4 Inflation

Inflation is a period there there was a de Sitter phase of exponential expansion preceding the radiation era, so $H \simeq$constant, and $a \sim e^{Ht}$. This simultaneously addresses the 3 puzzles below, and also accounts for the formation of inhomogeneity and structure very successfully. At the moment it is the best candidate for the behaviour of the very early universe.

4.1 3 problems

As things stand there are 3 problems associated to the hot big bang assuming that the radiation era started with a big bang. All three are of a similar nature. Let us focus on the 'horizon problem'.

Horizon problem

When we look in different directions in the sky, the relic CMB photons that last interacted at last scattering, $Z \sim 1100$, have very similar temperatures to 1 part in 10^5 (taking into account our peculiar motion).

Assuming the radiation universe starts at the big bang, consider how large a region was in causal contact at $Z \sim 1100$. Consider a flat universe, $k = 0$.

[**Recap:** consider a general $a = k\,t^p$ (in terms of equation of state, $p = 2/(3(1+w))$), so that $H = p/t$ - eg. for matter $p = 2/3$, for radiation $p = 1/2$. Then the comoving coordinate distance $R(t_1, t_2)$ traversed by a null ray from time t_1 to t_2, goes as,

$$R(t_1, t_2) = \int_{t_1}^{t_2} \frac{dt'}{a(t')} = \frac{p}{1-p} \left[\frac{1}{a(t)H(t)} \right]_{t_1}^{t_2} \tag{453}$$

Note that since $1/(aH) \sim t^{1-p}$, then for $p < 1$ (eg. matter, radiation) the upper limit will tend to dominate this integral.]

By propagating forward a null ray from a point at $R = 0$ at the big bang, we see it is in causal contact with a comoving radius,

$$R_{causal} = \int_0^{t_{lss}} \frac{dt'}{a(t')} \tag{454}$$

at last scattering. Now ignoring factors of order one, we can approximate this as,

$$R_{causal} \simeq \left. \frac{1}{aH} \right|_{lss} \tag{455}$$

Conversely let us consider us today at $R = 0$ and compute the comoving radius of the last scattering surface, R_{lss}, by propagating back a null ray. Then,

$$R_{lss} = \int_{t_{lss}}^{t_0} \frac{dt'}{a(t')} \tag{456}$$

Again, ignoring order one factors we can approximate this as,

$$R_{lss} \simeq \left[\frac{1}{a(t)H(t)} \right]_{t_{lss}}^{t_0} \sim \left. \frac{1}{aH} \right|_0 \tag{457}$$

using the fact that the upper limit dominates.

Now last scattering $Z_{lss} \sim 1100$ occurred in the matter epoch. Hence, ignoring the last short epoch of dark energy, we can compare the causal radius R_{causal} to the radius of the last scattering surface R_{lss} using the intervening matter domination, so $a \sim t^{\frac{2}{3}}$ and $H \sim 1/t$, and thus, $aH \sim a^{-1/2}$. Then,

$$\frac{R_{causal}}{R_{lss}} \sim \frac{(aH)_0}{(aH)_{lss}} \sim \left(\frac{a_0}{a_{lss}} \right)^{-\frac{1}{2}} \sim (1 + Z_{lss})^{-\frac{1}{2}} \sim \frac{1}{\sqrt{1100}} \sim \frac{1}{30} \tag{458}$$

Thus the size of a causal region at last scattering is a small fraction of the size of the last scattering surface itself.

[**A more precise version of this;** for a null ray,

$$R(Z_1, Z_2) = \frac{1}{a_0 H_0} \int_{\frac{1}{1+Z_1}}^{\frac{1}{1+Z_2}} \frac{dx}{x^2 \sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4}}} \tag{459}$$

where from before we have,

$$\Omega_\Lambda \sim 0.7 \ , \quad \Omega_M \sim 0.3 \ , \quad \Omega_R \sim 1.68 \times \Omega_\gamma = 1.68 \times 5 \times 10^{-5} \tag{460}$$

Then numerically performing the integrals;

$$R_{causal} = R(\infty, 1100) = \frac{0.06}{a_0 H_0} \ , \quad R_{lss} = R(1100, 0) = \frac{3.10}{a_0 H_0} \tag{461}$$

to give,

$$\frac{R_{causal}}{R_{lss}} \sim \frac{1}{50} \tag{462}$$

which is similar but more accurate than our estimate above.]

Thus we find that assuming a radiation universe back to a big bang gives a **last scattering surface made of many patches that are causally disconnected.** Each patch subtends an angle of $\sim 1/50$ radians which is ~ 1 degree on the sky (ie. approx twice sun/moon).

Since each patch is causally disconnected from the others, why do they have such similar temperatures?

Flatness problem

Suppose there is some spatial curvature, so that $k = \pm 1$. Then recall that Friedmann gives,

$$\frac{8\pi G_N}{3} H^2 = \rho - \frac{3}{8\pi G_N} \frac{k}{a^2} \tag{463}$$

Now today we know that $\rho \sim \rho_{crit} = \frac{8\pi G_N}{3} H_0^2$. Hence we know that today,

$$\left| \frac{3}{8\pi G_N} \frac{k}{a_0^2} \right| < 1 \tag{464}$$

Define the critical density at a time t as $\rho_{crit}(t) = \frac{8\pi G_N}{3} H(t)^2$, and the fraction of energy density in curvature at time t, $\Omega_k(t)$, from,

$$\rho_{crit}(t)\Omega_k(t) = -\frac{3}{8\pi G_N} \frac{k}{a^2(t)} \quad \implies \quad \Omega_k(t) = -\frac{k}{\left(a(t)H(t)\right)^2} \tag{465}$$

Then we have today $|\Omega_k(t_0)| < 1$. But we see that,

$$|\Omega_k| \sim \frac{1}{a^2(t)H^2(t)} \tag{466}$$

Thus for $a \sim t^p$ for $p < 1$ the curvature fraction tends to increase in time. Thus the problem is that for the curvature to be small today, it must have been **tiny** in the past.

Note that as for the horizon problem, it is again the quantity $1/aH$ that is entering our problem.

For the whole matter era (ignoring the recent dark energy epoch), starting at $Z \sim 3500$, then,

$$\frac{|\Omega_k|_{Z=3500}}{|\Omega_k|_0} \sim \left(\frac{(aH)_0}{(aH)_{Z=3500}} \right)^2 \sim \left(\frac{a_0}{a_{Z=3500}} \right)^{-1} \sim \frac{1}{3500} \tag{467}$$

Then in the radiation era before that $a \sim t^{1/2}$ and $H \sim 1/t$ so that $aH \sim a^{-1}$. Thus,

$$\frac{|\Omega_k|_Z}{|\Omega_k|_{Z=3500}} \sim \left(\frac{(aH)_{Z=3500}}{(aH)_Z} \right)^2 \sim \left(\frac{a_{Z=3500}}{a_Z} \right)^{-2} \sim \frac{3500^2}{Z^2} \tag{468}$$

In the radiation era, $Z \simeq T/T_{CMB}$, and using $|\Omega_{k,0}| \sim 1$ gives,

$$|\Omega_k(T)| \sim \frac{1}{3500} \frac{3500^2}{(T/T_{CMB})^2} \sim \left(\frac{160K}{T} \right)^2 , \qquad T > 10^4 K \tag{469}$$

Consider the epoch of nucleosynthesis - then $T \sim 10^{10}K$. The universe was certainly this hot in the past. Then the curvature fraction at this time would have to be,

$$|\Omega_k|_{nuc} \sim \left(\frac{100K}{10^{10}K} \right)^2 \sim 10^{-16} \tag{470}$$

So if there is curvature, it would for some reason have to have been extremely small in the early universe in order to be consistent with its small value today.

The earlier and hotter we believe the radiation era extends, the worse the problem. For example, for weak scale temperatures, $T \sim 10^{15}K$ then,

$$|\Omega_k|_{weak} \sim \left(\frac{100K}{10^{15}K} \right)^2 \sim 10^{-26} \tag{471}$$

Monopole problem

A similar problem that arises in models where defects can form in high temperature phase transitions - see Weinberg for details.

