Imperial College London

MSc EXAMINATION May 2017

This paper is also taken for the relevant Examination for the Associateship

PARTICLE COSMOLOGY

For Students in Quantum Fields and Fundamental Forces

Tuesday, 2nd May 2017: 14:00 to 17:00

Answer **THREE** *out of the following four questions.*
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature and, unless otherwise stated, choose units so that $\hbar = 1$ and $c = 1$.

You may find the following useful:

For the FRW metric,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

the Friedmann equation and conservation law are,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \qquad \dot{\rho} + 3H\left(\rho + P\right) = 0\,.$$

In SI units the following constants have values;

$$\begin{aligned}
\hbar = \frac{h}{2\pi} &= 1.05 \times 10^{-34}\ \mathrm{kg\,m^2\,s^{-1}} \\
c &= 3.00 \times 10^{8}\ \mathrm{m\,s^{-1}} \\
G &= 6.67 \times 10^{-11}\ \mathrm{kg^{-1}\,m^3\,s^{-2}} \\
k_B &= 1.38 \times 10^{-23}\ \mathrm{kg\,m^2\,s^{-2}\,K^{-1}}
\end{aligned}$$

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1. (i) Compute the energy density for a single bosonic degree of freedom in the high temperature relativistic limit to show that,

$$\rho_{boson} = \frac{1}{2} a\, T^4$$

where $a = \pi^2 k_B^4/15$ is the radiation constant (in units $c = \hbar = 1$).
Repeat the same calculation for a single fermion degree of freedom.

You may find the following integral useful; $\int_0^\infty dx \frac{x^3}{e^x \pm 1} = \frac{15 \mp 1}{240}\pi^4$.

[6 marks]

(ii) Show that for flat FRW spacetime in the radiation era, with the energy density dominated by g_{eff} bosonic relativistic degrees of freedom, the Hubble expansion is approximately given in SI units as,

$$H \sim \sqrt{g_{eff}} \left(\frac{T}{10^{10} K} \right)^2 sec^{-1} \ .$$

[6 marks]

(iii) Assume dark matter is a thermal relic formed during the radiation era from a scalar particle X and antiparticle $\bar{X}$ with mass M. Let these particles interact via a 2-particle process $X + \bar{X} \to$ products, with an averaged cross section $\langle \sigma v \rangle$, where the products are in thermal equilibrium. Assume the freezeout temperature, T_{freeze}, occurs when X, $\bar{X}$ are non-relativistic.

Define $x = \frac{k_B T_{freeze}}{M}$. Stating clearly any assumptions you make, show that in order for the relic density to account for the dark matter we require,

$$M \sim \frac{H_0^2}{G\,(k_B T_{CMB})^3} x^{3/2} e^{\frac{1}{x}} \ , \quad \langle \sigma v \rangle \sim \frac{1}{x} \frac{\sqrt{G^3 g_{eff}}}{H_0^2} (k_B T_{CMB})^3$$

where T_{CMB} and H_0 are the CMB temperature and Hubble constant today, and g_{eff} was the effective number of bosonic relativistic degrees of freedom at freezeout.

[You may find it useful to recall the number density for a non-relativistic species is,

$$n = \left(\frac{k_B m T}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{m}{k_B T}}$$

where m is the mass and there is no chemical potential.]

[6 marks]

(iv) Briefly describe the 'WIMP miracle'? (You do not have to perform calculations)

[2 marks]

[Total 20 marks]

2. (i) Consider a free particle with mass m, moving in a flat FRW spacetime so $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$. This moves on a geodesic $x^\mu(\tau) = (T(\tau), X^i(\tau))$ where τ is its proper time. Show that,

$$\left(\frac{dT}{d\tau}\right)^2 = \frac{1}{a^2(T)}u^i u^i + 1 \,, \quad a(T)^2\frac{dX^i}{d\tau} = u^i$$

where u^i are constants.

[6 marks]

(ii) Compute the energy E of the particle as seen by a comoving observer. Hence deduce the total momentum p of the particle, as seen by the same observer.

Show that $a(t)p$ =constant along the particle trajectory. What is the origin of this constant of the motion?

[4 marks]

(iii) Using the previous result, derive the Boltzmann equation for a homogeneous isotropic gas of these free particles in flat FRW spacetime,

$$\left.\frac{\partial}{\partial t}n\right|_p - Hp\left.\frac{\partial}{\partial p}n\right|_t = 0$$

where $n(t,p)$ is the phase space density.

[5 marks]

(iv) Photons are held in thermal equilibrium by Compton scattering above $T \sim 10^5 K$. What happens around this temperature? Use the Boltzmann equation (which is the same in the massless case as above) to show why we may give an 'effective temperature' to the photons after the universe has cooled past this point, such as when we discuss the CMB temperature. How does this effective temperature vary with the scale factor?

