu p_\mu$

since, $p^\mu = (p^0, p_i)$

$V^\mu = (1, 0, 0, 0)$

and $p^\mu V_\mu = -p^0$ $(- + + +)$

$\Rightarrow E = -p^\mu V_\mu$

So, $p^\mu = m\frac{dx^\mu}{d\lambda}$

spatial momentum? Project

$|\vec{p}| =$

$\rightarrow p^\mu_{spat} = p^\mu - (-p^\mu v_\mu) v^\mu$

so, $p_s^\mu p_{s\mu} = p^\mu p_\mu \mp 2(-p^\mu v_\mu)^2$
$+ (p^\mu v_\mu)^2 v^\mu v_\mu$

$v^\mu v_\mu = -1$

So, $p_s^\mu p_{s\mu} = p^\mu p_\mu \mp (-p^\mu v_\mu)^2$
$= m^2 \mp (p^0)^2$

$\Rightarrow |\vec{p}|^2 = m^2 \mp (p^0)^2$

and $|\vec{p}| = \sqrt{g_{ij} p^i p^j}$

p^0 ?

$p^\mu = \left(\frac{\nu}{a}, \frac{\nu}{a^2}\sqrt{1-kr^2}, 0, 0\right)$

so, $p^0 = \frac{\nu}{a}$

6. ~~4~~

Friedmann:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G_N}{3}\rho$$

if:

$$\rho = \rho_{\Lambda 0} + \rho_{m0}\left(\frac{a_0}{a}\right)^3$$

(ignore radiation)

let, $\rho_0 = \rho_{\Lambda 0} + \rho_{m0}$.

So, $H_0^2 = \frac{8\pi G_N}{3}\rho_0$.

$$\therefore H^2 = H_0^2\left(\frac{\rho}{\rho_0} - \frac{k}{H_0^2 a^2}\right)$$

let Ω_k $= \frac{-k}{H_0^2 a_0^2}$

$$\Rightarrow H^2 = H_0^2\left(\frac{\rho}{\rho_0} + \Omega_k\left(\frac{a_0}{a}\right)^2\right)$$

$$= H_0^2\left(\Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_k\left(\frac{a_0}{a}\right)^2\right),$$

now, where $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_0}$, $\Omega_m = \frac{\rho_{m0}}{\rho_0}$

note, $1+z = \frac{a_0}{a}$ (red shift)

$$H_0^2 = H_0^2\left(\Omega_\Lambda + \Omega_m (1+z)^3 + \Omega_k (1+z)^2\right)$$

$\Omega_k = 0$

$$\Rightarrow H^2 = H_0^2\left(\Omega_\Lambda + \Omega_m (1+z)^3\right)$$

if $1+z = \frac{a_0}{a}$

$$\dot{z} = -\frac{a_0}{a^2}\dot{a} = -\frac{\dot{a}}{a}(1+z)$$

so,

$$\frac{\dot{a}}{a} = -\frac{\dot{z}}{1+z}$$

$$\Rightarrow -\frac{\dot{z}}{1+z} = H_0\sqrt{\Omega_\Lambda + \Omega_m (1+z)^3}$$

$$\frac{dz}{(1+z)\left(\Omega_\Lambda + (1-\Omega_\Lambda)(1+z)^3\right)^{\frac{1}{2}}} = -H_0\,dt$$

~~let $z = y^{\frac{3}{2}}$~~

~~let $y = 1+z$~~

$$\cancel{\Rightarrow \frac{dy}{y\sqrt{\Omega_\Lambda + (1-\Omega_\Lambda)y^3}}}$$

Let $(1+z)^3(1-\Omega_\Lambda) = \Omega_\Lambda y^{-3}$

so, $3(1+z)^2(1-\Omega_\Lambda)dz = -3\Omega_\Lambda y^{-4} dy$

so:

$$dz = \frac{-\cancel{3}\Omega_\Lambda y^{-4} dy}{(1-\Omega_\Lambda)(1+z)^2}$$

$$\frac{dz}{1+z} = \frac{-\Omega_\Lambda y^{-4} dy}{(1-\Omega_\Lambda)(1+z)^3}$$

$$= \frac{-\Omega_\Lambda y^{-4} dy}{(1-\Omega_\Lambda)}\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}\right) y^3$$

$$= -\cancel{(1-\Omega_\Lambda)}\, y^{-1} dy$$

$$\Rightarrow \frac{1}{\sqrt{\Omega_\Lambda}} \frac{\cancel{(1-\Omega_\Lambda)}\, dy}{y\sqrt{1+y^{-3}}} = H_0 dt$$

$\therefore \cancel{\int d\omega}$

Limits? $z = 0 \Rightarrow y = \left(\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\right)^{\frac{1}{3}}$