4.2 Resolution from early exponential expansion

Suppose the radiation era started at some temperature T_{rad}, and before that was an 'inflationary' de Sitter phase of expansion where,

$$a = a_{rad} e^{H_{inf}(t-t_{rad})} \ , \quad H(t) = H_{inf} \tag{472}$$

with H_{inf} =const, and t_{rad}, a_{rad} the time and scale factor at the beginning of the radiation era. By continuity, H_{inf} must equal the Hubble rate at the start of the radiation era,

$$H_{inf} = H_{rad}(T = T_{rad}) = s^{-1}\left(\frac{T_{rad}}{10^{10}K}\right)^2 \tag{473}$$

Suppose the inflationary period starts at time t_{inf} and lasts for a time $\Delta t = t_{rad} - t_{inf}$. The scale factor increases by a ratio,

$$\frac{a_{rad}}{a_{inf}} = e^{H_{inf}\Delta t} = e^N \tag{474}$$

where $N = H_{inf}\Delta t$ =number of **e-foldings** of inflation.

Horizon problem: Now reconsider the horizon problem. Clearly R_{lss} is unchanged, but now since the big bang is now replaced by inflation, the causal radius at last scattering is changed. Now consider R_{causal} again, taking a null ray starting at $R = 0$ at the beginning of inflation. Thus,

$$R_{causal} \ = \ \int_{t_{inf}}^{t_{lss}} \frac{dt'}{a(t')} = R_{inf} + \int_{t_{rad}}^{t_{lss}} \frac{dt'}{a(t')} \tag{475}$$

Now the second term, given by the period of the radiation era,

$$\int_{t_{rad}}^{t_{lss}} \frac{dt'}{a(t')} \sim \left.\frac{1}{aH}\right|_{lss} \tag{476}$$

takes the same value as before, as it is dominated by its upper limit.

The key difference is that for exponential expansion the integral is not dominated by the upper limit any more. In fact,

$$\begin{aligned} R_{inf} \ &= \ \int_{t_{inf}}^{t_{rad}} \frac{dt'}{a(t')} = \frac{1}{a_{rad}} \int_{t_{inf}}^{t_{rad}} e^{-H_{inf}(t-t_{rad})} \\ &= \ -\frac{1}{a_{rad}H_{rad}} \left[e^{-H_{inf}(t-t_{rad})}\right]_{t_{inf}}^{t_{rad}} = \frac{1}{a_{rad}H_{rad}}\left(e^N - 1\right) \end{aligned} \tag{477}$$

Thus for a large number of e-folds, $N \gg 1$, we have,

$$R_{causal} = \frac{1}{a_{rad}H_{rad}}e^N + \frac{1}{a_{lss}H_{lss}} \tag{478}$$

Now for the radiation era $aH \sim 1/a \sim T$,

$$\frac{a_{lss}H_{lss}}{a_{rad}H_{rad}} \sim \frac{T_{lss}}{T_{rad}} \tag{479}$$

Hence,

$$R_{causal} = \frac{1}{a_{lss}H_{lss}}\left(\frac{T_{lss}}{T_{rad}}e^N + 1\right) \tag{480}$$

As before,

$$R_{lss} = \frac{1}{a_0H_0} \tag{481}$$

and we estimate in the matter era,

$$\frac{a_0H_0}{a_{lss}H_{lss}} \sim \left(\frac{a_0}{a_{lss}}\right)^{-\frac{1}{2}} \sim (1+Z_{lss})^{-\frac{1}{2}} \sim \frac{1}{\sqrt{1100}} \sim \frac{1}{30} \tag{482}$$

Note, a more accurate value is the $1/50$ we found before, correctly taking into account the last epoch of dark energy. However, the precise number makes little difference - let's stick with $1/30$. Hence, now the relative size of a causal region in the last scattering surface is,

$$\frac{R_{causal}}{R_{lss}} \sim \frac{1}{30}\left(\frac{T_{lss}}{T_{rad}}e^N + 1\right) \tag{483}$$

The key point is that now by choosing the inflation period to have lasted long enough, we can always arrange $R_{lss} \ll R_{causal}$. We see we require,

$$30\left(\frac{T_{rad}}{T_{lss}}\right) \ll e^N \tag{484}$$

Then there is no problem in principle with different patches of the last scattering surface having the same temperature, since they were in causal contact in the past.

We see the **minimum** number of e-folds required for inflation to solve the horizon problem is then determined by the point at which the radiation era started, ie. by the value T_{rad}. We certainly believe from nucleosynthesis that $T_{rad} > 10^{10}K$. Then we require

$$\log 30 \left(\frac{10^{10}K}{10^3K} \right) \simeq 20 \ll N \tag{485}$$

to solve the horizon problem.
However, if we believe the hot radiation epoch extended back to higher energies, then we require more inflation. For example, extending back to the weak scale $T_{rad} = 10^{15}K$ ($\sim 100GeV$) requires $N \sim 31$. Extending back at a GUT scale, $T_{rad} = 10^{29}K$ ($\sim 10^{16}GeV$) requires at least $N \sim 63$.

Flatness problem: During inflation the universe expands and so the effect of curvature is reduced. Recall,

$$|\Omega_k(t)| \sim \frac{1}{a(t)^2 H(t)^2} \tag{486}$$

Now since for inflation H =constant, then we can propagate Ω_k from its value at the start of inflation to its value at the end as,

$$\frac{|\Omega_k|_{rad}}{|\Omega_k|_{inf}} \sim \frac{a_{inf}^2}{a_{rad}^2} = e^{-2N} \tag{487}$$

Thus if curvature took a natural value at the start of inflation, so $|\Omega_k|_{inf} \sim 1$, then after many e-folds it would become extremely small.

Recall if the radiation era started at $T \sim 10^{10}K$, we computed earlier that,

$$|\Omega_k|_{nuc} \sim 10^{-16} \tag{488}$$

at that time, and we said this was unnaturally small. However, we now see that if inflation ended then, and there were enough e-folds so,

$$e^{-2N} \sim 10^{-16} \; , \quad N > 20 \tag{489}$$

then curvature would not have to be small at the beginning of inflation, and no fine tuning of curvature would be required.

If it extended back to the weak scale temperatures, $T \sim 10^{15} K$ recall we found,

$$|\Omega_k|_{weak} \sim 10^{-26} \tag{490}$$

Then we require,

$$e^{-2N} \sim 10^{-26} , \quad N > 30 \tag{491}$$

e-folds to make it natural that the curvature was so small at the weak scale.

In fact we can see that the condition on the number of e-folds from considering the curvature problem is precisely the same as that from the horizon problem. Indeed, it is also the same condition that the monopole problem requires to be solve.

As you may think, the idea of introducing a period of exponential expansion to solve the above problems seems quite drastic. Maybe there are other resolutions to these. However, in fact the real power of inflation is that having introduced it, it then naturally provides a mechanism to generate inhomogeneity. That is the principle reason we believe it (or something similar to it) occurred.

4.3 The inflaton

Recall we can obtain de Sitter expansion from a positive cosmological constant, Λ, so that,

$$a(t) = e^{\sqrt{\frac{\Lambda}{3}} t} , \quad H = \sqrt{\frac{\Lambda}{3}} = \text{const} \tag{492}$$

Recall that the energy density and pressure are,

$$\rho = -P = \frac{\Lambda}{8\pi G} \tag{493}$$

However, such an expansion would be forever present and would never allow a radiation and matter era. Thus we need an effective cosmological constant that then 'turns off'.

This can be achieved in **slow roll** inflation models, where there is a real scalar field ϕ, the inflaton, with a potential $V(\phi)$. The action for the scalar is,

$$I = -\int \sqrt{g} \left(\frac{1}{2} (\partial\phi)^2 + V(\phi) \right) \tag{494}$$

which leads to an equation of motion,

$$\nabla^2 \phi = V'(\phi) \tag{495}$$

and stress tensor,

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} (\partial \phi)^2 + V(\phi) \right) \tag{496}$$

Let us now consider the inflaton in a flat $k = 0$ FRW spacetime. One can compute (see Ex sheet 4) that the equation of motion yields,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{497}$$

where the second term is the 'Hubble damping' arising from the expansion of the universe. From the stress tensor in FRW we can determine the energy density ρ and pressure P finding,

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ P &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned} \tag{498}$$

The Friedmann equation is then,

$$H^2 = \frac{8\pi G}{3} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \tag{499}$$

as usual. By differentiating it we obtain another useful equation,

$$\begin{aligned} 2H\dot{H} &= \frac{8\pi G}{3} \left(\dot{\phi}\ddot{\phi} + \frac{dV(\phi)}{dt} \right) \\ &= \frac{8\pi G}{3} \dot{\phi} \left(\ddot{\phi} + V'(\phi) \right) \\ &= \frac{8\pi G}{3} \dot{\phi} \left(-3H\dot{\phi} \right) \end{aligned} \tag{500}$$

using the scalar equation of motion. Hence we find,

$$\dot{H} = -4\pi G \dot{\phi}^2 \tag{501}$$

Slow roll: We see that if the scalar field is rolling slowly, so that $\phi \simeq$constant and $\dot{\phi} \simeq 0$, then $\rho \simeq P \simeq V(\phi) \simeq$const, and $\dot{H} \simeq 0$, which is precisely what we have for de Sitter where $\Lambda = 8\pi G V$.

