3 The radiation era

We will now apply the ideas above to discuss the radiation era in more detail.

Epoch: $10^{10}K < T < 10^{12}K$

Let us start be considering the radiation era at temperatures $10^{10}K < T < 10^{12}K$ ($10^{10}K = 1MeV$, $10^{12}K = 100MeV$) where we have a thermal mix of $\gamma, e, \bar{e}, \nu, \bar{\nu}$ and the baryons n, p. For all the other particles we are far below their mass scale, and so unless they have left a relic (eg. dark matter), there will be effectively none.

Whilst we are also below the baryon mass scale, the conserved baryon number guarantees we still have these left over. However, their number is tiny compared to the number of relativistic species ($\sim 10^-8$). Since we will now discuss the evolution of the scale factor, we will ignore the contribution of these baryons.

Thus we consider a thermal mix of $\gamma, e, \bar{e}, \nu, \bar{\nu}$, all of which are relativistic. Thus we will have,

$$\begin{aligned} \rho &= \left(\frac{1}{2} \times 2 + \frac{7}{16} (2 \times 2 + 6 \times 1) \right) aT^4 \\ &= \frac{1}{2} g_{eff} aT^4 \end{aligned} \quad (369)$$

where the effective number of (bosonic) degrees of freedom g_{eff} is,

$$g_{eff} = \frac{43}{4} \quad (370)$$

given by 2 photon spins, the e and $\bar{e}$ each have 2 spins, and there are 3 generations of neutrinos and anti-neutrinos, but each have only one d.o.f. as they are left handed.

Thus (ignoring the baryons and dark matter) we have,

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho = \frac{43}{4} \times \frac{1}{2} \times \frac{8\pi G}{3} a_R T^4 \qquad 10^{10}K < kT < 10^{13}K \\ &\sim 0.5sec \left(\frac{T}{10^{10}K} \right)^2 \end{aligned} \quad (371)$$

where the radiation constant $a_R = \pi^2 k^4/15$ in units $\hbar = c = 1$. Recall the temperatures dependence on scale factor is $a \sim 1/T \sim t^{1/2}$.

Neutrino decoupling: $T \sim 10^{10}K$

Above $T \sim 10^{10}K$ the neutrinos are held in equilibrium only by the weak processes; $\nu + e \to \nu + e$. Since $kT > m_e$ then as we discussed before the Fermi approximation implies the weak interaction rate for the neutrinos is,

$$\Gamma \sim G_W^2 (kT)^5 \sim sec^{-1} \left(\frac{T}{10^{10}K} \right)^5 \tag{372}$$

recalling that $G_W \sim 10^{-5}/(GeV^2)$. Hence, since $H \sim 0.5sec\,(T/10^{10}K)^2$, then,

$$\frac{\Gamma}{H} \sim \left(\frac{T}{10^{10}K} \right)^3 \gg 1 \qquad \text{for} \quad T > 10^{10}K \tag{373}$$

However at $T \sim m_e \sim 10^{10}K$ the interaction rate drops to $\Gamma/H \sim 1$, and the neutrinos quickly decouple from the thermal mix of photons and electrons.

For $T < 10^{10}K$ the neutrinos free stream. As we previously discussed they subsequently maintain a fermi distribution, with an effective temperature $T_\nu \sim 1/a$.

Electron annhilation: $10^8K < T < 10^{10}K$

Just below $10^{10}K$ we have a thermal mix of electrons and photons, together with decoupled free streaming neutrinos. Unlike the baryons (which largely annihilated), these relic neutrinos are still important when considering the energy density. Thus we have,

$$\begin{aligned} \rho &= \left(\frac{1}{2} \times 2 + \frac{7}{16} (2 \times 2) \right) aT^4 + \frac{7}{16} (6 \times 1)\, aT_\nu^4 \quad T \sim 10^{10}K \\ &= \frac{1}{2} g_{eff} a T^4 \qquad g_{eff} = \frac{43}{4} \end{aligned} \tag{374}$$

with $T_\nu \simeq T$ since they have only just decoupled. For lower temperatures since $m_e \sim 512keV$, so $kT \sim m_e$ implies $T \sim 10^9K$, the electrons and anti-electrons become non-relativistic, their number density drops exponentially; $n_e, n_{\bar{e}} \sim (mkT)^{3/2} e^{-m/kT} \to 0$. By around 10^8K they are largely gone.

In fact since the universe is charge neutral and there are baryons carrying positive charge, there is a potential that means a small number of electrons are left over. However, as for the baryons this number is very small. If we consider their effect on the scale factor evolution it is negligible. However, their presence is very important as the remaining charged baryons and electrons are crucial in maintaining thermal equilibrium for the photons.