$z = \infty \Rightarrow y = 0$

so, $$\int_0^{\left(\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\right)^{\frac{1}{3}}} \frac{dy}{y\sqrt{1+y^{-3}}} = \sqrt{\Omega_\Lambda}\, H_0 \int_0^{t_0} dt$$

So,

let $u = y^{\frac{3}{2}}$

$du = \frac{3}{2} y^{\frac{1}{2}} dy$

so, $\frac{dy}{y} = \frac{2}{3u} du$

and $\sqrt{1+y^{-3}} = \sqrt{1+u^{-2}}$

$$\Rightarrow \frac{2}{3}\int_0^{\sqrt{\frac{\Omega_\Lambda}{1-\Omega_\Lambda}}} \frac{du}{\sqrt{u^2+1}} = H_0\sqrt{\Omega_\Lambda}\, t_0$$

$$= \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{1-\Omega_\Lambda}}\right)$$

$$\Rightarrow t_0 = \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{1-\Omega_\Lambda}}\right)$$

Λ_{CDM} model age of universe.

$\Omega_\Lambda = 0$

$$\Rightarrow t_0 = \frac{2}{3H_0} \lim_{x\to 0} \frac{1}{\sqrt{x}} \sinh^{-1}\left(\sqrt{\frac{x}{1-x}}\right)$$

so... $\sinh^{-1}\left(\sqrt{\frac{x}{1-x}}\right) \sim \sqrt{\frac{x}{1-x}}$

so, $\lim_{x\to 0} \frac{1}{\sqrt{x}} \sinh^{-1}\left(\sqrt{\frac{x}{1-x}}\right) = 1$ ✓

$$\therefore \; t_0 = \frac{2}{3H_0}$$

Matter domination? $\rightarrow t_0 = \frac{2}{3H_0}$ ✓

$w = -1$ gives $t_0 = \infty$!!!

here, $\Omega_\Lambda = 1$

$$\Rightarrow t = \lim_{\Omega_\Lambda \to 1} \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{1-\Omega_\Lambda}}\right)$$

$$= \infty \quad \text{as expected!}$$

7. Luminosity distance

I_0 intrinsic luminosity, $\quad I_{app} = \frac{I_0}{4\pi d_L^2}$

But in reality. $\quad I_{app} = \frac{I_0}{4\pi (a_0 R)^2 (1+z)^2}$

why? $\cdot \frac{1}{1+z}$ because less received in unit time.

$\cdot \frac{1}{1+z}$ from redshift of photons.

$$\therefore \; \underline{d_L = a_0 R(1+z)}$$

$\Omega_k = 0$,

~~Mass~~
<u>Matter dominated</u>

$\hookrightarrow H^2 = H_0^2\left(\Omega_\Lambda + (1-\Omega_\Lambda)\left(\frac{a_0}{a}\right)^3\right)$

So... what is R?

Null ray, $ds^2 = 0$

$$-dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega_2^2\right) = 0$$

$$dr^2 = \frac{dt^2}{a^2}$$

$$\therefore dr = -\frac{dt}{a(t)}$$

$$\Rightarrow \int_0^R dr = R = -\int_{t_0}^{t} \frac{dt}{a(t)}$$

$$= \int_t^{t_0} \frac{dt}{a(t)} = \int_a^{a_0} \frac{1}{Ha^2}\, da$$

convert to z!

$$1+z = \frac{a_0}{a} \qquad dz = \frac{-a_0}{a^2}\, da$$

$$\text{so, } \frac{1}{a_0}\int_0^z \frac{dz}{H} = R.$$

$$R = \frac{\cancel{a_0}}{\frac{1}{a_0}} \int_0^{z} \frac{dz}{H(z)}$$

$$= \frac{\cancel{a_0}}{a_0 H_0} \int_0^{z} \frac{dz}{\sqrt{\Omega_\Lambda + (1-\Omega_\Lambda)(1+z)^3}}$$

let $\cancel{y = (1+z)^{\frac{2}{3}}}$ $\qquad y = \sqrt{\frac{1-\Omega_\Lambda}{\Omega_\Lambda}}(1+z)^{\frac{3}{2}}$

$$(1-\Omega_\Lambda)(1+z)^3 = \Omega_\Lambda y^2$$

$$(1+z) = \left(\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\right)^{\frac{1}{3}} y^{\frac{2}{3}}$$

$$\text{so } dz = \frac{2}{3}\left(\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\right)^{\frac{1}{3}} y^{-\frac{1}{3}} dy$$

$$\Rightarrow R = \frac{2\cancel{a_0}}{3H_0\sqrt{\Omega_\Lambda}\, a_0}\left(\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\right)^{\frac{1}{3}} \int_{\sqrt{\frac{1-\Omega_\Lambda}{\Omega_\Lambda}}}^{\sqrt{\frac{1-\Omega_\Lambda}{\Omega_\Lambda}}(1+z)^{\frac{3}{2}}} \frac{dy}{y^{\frac{1}{3}}\sqrt{1+y^2}}$$