[5 marks]

[Total 20 marks]

3. (i) Consider our universe as a flat FRW model filled with non-relativistic matter, radiation and a positive cosmological constant. The matter and radiation energy density fractions today are Ω_m and Ω_r respectively, and the cosmological constant has recently come to dominate the energy density, with matter-Λ equality at redshift $Z_{m-\Lambda} < 1$. Suppose matter-radiation equality occurred at a high redshift $Z_{m-r} \gg 1$. Show that,

$$\Omega_m \simeq \frac{1}{1+(1+Z_{m-\Lambda})^3}\,, \qquad \Omega_r \simeq \frac{1}{Z_{m-r}}\frac{1}{(1+(1+Z_{m-\Lambda})^3)}\,.$$

[4 marks]

(ii) The scale factor and Hubble constant today are a_0 and H_0. Show that, between a redshift Z_1 and a later redshift Z_2, a null ray travels a comoving distance R given by,

$$a_0 R = \frac{1}{H_0}\int_{\frac{1}{1+Z_1}}^{\frac{1}{1+Z_2}} \frac{dx}{x^2\sqrt{(1-\Omega_m-\Omega_r)+\Omega_m x^{-3}+\Omega_r x^{-4}}}\,.$$

[6 marks]

(iii) Suppose $Z_{m-r} = 3500$ and $Z_{m-\Lambda} = 0.33$. Let photon last scattering occur at $Z_{ls} = 1100$. Assume that the radiation era extends back to the big bang. Neglecting the cosmological constant, show that the comoving size of the particle horizon at last scattering, $R_{horizon}$, is given by,

$$a_0 R_{horizon} \simeq \frac{2}{H_0\sqrt{\Omega_m}}\left(\sqrt{\frac{1}{Z_{ls}}+\frac{1}{Z_{m-r}}}-\sqrt{\frac{1}{Z_{m-r}}}\right)\,.$$

Now neglecting the contribution of radiation, show that the comoving distance a photon travels between last scattering and today, R_{lss}, is approximated by,

$$a_0 R_{lss} \simeq \frac{1}{H_0\sqrt{1-\Omega_m}} F\left(\frac{\Omega_m}{1-\Omega_m}\right)$$

where we have defined the function F as the definite integral,

$$F(y) = \int_1^\infty \frac{dq}{\sqrt{1+yq^3}}\,.$$

[7 marks]

(iv) Compute the ratio of the size of a region of the last scattering surface that has been in casual contact to the total size of the last scattering surface. You should use the following approximation; for $0 < y < 1$ the function $F(y)$ may be approximated as,

$$F(y) \simeq \frac{2.804}{y^{1/3}} - 1 + \frac{y}{8}\,.$$

[3 marks]

[Total 20 marks]

4. (i) Consider flat FRW spacetime, $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, and an inflaton potential $V(\phi) = m^2\phi^2$. Compute the slow roll function,

$$\epsilon(\phi) = \frac{1}{16\pi G}\frac{V'(\phi)^2}{V(\phi)^2}$$

for this potential. Let N be the number of e-folds of slow roll inflation. Compute N as a function of the starting position of the scalar when $N \gg 1$.
[Recall that $3H\dot{\phi} \simeq -V'(\phi)$ and $H^2 \simeq \frac{8\pi G}{3}V(\phi)$ for slow roll inflation.]

[6 marks]

(ii) Approximate the slow roll inflation by de Sitter expansion with $H =$ constant. Then fluctuations of the inflaton, $\delta\phi(t, x^i)$, obey,

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\delta\phi = 0\ .$$

Show that a solution with comoving wavenumber k_i may be written as,

$$\delta\phi(t,x) = \delta\phi_{k_j}(t)e^{-ik_ix^i} + c.c.\ , \quad \delta\phi_{k_i}(t) = c_{k_i}e^{+\frac{ik}{a(t)H}}\left(\frac{1}{a(t)} + \frac{iH}{k}\right)$$

where $k = \sqrt{\delta^{ij}k_ik_j}$ and c_{k_i} is a constant.

[4 marks]

(iii) We may quantize the inflaton taking,

$$\delta\hat{\phi}(t,x) = \int d^3k_i\left(\hat{a}_{k_i}\delta\phi_{k_i}(t)e^{-ik_ix^i} + \hat{a}^\dagger_{k_i}\delta\phi_{k_i}(t)^\star e^{+ik_ix^i}\right)$$

with creation and annihilation operators, $\hat{a}_{k_i}$ and $\hat{a}^\dagger_{k_i}$, obeying,

$$\left[\hat{a}_{k_i}, \hat{a}_{q_j}\right] = 0\ , \quad \left[\hat{a}^\dagger_{k_i}, \hat{a}^\dagger_{q_j}\right] = 0\ , \quad \left[\hat{a}_{k_i}, \hat{a}^\dagger_{q_j}\right] = \delta^{(3)}(k_i - q_i)\ .$$

The canonical momentum for the fluctuation $\delta\hat{\pi}(t,x) = a^3(t)\frac{d}{dt}\delta\hat{\phi}(t,x)$. Find an appropriate choice of the constants c_{k_i} above so that the equal time commutation relations of $\delta\hat{\phi}$ and $\delta\hat{\pi}$ have the correct form, namely,

$$\left[\delta\hat{\phi}(t,x), \delta\hat{\phi}(t,y)\right] = 0\,,\ [\delta\hat{\pi}(t,x), \delta\hat{\pi}(t,y)] = 0\,,\ \left[\delta\hat{\phi}(t,x), \delta\hat{\pi}(t,y)\right] = i\delta^{(3)}(x-y)\ .$$

[You may find it useful to recall that $(2\pi)^3\delta^{(3)}(x-y) = \int d^3k_ie^{-ik_i(x^i-y^i)}$.]

[7 marks]

(iv) We define the dimensionless power spectrum, $\Delta^2(t,k)$, as,

$$\langle 0|\delta\hat{\phi}(t,x)\delta\hat{\phi}(t,y)|0\rangle = \int \frac{d^3k_i}{k^3}\,\Delta^2(t,k)\,e^{-ik_i(x^i-y^i)}\ .$$

Explicitly compute this to show that on super horizon scales $\Delta^2(t,k)$ is constant in time and comoving wavenumber k.

[3 marks]

[Total 20 marks]

 End of examination paper