Thus for $T \sim 10^8 K$ from the point of view of the scale factor we can ignore the electrons, and have a thermal mix of photons temperature T, together with free streaming neutrinos, effective temperature T_ν. Thus we have,

$$\rho = \frac{1}{2} \times 2aT^4 + \frac{7}{16}(6 \times 1)\, aT_\nu^4 \quad T \sim 10^8 K \tag{375}$$

Now the electrons and photons interact very strongly via EM interactions which maintain equilibrium. Hence the annihilation occurs as an adiabatic process, and thus the entropy per comoving volume for the electron/photon mixture, sa^3 =constant. Recall that $s_\gamma = 4/3 \times aT^3$, and $s_{e,\bar{e}} = 7/3 \times aT^3$.

Furthermore recall that $T_\nu \sim 1/a$ and that $T \sim 10^{10} K$ the neutrinos are just stopping interacting, and hence $T_\nu = T$ then. Thus,

$$sa^3 = \text{const} \quad \Longrightarrow \quad \frac{s}{T_\nu^3} = \text{const} \tag{376}$$

Hence at $T \sim 10^{10} K$ we have,

$$\begin{aligned} \frac{s}{T_\nu^3} &\simeq \frac{s_\gamma + s_{e,\bar{e}}}{T_\nu^3} \\ &= \frac{4}{3}\left(1 + \frac{7}{4}\right) a_R \left(\frac{T}{T_\nu}\right)^3 = \frac{4}{3}\left(1 + \frac{7}{4}\right) a_R \quad T \sim 10^{10} K \end{aligned} \tag{377}$$

as $T \simeq T_\nu$. However, after the electrons have annihilated at $T \sim 10^8 K$ we have,

$$\frac{s}{T_\nu^3} \simeq \frac{s_\gamma}{T_\nu^3} = \frac{4}{3} a_R \left(\frac{T}{T_\nu}\right)^3 \quad T \sim 10^8 K \tag{378}$$

and thus equating these we find for $T \sim 10^8 K$;

$$\left(\frac{T}{T_\nu}\right)^3 \simeq 1 + \frac{7}{4} = \frac{11}{4} \tag{379}$$

However, both the photon temperature and neutrino temperature subsequently go as $T_\gamma \sim 1/a$ and $T_\nu \sim 1/a$ and so this ratio is maintained for ever after,

$$\frac{T_\gamma}{T_\nu} \simeq \left(\frac{11}{4}\right)^{\frac{1}{3}} \sim 1.4 \tag{380}$$

Today we know the photon temperature (which is the CMB temperature) is $2.7K$. Hence the neutrino temperature is $T_\nu \sim 1.9K$.

Note that while $T_\nu \sim 1/a$, over the period $10^8 K < T < 10^{10} K$, the photon temperature T does not go precisely as $1/a$. In fact as we have seen, the photons are 1.4 times hotter than if they had done so. The reason is due to the annihilation of the $e, \bar{e}$. As this occurred in equilibrium, the electrons transferred their entropy to the photons, and this injection of entropy heated the photons up.

Similarly the evolution of the scale factor does not go precisely as $H \sim T^2$ (and hence $a \sim t^{1/2}$) over this period. Recall above $T > 10^{10} K$ we have,

$$\rho = \frac{1}{2} g_{eff}\, a_R T^4 \qquad g_{eff} = \frac{43}{4} \sim 10.75 \tag{381}$$

but now below $T < 10^8 K$ we have,

$$\begin{aligned} \rho &= \frac{1}{2} \times 2 a_R T^4 + \frac{7}{16}\,(6 \times 1)\, a_R T_\nu^4 \\ &\simeq \frac{1}{2} \times 2 a_R T^4 + \frac{7}{16}\,(6 \times 1)\, a_R \left(\frac{T_\nu}{T}\right)^4 T^4 \\ &\simeq \frac{1}{2} g_{eff} a_R T^4 \end{aligned} \tag{382}$$

with,

$$g_{eff} = 2 + \frac{7 \times 3}{4} \times \left(\frac{4}{11}\right)^{\frac{4}{3}} \simeq 3.36 \tag{383}$$

Thus the Hubble rate, $H \sim \sqrt{g_{eff}} T^2$ drops by an addition factor $\sqrt{3.36/10.75} \sim 0.5$ during this temperature period, beyond its usual T^2 dependence.