Let us be more precise. In order that the stress tensor looks like a cosmological constant, we require,

$$\dot{\phi}^2 \ll V \tag{502}$$

which then leads to,

$$H^2 \simeq \frac{8\pi G}{3} V \tag{503}$$

Furthermore we require that it applies for a *period of time* as the scalar field rolls. We wish it to apply between two values of the scalar ϕ as it rolls down the potential, where the potential slowly changes from one value, say V_1 at ϕ_1 to a lower value V_2 at ϕ_2. Then we require,

$$\dot{\phi}^2 = \alpha V \ , \quad |\alpha(t)| \ll 1 \, \forall \, t \ s.t. \ V_2 < V(\phi(t)) < V_1 \tag{504}$$

In particular show roll to apply for $V_2/V_1 \sim O(1)$. This requires that the time derivative of the function α is also small. In particular it implies,

$$\left| V \frac{d\alpha}{dV} \right| \quad \ll \quad 1 \tag{505}$$

Then for an order $O(1)$ fractional change in V we will have only a small change in α so that if it is small initially it remains small. We can compute,

$$V \frac{d\alpha}{dV} \quad = \quad \frac{V}{V' \dot{\phi}} \frac{d}{dt} \left(\frac{\dot{\phi}^2}{V} \right) = 2 \frac{\ddot{\phi}}{V'} - \frac{\dot{\phi}^2}{V} = 2 \frac{\ddot{\phi}}{V'} - \alpha \tag{506}$$

Since the second term, $|\alpha| \ll 1$, then our condition $\left| V \frac{d\alpha}{dV} \right| \ll 1$ requires the first term also to be small, ie. it requires,

$$\left| \ddot{\phi} \right| \ll |V'(\phi)| \tag{507}$$

One could proceed and show that in addition higher derivatives, $V^n \frac{d\alpha}{dV^n}$, are also constrained to be small. [See example sheet question.]

This condition implies that we may ignore the $\ddot{\phi}$ term in the scalar equation. Hence the Hubble damping term dominates, giving a first order flow and together with the Freidmann equation above, give the slow roll equations of inflation;

$$\begin{aligned} H^2 &\simeq \frac{8\pi G}{3} V \\ 3H\dot{\phi} &\simeq -V'(\phi) \end{aligned} \quad (508)$$

Thus we see that slow roll is where Hubble damping dominates, and we expect this **requires sufficient flatness of the potential**.

Slow roll consistency for the potential:

Let us consider the potential and its gradients. Computing,

$$\frac{V'}{V} = \frac{-3H\dot{\phi}}{V} = \frac{-3\sqrt{\frac{8\pi G}{3}V}\dot{\phi}}{V} = -\sqrt{24\pi G}\sqrt{\frac{\dot{\phi}^2}{V}} \quad (509)$$

and recalling $\dot{\phi}^2 \ll V$ we see that we require,

$$\epsilon(\phi) \equiv \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = \frac{3}{2}\frac{\dot{\phi}^2}{V} \ll 1 \quad (510)$$

and ϵ is known as a the first **slow roll parameter**. It characterised the flatness of the potential, and we see indeed it must be small.
Now consider the second derivative of the potential.

$$\begin{aligned} \frac{V''}{V} &= \frac{1}{V\dot{\phi}}\frac{dV'}{dt} = \frac{1}{V\dot{\phi}}\frac{d}{dt}\left(-3H\dot{\phi}\right) = -\frac{3\dot{H}}{V} - \frac{3H\ddot{\phi}}{V\dot{\phi}} \\ &= 12\pi G\frac{\dot{\phi}^2}{V} - \frac{3H^2}{V}\frac{\ddot{\phi}}{H\dot{\phi}} = 12\pi G\frac{\dot{\phi}^2}{V} + 24\pi G\frac{\ddot{\phi}}{V'} \end{aligned} \quad (511)$$

and recalling that $\dot{\phi}^2 \ll V$ and $\left|\ddot{\phi}\right| \ll |V'(\phi)|$, then,

$$\eta(\phi) \equiv 8\pi G\left|\frac{V''}{V}\right| < \frac{3}{2}\left|\frac{\dot{\phi}^2}{V}\right| + 3\left|\frac{\ddot{\phi}}{V'}\right| \ll 1 \quad (512)$$

Now η is known as a the second **slow roll parameter**, and again characterizes how flat the potential is in a neighborhood of where the scalar it.
Now for a potential $V(\phi)$ we may compute the slow roll parameters $\epsilon(\phi), \eta(\phi)$. If we require slow roll inflation for some range of field values, say $\phi_1 < \phi < \phi_2$, then a necessary condition is that $\epsilon(\phi) \ll 1$ over this range of ϕ. Provided it holds over the range, then it must also be true that $\eta(\phi) \ll 1$ too.

[People often discuss having ϵ and η small being the condition for slow roll, but this is misleading. For what value of ϕ are they requiring this? Really they mean ϵ small over a range of ϕ which implies η is small too, and indeed higher derivatives of the potential are also appropriately small (see example sheet 4, Qu 4)]

E-foldings

As defined earlier, the number of e-foldings during inflation is,

$$e^N = \frac{a_{finish}}{a_{start}} , \quad N = \log a_{finish} - \log a_{start} \tag{513}$$

Hence we can express this as the integral,

$$N = \int_{a_{start}}^{a_{finish}} \frac{da}{a} \tag{514}$$

Now we can use our slow roll equations,

$$H^2 \simeq \frac{8\pi G}{3} V , \quad \dot{\phi} = -\frac{V'}{3H} \tag{515}$$

to find an equation for N in terms of the potential and range of scalar traversed;

$$\begin{aligned} N &= \int_{\phi_{start}}^{\phi_{finish}} \frac{1}{a}\frac{da}{dt}\frac{dt}{d\phi} d\phi = \int_{\phi_{start}}^{\phi_{finish}} \frac{H}{\dot{\phi}} d\phi \\ &\simeq -\int_{\phi_{start}}^{\phi_{finish}} \frac{3H^2}{V'} d\phi \\ &\simeq -8\pi G \int_{\phi_{start}}^{\phi_{finish}} \frac{V}{V'} d\phi \end{aligned} \tag{516}$$

The sign is due to the fact $V' < 0$ leads to slow roll in the positive ϕ direction ($\dot{\phi} = -V'/3H$) and $V' > 0$ to negative rolling.

Noting the slow roll condition $|V'/V| \ll \sqrt{G}$ this implies,

$$
\begin{aligned}
N &\gg 8\pi G \left| \int_{\phi_{start}}^{\phi_{finish}} \frac{1}{\sqrt{G}} d\phi \right| \\
&\gg 8\pi \sqrt{G} |\Delta\phi| \qquad (517)
\end{aligned}
$$

Thus having a large number of e-folds of inflation requires the inflaton field to move a long way in terms of the gravitational scale G. This can lead to potential problems with effective field theory.

End of inflation

The idea of inflation is to have a potential that is sufficiently flat to allow slow roll. However, eventually the scalar reaches a point in the potential, say near the minimum of the potential (which must be zero - up to a possible tiny dark energy component), where $\epsilon \not\ll 1$ anymore and the dynamics ceases to be slow roll.

