Imperial College London

MSc EXAMINATION May 2018

This paper is also taken for the relevant Examination for the Associateship

PARTICLE COSMOLOGY

For Students in Quantum Fields and Fundamental Forces

Friday, 11th May 2018: 14:00 to 17:00

Answer **THREE** *out of the following four questions.*
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

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Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature and, unless otherwise stated, choose units so that $\hbar = 1$ and $c = 1$.

In SI units the following constants have values;

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.05 \times 10^{-34}\ \mathrm{kg\,m^2\,s^{-1}} \\ c &= 3.00 \times 10^{8}\ \mathrm{m\,s^{-1}} \\ G &= 6.67 \times 10^{-11}\ \mathrm{kg^{-1}\,m^3\,s^{-2}} \\ k_B &= 1.38 \times 10^{-23}\ \mathrm{kg\,m^2\,s^{-2}\,K^{-1}} \end{aligned}$$

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1. (i) Consider a photon following a null geodesic in a spatially flat expanding FRW spacetime. By considering this *single* null geodesic show how the energy of the photon emitted by a comoving source is redshifted by the time it is measured by a comoving observer.

[6 marks]

(ii) Suppose a source has a known luminosity L and redshift Z, and its light is observed by a detector to have an apparent energy flux f. Define its luminosity distance d_L in terms of these quantities.

Suppose the universe is spatially flat FRW and is dominated by a cosmological constant and non-relativistic matter. A comoving observer living at time t_o sees the non-relativistic matter to have energy fraction Ω_m, and measures the Hubble constant to be H_o. This observer measures the apparent energy flux and redshift of comoving sources with known luminosity. Assuming these give off their light isotropically show they have a luminosity distance,

$$d_L \;=\; \frac{1+Z}{H_o}\int_1^{1+Z}\frac{1}{\sqrt{(1-\Omega_m)+\Omega_m x^3}}dx\ .$$

[7 marks]

(iii) Consider comoving cosmologists in a very advanced civilization far in our future – the Ω_m they measure is much smaller than ours today. Evaluate the expression for d_L in the approximation that $|\Omega_m| \ll 1$; Taylor expand the integrand up to, and including linear order in Ω_m, and integrate the resulting terms.

Use the following observational data of two comoving supernovae that these advanced cosmologists have measured to estimate their values of Ω_m and H_o. Suppose these cosmologists are just receiving television signals from our current 'civilization' - how redshifted will these be?

Redshift Z	Luminosity $L/\,(kg\,m^2\,s^{-3})$	Observed energy flux $f/\,(kg\,s^{-3})$
0.91	1.57×10^{36}	1.72×10^{-18}
2.23	2.00×10^{36}	1.34×10^{-19}

[7 marks]

[Total 20 marks]

2. (i) Consider a cosmology that maintains homogeneity, but breaks isotropy by having a preferred direction, and has spacetime metric,

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2) + b(t)^2 dz^2$$

where the preferred direction, z, has its own scale factor $b(t)$. Parameterize a timelike geodesic curve as $x^\mu = (T(\tau), X(\tau), Y(\tau), Z(\tau))$, with τ an affine parameter. Compute the geodesic equations to show;

$$X'(\tau) = \frac{v_x}{a(T)^2}\,,\ Y'(\tau) = \frac{v_y}{a(T)^2}\,,\ Z'(\tau) = \frac{v_z}{b(T)^2}$$

for constants v_x, v_y and v_z. Determine also an equation for $T'(\tau)$.

[5 marks]

(ii) Suppose a massive particle, mass m, follows the above timelike geodesic. Compute the energy of the particle as seen by a comoving observer. (As for FRW a comoving observer has constant x, y, z coordinates.)

From the form of this energy deduce the magnitude of the momentum seen in the x-y plane. We shall call this p. Likewise deduce the observed magnitude of the momentum in the z direction. We call this q.

Show that $a\,p$ and $b\,q$ are conserved quantities along the particle's geodesic trajectory. What is the origin of this conservation?

[5 marks]

(iii) Suppose we have a gas of these massive particles which we treat by giving a smooth phase space distribution function, N. Let this gas preserve the symmetries of the spacetime so we may take $N = N(t, p, q)$. Show that the Boltzmann equation for *free* particle motion is;

$$\left(\left.\frac{\partial}{\partial t}\right|_{p,q} - \frac{\dot{a}}{a} p \left.\frac{\partial}{\partial p}\right|_{t,q} - \frac{\dot{b}}{b} q \left.\frac{\partial}{\partial q}\right|_{t,p} \right) N = 0$$

(here $\dot{} = d/dt$ as usual).