Last phase of equilibruim: $10^5K < T < 10^8K$

The photons remain in equilibrium due to the small number of electrons. Consider Compton scattering $e + \gamma \to e + \gamma$. For $kT \ll m_e$ this reduces to Thomson scattering, where the electrons are non-relativistic. Consider the rate for an electron to interact, Γ_e, and for a photon to interact, Γ_γ. Following our previous discussion this rate goes as,

$$\Gamma_e = \frac{\alpha^2}{m_e^2} n_\gamma \;, \quad \Gamma_\gamma = \frac{\alpha^2}{m_e^2} n_e \tag{384}$$

where $\alpha \sim 10^{-2}$. Note that there are also interactions with the protons, but since these go as α^2/m_p^2 these are much suppressed. Recall that $n_\gamma \sim 10^8 n_e$, and due to charge conservation and baryon conservation,

$$n_e \simeq n_B = n_{B,0} \left(\frac{T_\gamma}{T_{\gamma,0}} \right)^3 \tag{385}$$

assuming $T_\gamma \sim 1/a$ (which we will see it does) and $T_{\gamma,0} \sim 2.7K$. Then, using $n_B \sim 1m^{-3}$,

$$\Gamma_\gamma = \frac{\alpha^2}{m_e^2} n_{B,0} \left(\frac{T_\gamma}{T_{\gamma,0}} \right)^3 \simeq 10^{-20} sec^{-1} \left(\frac{T_\gamma}{T_{\gamma,0}} \right)^3 \tag{386}$$

Likewise,

$$H = \sqrt{\frac{8\pi G}{3} a_R T^2} \sim 10^{-20} sec^{-1} \left(\frac{T_\gamma}{T_{\gamma,0}} \right)^2 \tag{387}$$

Hence we see that naively we have $\Gamma/H \ll 1$ right down to CMB temperatures - assuming that the electron charges remain exposed (which they don't as they combine to neutral atoms).

In fact this simple estimate is naive, as Thomson scattering is elastic. Recall, an elastic process does not change the particle energy. Thus whilst there would be strong scattering of photons by electrons (if they were exposed), in fact this would not maintain photons in thermal equilibrium as they could not scatter between momentum shells, only within them.

In fact, the rate of scattering between energy shells is in fact reduced by a factor kT/m_e. Take the initial state electron rest frame, so $E_e = m_e$, with the photon with momentum p. Consider the final state where the election has energy E', momentum k and the photon has momentum p'. Then $k = p' - p$, and,

$$E' \simeq m_e + \frac{k^2}{2m_e} \quad \implies \quad E' - E \sim \frac{p^2}{m_e} \tag{388}$$

and the initial photon momentum $p \sim kT$. Hence in each interaction, rather than having a change in electron energy $\sim kT$, instead we have only a change $(kT)(kT/m_e)$, which is down by a factor of kT/m_e.

We may more accurately estimate when equilibrium is maintained, as when $\Gamma^\star/H \gg 1$ where $\Gamma^\star$ is not the rate of interactions, but rather the rate at a particles energy is changed by kT. Hence, here, we may estimate,

$$\Gamma^\star \simeq \Gamma \frac{kT}{m_e} \tag{389}$$

Then,

$$\Gamma^\star_\gamma \simeq 10^{-30} sec^{-1} \left(\frac{T_\gamma}{T_{\gamma,0}} \right)^4 \tag{390}$$

Then $\Gamma^\star_\gamma/H \sim 1$ at $T \sim 10^5 K$.

Thus we conclude that thermal equilibrium for the photons and electrons is actually maintained despite down to $10^5 K$ despite the scattering being highly elastic.

Note that $T \sim 10^5 K$ is still in the radiation domination. The transition to matter domination occurs at $Z \sim 3600$ or $T \sim 10^4 K$. Thus from $T \sim 10^8 K$ down to $T \sim 10^5 K$ we have a perfect radiation universe, with $T \sim 1/a$.

Below $10^5 K$ temperature there remains strong elastic scattering, but it is not sufficient to maintain equilibrium. In fact the charges neutralise each other around $10^3 K$ anyway and then the photons free stream. But even before that, at $T \sim 10^5 K$ the collision term in the Boltzmann equation effectively vanishes due to the elastic scattering. From that point on, the photons maintain an effective bose distribution with $T \sim 1/a$, even though the universe enters the matter era around $T \sim 10^4 K$.

3.1 Earlier epochs; $T > 10^{12} K$

We have now accounted for the thermal behaviour of the universe from $10^{13} K$ down to the CMB temperature today. At even earlier times the picture was similar. The universe is well described by a radiation model, where,

$$\rho = \frac{1}{2} g_{eff}(T) a_R T^4 \ , \quad s = \frac{4\rho}{3T} = \frac{2}{3} g_{eff}(T) a_R T^3 \tag{391}$$

and we may take $g_{eff}(T)$ to be constant in time, except at transition temperatures T_i, when it jumps as a massive species satisfies $kT_i \sim m_i$ and hence goes from being relativistic and having $n \sim T^3$ to being exponentially suppressed so $n \sim e^{-m/kT}$.