As it descends to the bottom of the potential the slow roll conditions will be violated, and the scalar will then roll back and forth, gradually slowing down due to the Hubble damping. In order to make contact with the hot big bang, much of the energy in the scalar as it rolls must be converted to SM particles - a process called **reheating**. Via small coupling to the SM fields, high energy SM particles will be produced that then thermalize to start the hot big bang radiation era. The temperature scale at which this happens is unknown, but must be higher than that of nucleosynthesis - $\gg 10^{10}K$.

Scales during inflation

The basic physics of inflation is that a fixed comoving scale, radius R that was initially within causal contact of some comoving observer then later during inflation leaves their causal domain, in the sense that they cannot causally affect the observer for the rest of the inflation era. However, later in the decelerated expansion of the radiation and matter era, this comoving scale will then reestablish causal contact again. The larger the scale initally, the longer one has to wait to see this scale again. (In fact since we are now dark energy dominated, there are likely to be scales that we will never see).

It is usually described in the following way; during inflation a scale 'leaves the horizon' and then later during the radiation/matter era the scale 'reenters the horizon'. This is technically a little inaccurate, but physically is a reasonable way to describe things.

What is meant is that during inflation, one effectively has de Sitter which has an **event horizon**. As discussed earlier, this event horizon has proper size,

$$d_{event}(t) = \frac{1}{H_{inf}} \tag{518}$$

(Note - since inflation ends, there isn't actually an event horizon - hence the technical inaccuracy).

On the other hand, during the matter and radiation era, we can discuss a particle horizon, as if there was a big bang with the radiation era extending back to $a = 0$. This has approximate proper size,

$$d_{particle}(t) \sim \frac{1}{H(t)} \sim t \tag{519}$$

(Of course, since there was no big bang, but rather an inflation epoch, there isn't really a particle horizon - hence again the technical inaccuracy).

During inflation a comoving scale, radius R, has proper size,

$$d_R(t) = a(t)R \sim a_{inf} e^{H_{inf} t} R \tag{520}$$

and so quickly becomes bigger than d_{event} which is roughly constant. However, later in the radiation era,

$$d_R(t) = a(t)R \sim a_{rad} \left(\frac{t}{t_{rad}} \right)^{1/2} R \quad , \qquad a_{rad} = a_{inf} e^N \tag{521}$$

so the proper size of a comoving scale still increases, but now slower than the particle horizon size. Similarly in the matter era. Thus eventually scales outside the 'particle horizon' at early times re-enter it.

Given the temperature at which the radiation era began, T_{rad}, we have computed the number of e-folds required to solve the horizon problem. Recall we found,

$$\begin{aligned} T_{rad} = 10^{10} K \ (\sim MeV) \ , &\quad N_{min} \sim 20 \\ T_{rad} = 10^{15} K \ (\sim 100 GeV) \ , &\quad N_{min} \sim 31 \\ T_{rad} = 10^{29} K \ (\sim 10^{16} GeV), &\quad N_{min} \sim 63 \end{aligned} \tag{522}$$

An important consequence is that however long inflation goes on (e.g. if $N \gg N_{min}$), we currently can only see the scales leaving the inflationary 'horizon' during the last N_{min} e-folds. Scales leaving before that are necessarily still outside our 'horizon' today - and in fact we will never see them due to dark energy.
We can ask when a comoving scale R associated to a physical size today left the horizon during inflation. See ex sheet 4; for example, taking a GUT scale $T_{rad} = 10^{29} K$, then one finds,

- Largest scales today, $\sim 10 Gpc$, left ~ 63 e-folds before the end of inflation
- $\sim 1^o$ on the sky at last scattering ie. patch that is casual in radiation era, $\sim 100 Mpc$, left ~ 58 e-folds before the end of inflation
- galaxy cluster scale, $\sim 10 Mpc$, left ~ 56 e-folds before the end of inflation
- galaxy scale, $\sim kpc$, left ~ 49 e-folds before the end of inflation
- solar system scale, $\sim 10^{12} m$, left ~ 29 e-folds before the end of inflation

One can see that only the most recent scales to reenter, corresponding to the e-foldings $\sim 56 - 63$ before the end of inflation, have simple physics with approximate FRW evolution. For scales that reentered earlier, < 56 e-folds, they have subsequently undergone complicated non-linear gravitational collapse.

4.4 Inhomogeneity from inflation

We now consider how inhomogeneity is naturally created during inflation from quantum mechanics. During inflation this inhomogeneity is formed in the inflaton, and only later at reheating, is transferred to the metric and other matter fields. Hence it is a good approximation to treat the inflaton as inhomogeneous, but in a homogeneous flat FRW background.

In this case the inflaton is $\phi = \phi(t, x)$ and the scalar equation $\nabla^2 \phi = V'(\phi)$ becomes;

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi + V'(\phi) = 0 \tag{523}$$

We decompose the inflation into a classical homogeneous slow roll part, $\phi_{cl}(t)$, and small inhomogeneous fluctuations $\delta\phi(t, x)$, so,

$$\phi = \phi_{cl}(t) + \delta\phi(t, x) \tag{524}$$

To zeroth order in the fluctuation (ie. ignoring them) we have our usual,

$$3H\dot{\phi}_{cl} = -V'(\phi_{cl}) \tag{525}$$

Then to leading non-trivial order (ie. to linear order) we obtain,

$$\delta\ddot{\phi} + 3\frac{\dot{a}}{a}\delta\dot{\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\delta\phi + V''(\phi_{cl})\delta\phi = 0 \tag{526}$$

where we have used $V'(\phi_{cl} + \delta\phi) = V'(\phi_{cl}) + \delta\phi V''(\phi_{cl}) + \ldots$.

Consider first classical solutions for $\delta\phi$ at this leading order. We will later build out quantum theory from them. Since we have a flat spatial section,

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j \tag{527}$$

we may Fourier decompose and consider a single mode,

$$\delta\phi(t, x) = \delta_{k^i}\phi(t)\, e^{-ik_i x^i} \tag{528}$$

where $\vec{k}^i$ is the **comoving wavenumber**. Let us for convenience use the notation $x^i = \vec{x}$ and $k_i = \vec{k}$. Then we may write,

$$\delta\phi(t, x) = \delta_{\vec{k}}\phi(t)\, e^{-i\vec{k}\cdot\vec{x}} \tag{529}$$

The comoving coordinate wavelength is the comoving scale $R_k = 2\pi/k$, where $k = \sqrt{\delta^{ij}k_i k_j}$, and hence the physical wavelength of this mode at time t is,

$$\lambda_{phy}(t) = a(t)R_k = \frac{2\pi}{k}a(t) \tag{530}$$

Now this comoving wave mode obeys,

$$\delta_{\vec{k}}\ddot{\phi} + 3H\delta_{\vec{k}}\dot{\phi} + \left(\frac{k^2}{a^2} + V''(\phi_{cl})\right)\delta_{\vec{k}}\phi = 0 \tag{531}$$

However the coefficients of this are time dependent. We will need to link the behaviour of $\delta_{\vec{k}}\phi$ at early times in inflation to that at late times.

Slow roll mode functions

Let us assume slow roll, so that the geometry looks very much like de Sitter, and we have $a \sim e^{Ht}$ with $H \simeq$const and V is very flat, so we may approximate it as linear, so $V'' \simeq 0$. Then the equation simplifies to,

$$\delta_{\vec{k}}\ddot{\phi} + 3H\delta_{\vec{k}}\dot{\phi} + \frac{k^2}{a^2}\delta_{\vec{k}}\phi = 0 \tag{532}$$

which we can exactly solve, having positive energy solution;

$$\delta_{\vec{k}}\phi(t) = \frac{c_{\vec{k}}}{a(t)}\, e^{+\frac{ik}{a(t)H}}\left(1 + \frac{ia(t)H}{k}\right) \tag{533}$$

There are two interesting limits corresponding to short or large comoving wavenumber.

Early times - sub horizon mode, wavelength $\ll 1/H$

For wavelengths which are well within the Hubble horizon (a sub horizon mode) inflation we have,

$$1 \ll \frac{k}{a(t)H} \qquad \Longrightarrow \qquad \delta_{\vec{k}}\phi(t) \simeq \frac{c_{\vec{k}}}{a(t)}\, e^{+\frac{ik}{a(t)H}} \tag{534}$$

where we recognize $e^{+\frac{ik}{a(t)H}} = e^{ikR(t)}$ since $R = \int dt/a(t) \sim 1/(aH)$. Hence we see that the phase of the mode is oscillatory, just as for wave modes in flat space. However, since a is increasingly exponentially with $H \simeq$const, typically a mode which is inside the horizon will only remain inside for some finite time, before exiting.