if $$\int \frac{dy}{y^{\frac{1}{3}}\sqrt{1+y^2}} = c\,\frac{3}{2} y^{\frac{2}{3}} F(y^2)$$

$$\Rightarrow R = \frac{2\cancel{a_0}}{a_0 3H_0\sqrt{\Omega_\Lambda}}\left(\frac{\Omega_\Lambda}{1-\Omega_\Lambda}\right)^{\frac{1}{3}}\left[\frac{3}{2}\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}\right)^{\frac{1}{3}} \cancel{B}(1+z)\, F\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}(1+z)\right.\right.$$

$$\left.- \frac{3}{2}\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}\right)^{\frac{1}{3}} F\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}\right)\right]$$

So,

$$R = \frac{1}{a_0 H_0 \sqrt{\Omega_\Lambda}} \left[(1+z)\, F\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda} (1+z)^3 \right) - F\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda} \right) \right]$$

Hence,

$$d_L = a_0 R (1+z)$$

$$\Rightarrow d_L(z) = \frac{(1+z)}{H_0 \sqrt{\Omega_\Lambda}} \left[(1+z)\, F\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda} (1+z)^3 \right) - F\left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda} \right) \right]$$

Aside; which Hypergeometric function is this?

$$\frac{d}{dy}\left(\frac{3}{2}y^{\frac{2}{3}}F(y^2)\right)$$

$$= y^{-\frac{1}{3}}F(y^2) + \frac{3}{2}y^{\frac{2}{3}}F'(y^2)\times 2y$$

$$= y^{-\frac{1}{3}}F(y^2) + 3y^{\frac{5}{3}}F'(y^2)$$

$$= y^{-\frac{1}{3}}\left(F(y^2) + 3y^2F'(y^2)\right)$$

so, we need $F(y^2) + 3y^2F'(y^2) = \dfrac{1}{\sqrt{1+y^2}}$

$$F(y) + 3yF'(y) - \frac{1}{\sqrt{1+y}} = 0$$

$$\text{sol} \Rightarrow F(y) = \frac{Beta(-y, \frac{1}{3}, \frac{1}{2})}{3(-y)^{\frac{1}{3}}} + \frac{C}{y^{\frac{1}{3}}}$$

incomplete beta function... (see next page)

$$B_x(p,q) = \frac{x^p}{p}\,{}_2F_1(p, 1-q, p+1, x)$$

so,

$$B_{-y}\left(\tfrac{1}{3}, \tfrac{1}{2}\right) = 3(-y)^{\frac{1}{3}}\,{}_2F_1\left(\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{4}{3}, -y\right)$$

so, $F(y) = {}_2F_1\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, -y\right) + \dfrac{C}{y^{\frac{1}{3}}}$

$F(y^2) = Cy^{-\frac{2}{3}}$

$\Rightarrow \frac{3}{2}y^{\frac{2}{3}}F(y^2) = \frac{3}{2}C$

$$F(y) = Cy^{-\frac{1}{3}}$$

$$F'(y^{2}) = -\frac{C}{3}y^{-\frac{4}{3}}$$

$$F(y^2) + 3y^2 F'(y^2)$$

$$= Cy^{-\frac{2}{3}} \cancel{+} \frac{3y^2}{3} C y^{-\frac{4}{3}} = 0$$

so, irrelevant'

$$\therefore \quad F(y^2) = {}_2F_1\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, -y^2\right)$$

where

$${}_2F_1(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} z^n$$

$$\text{and} \quad (a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)}$$

To solve:

$$F(y) + 3yF'(y) - \frac{1}{\sqrt{1+y}} = 0$$

$$\text{let } v(y) = y^{\frac{1}{3}} F(y) \Rightarrow F'(y) = v'(y) y^{-\frac{1}{3}} - \frac{1}{3} y^{-\frac{4}{3}} v(y)$$

$$\Rightarrow 3v'(y)\, y^{\frac{2}{3}} = (1+y)^{-\frac{1}{2}}$$

$$\text{so, } v(y) = \frac{1}{3}\int_0^y t^{-\frac{2}{3}} (1+t)^{-\frac{1}{2}} dt + C = \frac{(-1)^{\frac{1}{3}}}{3} \int_0^{-y} t^{-\frac{2}{3}} (1-t)^{-\frac{1}{2}} dt + C$$

$$\therefore v(y) = \frac{(-1)^{\frac{1}{3}}}{3} Beta\left(y, \frac{1}{3}, \frac{1}{2}\right) + C \longleftarrow \text{"Incomplete Beta function."}$$

$$\Rightarrow F(y) = \frac{Bet\left(y, \frac{1}{3}, \frac{1}{2}\right)}{3(-y)^{\frac{1}{3}}} + \frac{C}{y^{\frac{1}{3}}}$$