[5 marks]

(iv) Early in this anisotropic universe these massive particles are in thermal equilibrium, with temperature $T(t)$. At a time t_{freeze} they instantaneously freeze out and travel freely for later times. Assuming these particles are bosonic and have no chemical potential, compute the form of the phase space distribution function N for times $t > t_{freeze}$.

[5 marks]

[Total 20 marks]

3. (i) Consider a homogeneous isotropic thermal distribution of a single fermionic degree of freedom, with temperature T, mass m and chemical potential μ, in spatially flat FRW spacetime. State the phase space distribution function for this fermionic species and define any other variables it depends on.

Show that in the relativistic limit the energy density ρ behaves as,

$$\rho \simeq \frac{7}{240}\pi^2 (k_B T)^4$$

and in the non-relativistic limit,

$$\rho \simeq \left(m + \frac{3}{2} k_B T \right) n$$

where n is the real space number density.
You may find the following integral useful; $\int_0^\infty dx \frac{x^3}{e^x+1} = \frac{7\pi^4}{120}$.

[8 marks]

(ii) Show by calculation that the temperature of the relic neutrinos in the universe today, T_ν, is related to that of the relic photons, $T_\gamma = 2.7K$, as,

$$T_\nu = \left(\frac{4}{11} \right)^{\frac{1}{3}} T_\gamma \ .$$

You should take the neutrinos to be massless, and carefully state any other assumptions used to arrive at this result.
[Hints: In the relativistic limit, entropy density $s = \frac{4\rho}{3T}$. The energy density of a fermionic degree of freedom is 7/8 times that of a bosonic one.]

[6 marks]

(iii) Now assume that neutrinos are not massless, but have small masses. For simplicity take these masses all to be the same. Suppose this mass is $m = 100eV$, then today these free streaming neutrinos would be non-relativistic.

Carefully compute the energy fraction Ω_ν in these relic neutrinos today taking the Hubble constant to be $H_0 = 70kms^{-1}Mpc^{-1}$. Briefly comment on the value you obtain.

[You may find the following useful; $\int_0^\infty dx \frac{x^2}{e^x+1} \simeq 1.80$.
Also $1pc = 3.1 \times 10^{16} m$ and $1eV = 1.6 \times 10^{-19} kg\, m^2\, s^{-2}$.]

[6 marks]

[Total 20 marks]

4. (i) Approximating inflation as deSitter expansion, so H = constant, inhomogeneous inflaton fluctuations can be quantized as,

$$\delta\hat{\phi}(t,x) = \int d^3k_i \left(\hat{a}_{k_i}\delta_{k_i}\phi(t)e^{-ik_ix^i} + \hat{a}^\dagger_{k_i}\delta_{k_i}\phi(t)^\star e^{+ik_ix^i} \right)$$

where

$$\delta_{k_i}\phi(t) = \frac{1}{\sqrt{2(2\pi)^3k}} e^{+\frac{ik}{a(t)H}} \left(\frac{1}{a(t)} + \frac{iH}{k} \right)$$

and $k = \sqrt{\delta^{ij}k_ik_j}$ with standard commutation relations for the creation and annhilation operators, $\hat{a}_{k_i}$ and $\hat{a}^\dagger_{k_i}$. Compute the fluctuation two point function to determine the 'dimensionless power spectrum' $\Delta^2(t,k)$ defined by,

$$\langle 0|\delta\hat{\phi}(t,x)\delta\hat{\phi}(t,y)|0\rangle = \int \frac{d^3k_i}{k^3}\Delta^2(t,k)e^{-ik_i(x^i-y^i)} \ .$$

Determine the superhorizon behaviour of $\Delta^2(t,k)$, showing it tends to a constant that is independent of time t and wavenumber k_i.

[7 marks]

(ii) Recall that in the slow roll approximation the inflaton equation of motion and Friedmann equation are $3H\dot{\phi} = -V'(\phi)$ and $H^2 = \frac{8\pi G}{3}V(\phi)$, where $V(\phi)$ is the inflaton potential. The potential is required to obey $\epsilon(\phi) = \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll 1$ for slow roll consistency.
Show that as the scalar slowly rolls from a value ϕ_1 to ϕ_2, with $\phi_1 < \phi_2$, the number of efolds during this period is,

$$N = 2\sqrt{\pi G}\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{\epsilon(\phi)}} \ .$$

For a sextic potential $V(\phi) = \lambda\phi^6$ compute the range of scalar field values traversed during the *last* N e-folds of inflation assuming $N \gg 1$.

[6 marks]

(iii) Suppose a spatially flat FRW universe reheats after inflation to a temperature $T_{reheat} = 10^{26}K$. Derive an estimate for the minimum number of e-folds of inflation, N_{min}, that are required to solve the 'horizon problem'. You may neglect the recent domination of dark energy in your approximation.

[7 marks]

[Total 20 marks]

 End of examination paper