Note that this mode is indeed of WKB form,

$$\begin{aligned}\delta_k\phi(t) &\simeq \frac{c_k}{a(t)}\, e^{-i\omega(t)} \\ \omega(t) &= k\int \frac{dt}{a(t)} = k\int dt e^{-Ht} = -\frac{k}{H}e^{-Ht} = -\frac{k}{aH}\end{aligned} \tag{535}$$

and hence looks like a scalar in flat space.

Late times - super horizon mode, wavelength $\gg 1/H$
For wavelengths which are outside the Hubble horizon (a super horizon mode) we have,

$$\frac{k}{a(t)H} \ll 1 \qquad \Longrightarrow \qquad \delta_{\vec{k}}\phi(t) \simeq \frac{ic_{\vec{k}}H}{k} \tag{536}$$

We see that outside the horizon the mode is **frozen**, having no time dependence since H ~const.

This is a general phenomena for matter. Once a perturbation is outside the horizon, its evolution tends to be frozen in time.

Quantum Field

Consider the scalar action in FRW,

$$\begin{aligned} I &= -\int dtd^3x\,\sqrt{g}\left(\frac{1}{2}(\partial\phi)^2 + V(\phi)\right) \\ &= \int dtd^3x\,a^3(t)\left(\frac{1}{2}\dot{\phi}^2 + \dots\right) \end{aligned} \tag{537}$$

so we see the canonical momentum for the scalar,

$$\pi(t,x) = \frac{\partial I}{\partial\dot{\phi}(t,x)} = a^3(t)\dot{\phi}(t,x) \tag{538}$$

Using our modes above we can construct the general classical solution for the real scalar fluctuation can then be written as a sum of these positive frequency modes, together with an appropriate negative energy contribution to ensure reality;

$$\delta\phi(t,x) = \int d^3\vec{k}\left(\delta_{\vec{k}}\phi(t)\,e^{+i\vec{k}\cdot\vec{x}} + \delta_{\vec{k}}\phi(t)^\star e^{-i\vec{k}\cdot\vec{x}}\right) \tag{539}$$

Note that each mode contains a constant of integration $c_{\vec{k}}$ which we have so far not determined. These data parameterize the particular solution to the equations.

Now having found the classical solution for the fluctuation we proceed to quantise the inflaton. Now the field and its momentum becomes an operators,

$$\hat{\phi} = \phi_{cl}(t) + \delta\hat{\phi}(t,x) \ , \qquad \hat{\pi} = a^3\dot{\phi_{cl}}(t) + \delta\hat{\pi}(t,x) \tag{540}$$

which we require to obey equal time commutation relations,

$$\begin{aligned} \left[\hat{\phi}(t,x), \hat{\phi}(t,y)\right] &= [\hat{\pi}(t,x), \hat{\pi}(t,y)] = 0 \\ \left[\hat{\phi}(t,x), \hat{\pi}(t,y)\right] &= i\,\delta^3(\vec{x}-\vec{y}) \end{aligned} \tag{541}$$

We achieve this by writing the operator for the fluctuation in terms of our modes,

$$\delta\hat{\phi}(t,x) = \int d^3\vec{k}\left(\delta_{\vec{k}}\phi(t)\, e^{+i\vec{k}\cdot\vec{x}}\hat{a}_{\vec{k}} + \delta_{\vec{k}}\phi(t)^\star e^{-i\vec{k}\cdot\vec{x}}\hat{a}^\dagger_{\vec{k}}\right) \tag{542}$$

and introduce creation/annhilation operators with the usual commutation relations,

$$\begin{aligned} \left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] &= 0 \\ \left[\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] &= 0 \\ \left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] &= \delta^3(\vec{k}-\vec{k'}) \end{aligned} \tag{543}$$

and defining the vacuum state $|0\rangle$ as,

$$\hat{a}_{\vec{k}}|0\rangle = 0 \ , \qquad \langle 0|0\rangle = 1 \tag{544}$$

We must then specify the wave functions $\delta_{\vec{k}}\phi(t)$ by giving the data $c_{\vec{k}}$ so that we obtain the required equal time commutation relations.

A standard calculations yields,

$$\begin{aligned} \left[\hat{\phi}(t,x), \hat{\phi}(t,y)\right] &= \int d^3\vec{k}\int d^3\vec{q}\left(\delta_{\vec{k}}\phi(t)\,\delta_{\vec{q}}\phi(t)^\star\, e^{+i\vec{k}\cdot\vec{x}-i\vec{q}\cdot\vec{y}}\left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{q}}\right]\right. \\ &\qquad\qquad \left. + \,\delta_{\vec{k}}\phi(t)^\star\,\delta_{\vec{q}}\phi(t)\, e^{+i\vec{q}\cdot\vec{y}-i\vec{k}\cdot\vec{x}}\left[\hat{a}^\dagger_{\vec{k}}, \hat{a}_{\vec{q}}\right]\right) \\ &= \int d^3\vec{k}\left(\left|\delta_{\vec{k}}\phi\right|^2\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - \left|\delta_{\vec{k}}\phi\right|^2\, e^{i\vec{k}\cdot(\vec{y}-\vec{x})}\right) \\ &= \int d^3\vec{k}\left(\left|\delta_{\vec{k}}\phi\right|^2 - \left|\delta_{-\vec{k}}\phi\right|^2\right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \end{aligned} \tag{545}$$

using $\int d^3\vec{k}e^{i\vec{k}\cdot\vec{w}} = +\int d^3\vec{k}e^{-i\vec{k}\cdot\vec{w}}$ where one must take care with the limits on the integrals. Similarly,

$$[\hat{\pi}(t,x), \hat{\pi}(t,y)] = a^3(t) \int d^3\vec{k} \left(\left|\delta_{\vec{k}}\dot{\phi}\right|^2 - \left|\delta_{-\vec{k}}\dot{\phi}\right|^2 \right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \qquad (546)$$

Finally,

$$\begin{aligned}
\left[\hat{\phi}(t,x), \hat{\pi}(t,y)\right] &= a^3(t) \left[\hat{\phi}(t,x), \dot{\hat{\phi}}(t,y)\right] \\
&= a^3(t) \int d^3\vec{k} \int d^3\vec{q} \left(\delta_{\vec{k}}\phi(t)\, \delta_{\vec{q}}\dot{\phi}(t)^\star\, e^{+i\vec{k}\cdot\vec{x}-i\vec{q}\cdot\vec{y}} \left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{q}}\right] \right. \\
&\qquad \left. + \delta_{\vec{k}}\phi(t)^\star\, \delta_{\vec{q}}\dot{\phi}(t)\, e^{+i\vec{q}\cdot\vec{y}-i\vec{k}\cdot\vec{x}} \left[\hat{a}^\dagger_{\vec{k}}, \hat{a}_{\vec{q}}\right] \right) \\
&= a^3(t) \int d^3\vec{k} \left(\delta_{\vec{k}}\phi\, \delta_{\vec{k}}\dot{\phi}^\star\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - \delta_{\vec{k}}\phi^\star\, \delta_{\vec{k}}\dot{\phi}\, e^{i\vec{k}\cdot(\vec{y}-\vec{x})} \right) \\
&= a^3(t) \int d^3\vec{k} \left(\delta_{\vec{k}}\phi\, \delta_{\vec{k}}\dot{\phi}^\star - \delta_{-\vec{k}}\phi^\star\, \delta_{-\vec{k}}\dot{\phi} \right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \qquad (547)
\end{aligned}$$

Now the requirement that $\left[\hat{\phi}(t,x), \hat{\phi}(t,y)\right] = [\hat{\pi}(t,x), \hat{\pi}(t,y)] = 0$ implies we choose the mode functions so that,

$$\left|\delta_{\vec{k}}\phi\right| = \left|\delta_{-\vec{k}}\phi\right| \;, \quad \left|\delta_{\vec{k}}\dot{\phi}\right| = \left|\delta_{-\vec{k}}\dot{\phi}\right| \quad \Longrightarrow \quad |c_{\vec{k}}| = |c_{-\vec{k}}| \qquad (548)$$

recalling the form of the modes in equation (533). Let us make a parity invariant choice, so that $c_{\vec{k}} = c_{-\vec{k}}$, so this is automatically satisfied.

Then we have,

$$\begin{aligned}
\left[\hat{\phi}(t,x), \hat{\phi}(t,y)\right] &= [\hat{\pi}(t,x), \hat{\pi}(t,y)] = 0 \\
\left[\hat{\phi}(t,x), \hat{\pi}(t,y)\right] &= a^3(t) \int d^3\vec{k} \left(\delta_{\vec{k}}\phi\, \delta_{\vec{k}}\dot{\phi}^\star - \delta_{\vec{k}}\phi^\star\, \delta_{\vec{k}}\dot{\phi} \right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \qquad (549)
\end{aligned}$$

Now consider the behaviour of $\delta_{\vec{k}}\phi\, \delta_{\vec{k}}\dot{\phi}^\star$ for our mode functions in equation

(533), so;

$$
\begin{aligned}
\delta_{\vec{k}}\phi\,\delta_{\vec{k}}\dot{\phi}^{\star} &= c_{\vec{k}}\,e^{\frac{ik}{aH}}\left(\frac{1}{a}+\frac{iH}{k}\right)\times\frac{d}{dt}\left(c_{\vec{k}}\,e^{+\frac{ik}{aH}}\left(\frac{1}{a}+\frac{iH}{k}\right)\right)^{\star} \\
&= |c_{\vec{k}}|^2 e^{\frac{ik}{aH}}\left(\frac{1}{a}+\frac{iH}{k}\right)\times\left(-\frac{ik}{a^2}e^{\frac{ik}{aH}}\right)^{\star} \\
&= |c_{\vec{k}}|^2\frac{1}{a^3}\left(-aH+ik\right)
\end{aligned} \tag{550}
$$

so that,

$$
\delta_{\vec{k}}\phi\,\delta_{\vec{k}}\dot{\phi}^{\star}-\delta_{\vec{k}}\phi^{\star}\,\delta_{\vec{k}}\dot{\phi} = \frac{2ik}{a^3}|c_{\vec{k}}|^2 \tag{551}
$$

and then we see,

$$
\left[\hat{\phi}(t,x),\hat{\pi}(t,y)\right] = i\int d^3\vec{k}\,\left(2k|c_{\vec{k}}|^2\right)e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \tag{552}
$$

Now recalling that,

$$
\int d^3\vec{k}\,e^{i\vec{k}\cdot(\vec{x}-\vec{y})}=(2\pi)^3\delta^3(\vec{x}-\vec{y}) \tag{553}
$$

then we see requiring $\left[\hat{\phi}(t,x),\hat{\pi}(t,y)\right]=i\,\delta^3(\vec{x}-\vec{y})$ implies we take,

$$
|c_{\vec{k}}|^2=\frac{1}{2k(2\pi)^3} \tag{554}
$$

We may choose the $c_{\vec{k}}$ to be real, and then,

$$
c_{\vec{k}}=\frac{1}{\sqrt{2k(2\pi)^3}} \tag{555}
$$

Thus finally we see that our quantum field satisfies the correct equal time commutation relations, taking the mode functions to be,

$$
\delta_{\vec{k}}\phi(t)=\frac{1}{a(t)\sqrt{2k(2\pi)^3}}\,e^{+\frac{ik}{a(t)H}}\left(1+\frac{ia(t)H}{k}\right) \tag{556}
$$

Then we take the initial state of the scalar to be the usual vacuum state $|0\rangle$. This is called the **Bunch-Davies** vacuum.

Quantum Fluctuations

Now let us consider a 2-point function for our scalar in the Bunch-Davis vacuum;

$$\begin{aligned}\langle 0|\delta\hat{\phi}(t,x)\,\delta\hat{\phi}(t,y)|0\rangle &= \int d^3\vec{k}d^3\vec{q}\,\langle 0|\hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{q}}|0\rangle\,\delta_{\vec{k}}\phi e^{+i\vec{k}\cdot\vec{x}}\,\delta_{\vec{q}}\phi^\star e^{-i\vec{q}\cdot\vec{y}} \\ &= \int d^3\vec{k}d^3\vec{q}\,\langle 0|\hat{a}^\dagger_{\vec{q}}\hat{a}_{\vec{k}}+\left[\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{q}}\right]|0\rangle\,\delta_{\vec{k}}\phi e^{i\vec{k}\cdot\vec{x}}\,\delta_{\vec{q}}\phi^\star e^{-i\vec{q}\cdot\vec{y}}\end{aligned} \tag{557}$$

Now we use,

$$\langle 0|\hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{q}}|0\rangle = \langle 0|\hat{a}^\dagger_{\vec{q}}\hat{a}_{\vec{k}}+\left[\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{q}}\right]|0\rangle = \langle 0|\left[\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{q}}\right]|0\rangle \tag{558}$$

so,

$$\begin{aligned}\langle 0|\delta\hat{\phi}(t,x)\,\delta\hat{\phi}(t,y)|0\rangle &= \int d^3\vec{k}d^3\vec{q}\,\delta^3(\vec{k}-\vec{q})\,\delta_{\vec{k}}\phi e^{i\vec{k}\cdot\vec{x}}\,\delta_{\vec{q}}\phi^\star e^{-i\vec{q}\cdot\vec{y}} \\ &= \int d^3\vec{k}\,\left|\delta_{\vec{k}}\phi\right|^2 e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\end{aligned} \tag{559}$$

Hence we see that $\left|\delta_{\vec{k}}\phi\right|^2$ is the Fourier transform of the 2-point function $\langle 0|\delta\phi(t,x)\,\delta\phi(t,y)|0\rangle$. At early times, the modes are sub horizon and recall,

$$\delta_{\vec{k}}\phi \simeq \frac{1}{a\sqrt{2k(2\pi)^3}}e^{\frac{ik}{aH}} \qquad \Longrightarrow \qquad \left|\delta_{\vec{k}}\phi\right|^2 \simeq \frac{1}{2(2\pi)^3}\frac{1}{ka^2(t)} \tag{560}$$

and as in flat space the two-point function goes as,

$$\langle 0|\delta\phi(t,x)\,\delta\phi(t,y)|0\rangle \sim \frac{1}{a^2|\vec{x}-\vec{y}|^2} \tag{561}$$

However at late times when the modes are super horizon, recall,

$$\delta_{\vec{k}}\phi \simeq \frac{iH}{k\sqrt{2k(2\pi)^3}} \qquad \Longrightarrow \qquad \left|\delta_{\vec{k}}\phi\right|^2 \simeq \frac{1}{2(2\pi)^3}\frac{H^2}{k^3} \simeq \frac{4\pi G}{(2\pi)^3}\frac{V}{k^3} \tag{562}$$

corresponding to a real space behaviour of the 2-point function,

$$\langle 0|\delta\phi(t,x)\,\delta\phi(t,y)|0\rangle \quad \sim \quad H^2 \log|\vec{x}-\vec{y}| \tag{563}$$

Note that here the mode is constant in time - it doesn't evolve.

Generation of inhomogeneity

Thus towards the end of inflation the super horizon modes have fluctuations $< \delta\phi^2 > \sim H^2$. These lead to fluctuations in the time that inflation finishes at different spatial locations, and consequently the time reheating and the radiation era starts. We may give a simple picture using the so-called δN **approximation**.

Let us again assume a slow roll period of inflation with a very shallow linear potential so, $H_{inf}^2 = \frac{8\pi G}{3} V \simeq$ const, $V' \simeq$const, $V'' \simeq 0$, and an immediate transition to the radiation era, at time t_{rad}, so that the initial Hubble parameter in the radiation era is $H_{rad} \simeq H_{inf}$.

An approximation to the spatial variation in the reheat time $\delta t(x)$ is,

$$\delta t(x) \sim \frac{1}{\dot{\phi}} \delta\phi(x) \tag{564}$$

During inflation we assumed exactly flat FRW with $a \sim e^{H_{inf} t}$ and hence this time difference creates a change in the number of enfolds N from one location to another. Since $N \simeq \Delta t\, H_{inf}$, then if the end time changes,

$$\delta N(x) \sim H_{inf} \delta t(x) \sim \frac{H_{inf}}{\dot{\phi}} \delta\phi(x) \tag{565}$$

and implies a local variation in the number of e-folds, and hence also the scale factor,

$$\frac{\delta a(x)}{a} \sim \delta N(x) \tag{566}$$

at the end of inflation.

Now consider a specific solution to the flat FRW Friedmann equations for specific matter - such as in our radiation/matter/dark energy universe.

$$a(t) = f(t) \tag{567}$$

Then we can generate a family of solutions using spatial scale invariance;

$$a(t) = a_i f(t) \tag{568}$$

for exactly the same function f, and a_i a constant. Then perturbing a_i, one finds,

$$\frac{\delta a(t)}{a(t)} = \frac{\delta a_i}{a_i} = \text{constant} \tag{569}$$

On **super horizon scales** we can think of causally disconnected patches of FRW, each with the same matter, evolving approximately homogeneously. Thus the variation in $\delta a/a$ at the end of inflation on super horizon scales is actually preserved through to last scattering. The photons produced at last scattering have $T \sim 1/a$, and hence have temperature variations,

$$\begin{aligned} \left.\frac{\delta T(x)}{T}\right|_{lss} &\simeq -\left.\frac{\delta a(x)}{a}\right|_{lss} \\ &\simeq -\left.\frac{\delta a(x)}{a}\right|_{\text{end inflation}} \simeq -\delta N(x) \simeq -\frac{H_{inf}}{\dot{\phi}}\delta\phi(x) \\ &\sim \frac{H_{inf}^2}{V'}\delta\phi(x) \sim \frac{GV}{V'}\delta\phi(x) \end{aligned} \tag{570}$$

Recall that super horizon scales at last scattering correspond to around 1^o on the sky today. Thus we expect the above relation to hold on scales in the CMB corresponding to angles $> 1^o$. Thus on these scales we expect,

$$\left.\langle\frac{\delta T(x)}{T}\frac{\delta T(y)}{T}\rangle\right|_{lss,>1^o} \simeq \left(\frac{GV}{V'}\right)^2 \langle\delta\phi(x)\delta\phi(y)\rangle \tag{571}$$

Recall from above on super horizon scales we have,

$$\langle\delta\phi(x)\,\delta\phi(y)\rangle \sim \int d^3\vec{k}\,\frac{GV}{k^3}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \tag{572}$$

Now defining the Fourier transform,

$$\langle \frac{\delta T(x)}{T} \frac{\delta T(y)}{T} \rangle \Big|_{lss} = \int \frac{d^3\vec{k}}{k^3} \, \Delta^2_{(T)}(\vec{k}) \, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \tag{573}$$

then the quantity $\Delta^2_{(T)}$ is called the 'dimensionless power spectrum'. Note the extra factor of k^{-3} in the integral means it is indeed dimensionless.

Then on super horizon scales at last scattering inflation predicts,

$$\Delta^2_{(T)} \sim \frac{G^3 V^3}{V'^2} \tag{574}$$

where we note that the behaviour is constant in k. This is a 'so-called' scale invariant power spectrum.

The quantity $\Delta^2_{(T)}$ is measured directly from the CMB photons. On finds the fractional temperature fluctuation $\delta T/T \sim 10^{-5}$, so that,

$$\Delta^2_{(T)} \sim (10^{-5})^2 \sim 10^{-10} \tag{575}$$

The fact that this is related to V^3/V'^2 then constraints any inflationary model potential.

The matter era

Another longer story is to take the primordial inflationary fluctuations and transfer them forward to today. The fluctuations in the radiation reheating temperature give rise to fluctuations in energy density. Gravity then causes the amplification of these fluctuations when they reenter the our matter era horizon. The spectrum of these density fluctuations can then be compared to the observed matter power spectrum, and we see very good agreement. On cluster scales today, our linear approximation breaks down and we require a non-linear analysis.

End of course material

4.5 Extra material - not covered in course but included for interest

3D Power Spectra

Consider a real quantity $\delta(\vec{x})$, with Fourier transform,

$$\delta(\vec{x}) = \int d^3\vec{k}\, \delta_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}} \tag{576}$$

so $\delta_{-\vec{k}} = \delta^\star_{\vec{k}}$. Consider the two point function of this quantity, and assume that its expectation value is homogeneous and isotropic. These symmetries imply it must take the form,

$$\langle \delta(\vec{x})\delta(\vec{y}) \rangle = f(|\vec{x} - \vec{y}|) \tag{577}$$

but we have,

$$\langle \delta(\vec{x})\delta(\vec{y}) \rangle = \int d^3\vec{k} \int d^3\vec{k}' \langle \delta_{\vec{k}}\delta_{\vec{k}'} \rangle e^{-i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{y}} \tag{578}$$

and hence we see these symmetries imply,

$$\langle \delta_{\vec{k}}\delta_{\vec{k}'} \rangle = p(k)\delta^3(\vec{k} + \vec{k}') \tag{579}$$

for the 2-point function in Fourier space, where $k = |\vec{k}|$. Then,

$$\langle \delta(\vec{x})\delta(\vec{y}) \rangle = \int d^3\vec{k}\, p(k) e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \tag{580}$$

Now $p(k)$ is called the 3D power spectrum (and is the Fourier transform of f above. However note that $p(k)$ has dimensions of volume (as dkk^2 has inverse volume dimensions). Due to isotropy we may perform the angular Fourier integrals directly,

$$\begin{aligned} \sigma(r) = \langle \delta(\vec{x})\delta(\vec{x} + \vec{r}) \rangle &= \int d^3\vec{k}\, p(k)\, e^{-i\vec{k}\cdot\vec{r}} \\ &= \int_0^\infty dk\, 2\pi k^2\, p(k) \int_{-1}^{1} d(\cos\theta)\, e^{-ikr\cos\theta} \\ &= \int_0^\infty dk\, 4\pi k^2\, p(k)\, \frac{\sin kr}{kr} \end{aligned} \tag{581}$$

where $r = |\vec{r}|$. It is then convenient to define the dimensionless 3D power spectrum $\Delta^2(k)$ as,

$$\sigma(r) = \int_{-\infty}^{\infty} (d\log k)\, \Delta^2(k)\, \frac{\sin kr}{kr} \qquad (582)$$

so that,

$$\Delta^2(k) = 4\pi k^3\, p(k) \qquad (583)$$

Hence Δ^2 gives the power in an interval of log k.

3D temperature power spectrum
In the case of the temperature fluctuations, so $\delta = \delta T/T$, we have,

$$\Delta^2(k) \simeq \frac{G^3 V^3}{V'^2} \qquad (584)$$

which is **roughly constant**, as we have shown, on scales large enough that at last scattering they were causally disconnected - ie. $> 1^o$ on the sky today.

In terms of the dimensionless power spectrum $\Delta^2 = 4\pi k^3 p(k)$, then,

$$\begin{aligned} \langle \frac{\delta T}{T}(\vec{k}) \frac{\delta T}{T}(\vec{k}')\rangle &= p(k)\delta^3(\vec{k}+\vec{k}') \\ &= \frac{\Delta^2(k)}{4\pi k^3}\delta^3(\vec{k}+\vec{k}') \end{aligned} \qquad (585)$$

for the 2-point function in Fourier space, where $k = |\vec{k}|$, and in real space we have,

$$\begin{aligned} \langle \frac{\delta T}{T}(\vec{x}) \frac{\delta T}{T}(\vec{y})\rangle &= \int d^3\vec{k}\, p(k) e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \\ &= \int \frac{d^3\vec{k}}{4\pi k^3}\, \Delta^2(k) e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \end{aligned} \qquad (586)$$

Projection on the 2D sky

When we observe the temperature fluctuations of course we only observe them on the surface of last scattering - we cannot access the full 3-d function

$\delta T(x)$. Instead we observe, $\delta T(\hat{q})|_{lss}$ where $\hat{q}$ is a unit vector specifying the direction on the sky.

If we use spherical coordinates θ (polar), ϕ (azimuthal) on the sky, then it is natural to decompose a function on the sky in **spherical harmonics**, so,

$$\delta T(\hat{q})|_{lss} = \sum_{l,m} a_{lm} \, Y_{lm}(\hat{q}) \tag{587}$$

where we observe the a_{lm}.

Recall from QM that these are harmonic functions on a unit 2-sphere labelled by l, m which specify their representation under $SO(3)$. Recall that $0 \leq l$ and $m = -l, \ldots, +l$, and that,

$$\begin{aligned} Y_{lm} &\propto e^{im\phi} P_{lm}(\cos\theta) \\ P_{lm}(x) &= (-1)^m \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x) \end{aligned} \tag{588}$$

where $P_l(x)$ is a Legendre polynomial. Then

$$\begin{aligned} (1 - x^2) P''_{lm} - 2x P'_{lm} + \left(l(l+1) - \frac{m^2}{1 - x^2} \right) P_{lm} &= 0 \\ (1 - x^2) P''_l - 2x P'_l + l(l+1) P_l &= 0 \end{aligned} \tag{589}$$

Note that $P_l(x)$ is a polynomial of degree l. Slightly confusingly, $P_{lm}(x)$, are referred to as the associated Legendre polynomials, but are not generally polynomials as we can see from above!

We take the Y_{lm} to be normalised as,

$$\int Y_{lm} Y^\star_{l'm'} d\Omega = \delta_{ll'} \delta_{mm'} \tag{590}$$

where $d\Omega = \sin\theta d\theta d\phi$. Then one can write the inverse transform as,

$$a_{lm} = \int f(\theta, \phi) Y_{lm}(\theta, \phi)^\star d\Omega \tag{591}$$

Another important identity is,

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\hat{q}) Y_{lm}(\hat{q}')^\star \tag{592}$$

where θ is the angle between the two directions $\hat{q}$, $\hat{q}'$, so we might write $\cos\theta = \hat{q}\cdot\hat{q}'$ thinking of a dot product in the Euclidean space in which we embed the unit 2-sphere.

The question is what does inflation predict for these a_{lm}. Firstly by isotropy, it predicts $< a_{lm} >= 0$. Thus we have to look at a two point function in order to see something non-trivial.

Now isotropy dictates that,

$$\langle a_{lm} a_{l'm'} \rangle = \delta_{l\,,l'} \delta_{m\,,-m'} C_l \tag{593}$$

as then,

$$\begin{aligned}
\langle \delta T(\hat{q}) \delta T(\hat{q}') \rangle_{lss} &= \sum_{l,m} \sum_{l',m'} \langle a_{lm} a_{l'm'} \rangle Y_{lm}(\hat{q}) Y_{l'm'}(\hat{q}') \\
&= \sum_{l,m} C_l Y_{lm}(\hat{q}) Y_{l\,-m}(\hat{q}) \\
&= \sum_{l} \left(\frac{2l+1}{4\pi} \right) P_l(\hat{q}\cdot\hat{q}') C_l
\end{aligned} \tag{594}$$

is rotationally invariant. These coefficients C_l are measured directly by the CMB experiments.

Now we may take,

$$\delta T(\vec{x}) = \int d^3\vec{k}\, \delta_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} \tag{595}$$

and consider $\vec{k} = k\hat{k}$ and $\vec{x} = r\hat{n}$ where $\hat{k}$ and $\hat{n}$ are unit vectors. Recall from above we defined,

$$\begin{aligned}
\frac{1}{T^2} \langle \delta_{\vec{k}} \delta_{\vec{k}'} \rangle &= p(k) \delta^3(\vec{k} + \vec{k}') \\
&= \frac{\Delta^2(k)}{4\pi k^3} \delta^3(\vec{k} + \vec{k}')
\end{aligned} \tag{596}$$

Then to proceed we use the identity,

$$e^{i\vec{k}\cdot\vec{x}} = \sum_l (2l+1) i^l P_l(\hat{k}\cdot\hat{n}) j_l(kr) \tag{597}$$

where j_l is a spherical Bessel function to obtain,

$$\delta T(r,\hat{n}) \;=\; \sum_l (2l+1) i^l \int d^3\vec{k}\, \delta_{\vec{k}}\, P_l(\hat{k}\cdot\hat{n}) j_l(k\,r) \tag{598}$$

Now we compute,

$$\begin{aligned}
\langle \delta T(r,\hat{n})\delta T(r,\hat{n}')\rangle &= \sum_{l,l'}(2l+1)(2l'+1)i^{l+l'}\int d^3\vec{k}d^3\vec{k}'\langle \delta_{\vec{k}}\,\delta_{\vec{k}'}\rangle P_l(\hat{k}\cdot\hat{n})P_{l'}(\hat{k}'\cdot\hat{n}')j_l(k\,r)j_{l'}(k'\,r)\\
&= \sum_{l,l'}(2l+1)(2l'+1)i^{l+l'}\int d^3\vec{k}\,\left(T^2 p(k)\right)\, P_l(\hat{k}\cdot\hat{n})P_{l'}(-\hat{k}\cdot\hat{n}')j_l(k\,r)j_{l'}(k\,r)\\
&= T^2\sum_{l,l'}(2l+1)(2l'+1)i^{l+l'}\int_0^\infty dk\int d\Omega_{\hat{k}}\, P_l(\hat{k}\cdot\hat{n})P_{l'}(-\hat{k}\cdot\hat{n}')\, k^2\, p(k) j_l(k\,r)j_{l'}(k\,r)
\end{aligned}$$

and using the orthogonality of P_l Legendre polynomials,

$$\int d\Omega_{\hat{q}} P_l(\hat{n}\cdot\hat{q})P_{l'}(\hat{n}'\cdot\hat{q}) = \left(\frac{4\pi}{2l+1}\right)\delta_{ll'}P_l(\hat{n}\cdot\hat{n}') \tag{599}$$

then,

$$\begin{aligned}
\langle \delta T(r,\hat{n})\delta T(r,\hat{n}')\rangle &= T^2\sum_l 4\pi(2l+1)i^{2l}P_l(-\hat{n}\cdot\hat{n}')\int_0^\infty dk\, k^2\, p(k)\,(j_l(k\,r))^2\\
&= T^2\sum_l 4\pi(2l+1)(-1)^l P_l(-\hat{n}\cdot\hat{n}')\int_0^\infty dk\, k^2\, p(k)\,(j_l(k\,r))^2
\end{aligned} \tag{600}$$

and since $(-1)^l P_l(-x) = P_l(x)$, and,

$$\begin{aligned}
\langle \delta T(r,\hat{n})\delta T(r,\hat{n}')\rangle &= T^2\sum_l 4\pi(2l+1)P_l(\hat{n}\cdot\hat{n}')\int_0^\infty dk\, k^2\, p(k)\,(j_l(k\,r))^2\\
&= \sum_l\left(\frac{2l+1}{4\pi}\right)P_l(\hat{n}\cdot\hat{n}')C_l
\end{aligned} \tag{601}$$

Hence we see,

$$\begin{aligned}
C_l &= (4\pi)^2T^2\int_0^\infty dk\, k^2\, p(k)\,(j_l(k\,r))^2\\
&= (4\pi)T^2\int_0^\infty \frac{dk}{k}\,\Delta^2(k)\,(j_l(k\,r))^2
\end{aligned} \tag{602}$$

Now we have seen that $\Delta^2(k) \simeq G^3V^3/V'^2 \simeq$const. Hence our inflation model predicts,

$$\begin{aligned}\frac{1}{T^2}C_l &= (4\pi)\frac{G^3V^3}{V'^2}\int_0^\infty \frac{dk}{k}\,(j_l(k\,r))^2 \\ &= (4\pi)\frac{G^3V^3}{V'^2}\int_0^\infty \frac{dx}{x}\,(j_l(x))^2 \\ &= \frac{(2\pi)}{l(l+1)}\frac{G^3V^3}{V'^2} \end{aligned} \tag{603}$$

using,

$$\int_0^\infty \frac{dx}{x}\,(j_l(x))^2 = \frac{1}{2l(l+1)} \tag{604}$$

on scales $> 1^o$ where the last scattering surface can be treated as causally disconnected.

This is a very good approximation to what is seen. For $l < 50$ the large scales of the CMB show that,

$$l(l+1)C_l \sim \text{constant} \tag{605}$$

This part of the CMB is termed the **Sachs-Wolfe** plateau. Experimentally the average CMB temperature $T \simeq 2.725K$ and one finds,

$$l(l+1)C_l \sim \left(10^{-5}T\right)^2 \tag{606}$$

These large scale anisotropies were first measured by the COBE satellite in 1992, and later refined by COBE, and the later WMAP and PLANCK satellites.