In the approximation that these species remain in equilibrium as they disappear, we may imagine that they quickly inject their entropy into the remaining species. Thus suppose, $g_{eff} = g_1$ for $T > T_c$ to $g_{eff} = g_2$ for $T \ll T_c$. Further suppose that,

$$\frac{T}{T_c} = \frac{a_c}{a} \tag{392}$$

for $T > T_c$. Then, conserving entropy we have,

$$s a^3 = \text{const} \qquad \Longrightarrow \qquad g_{eff} (aT)^3 = \text{const} \tag{393}$$

Hence for $T \ll T_c$ we have,

$$g_2 (aT)^3 = g_1 (a_c T_c)^3 \qquad \Longrightarrow \qquad \frac{T}{T_c} = \left(\frac{g_1}{g_2} \right)^{\frac{1}{3}} \frac{a_c}{a} > \frac{a_c}{a} \tag{394}$$

Thus after the transition, for a given size a the universe is hotter than it would have been if the transition did not occur, as fewer particles carry the same total entropy.

We see that the photon temperature has always gone as $T \sim 1/a$, even during the matter era, since the photons had by then decoupled. Thus we may relate the CMB temperature and scale factor today a_0 to the temperature T and scale factor a in some early epoch as,

$$\frac{T_{CMB}}{T} = \left(\frac{g_0}{g(T)} \right)^{\frac{1}{3}} \frac{a}{a_0} \tag{395}$$

where $g_0 \sim 3.36$ today, and $g(T)$ is the effective number of degrees of freedom at temperature T.

For $T \gg 10^{15} K$ (ie. $kT \sim 100 GeV$), then we should include all the SM fields and I believe one gets $g_{eff} \sim 301/4$.

3.2 WIMP dark matter

During this early radiation era, we may find that (like the neutrinos) instead of a species going to zero density below its mass scale, instead it decouples due to weakening interactions and leaves a relic density behind. This is the idea behind dark matter.

An important point is that dark matter should not decouple when it is still relativistic. That would lead to a huge amount of matter that would subsequently dominate the density of the universe as soon as the temperature reached its mass scale. Instead the matter should almost entirely annihilate, but just leave a tiny amount left over. This then is irrelevant during the radiation era in terms of energy density, but becomes important in the later matter era.

Weakly interacting massive particle (WIMP) dark matter is just such a model. Consider the dark matter to be a particle X with antiparticle $\bar{X}$, mass M, which annihilates in a 2-body tree level process as in the above example. Take at early times the X are in thermal equilibrium and then consider relics in the model. Consider the case $n = n_X = n_{\bar{X}}$ (ie. no potential, and interactions that preserve the particle number). Then the Boltzmann equation is;

$$\frac{d \ln N}{d \ln T} = \frac{\Gamma}{H} \left(1 - \frac{N_{eq}^2}{N^2} \right) \ , \quad \Gamma = < \sigma v > n \tag{396}$$

In order for us not to produce much too much dark energy, the interaction should turn off below the particle mass scale, so that the particles have largely annihilated. Then we may treat the X particles as non-relativistic.

In our scalar field model, in the non-relativistic limit we computed a two body decay and found,

$$\langle \sigma v \rangle = \frac{\lambda^2}{16\pi M^2} \tag{397}$$

However, let us leave this cross section free for now.

The equilibrium density is,

$$n_{eq} = \left(\frac{MkT}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{M}{kT}} \tag{398}$$

and we see the important exponential suppression due to being below the mass scale. This is crucial for having a viable model.

The freeze out temperature occurs at $\Gamma/H \sim 1$. In the radiation era we have,

$$H = \sqrt{\frac{8\pi G}{3}\rho_{rad}} \sim \sqrt{Ga}T^2 \sim sec^{-1} \left(\frac{T}{10^{10}K} \right)^2 \tag{399}$$

We could be more accurate and include a factor $g_{eff}(T)$ but we shall leave that out.

Define $x = kT_{freeze}/M$, with $x \ll 1$ so that the X are non-relativistic. Then,

$$\langle \sigma v \rangle M^3 x^{\frac{3}{2}} e^{-\frac{1}{x}} \sim \sqrt{Ga}T^2_{freeze} = \sqrt{Ga} \left(\frac{M}{k} \right)^2 x^2 \tag{400}$$

Now $[\langle \sigma v \rangle] = -2$ so $[M\langle \sigma v \rangle] = -1$. Then,

$$x^{-\frac{1}{2}} e^{-\frac{1}{x}} \sim \frac{\sqrt{Ga}}{k^2} \frac{1}{M\langle \sigma v \rangle} \sim 10^{-17} \left(\frac{100GeV}{M} \right) \left(\frac{1}{\langle \sigma v \rangle \times (100GeV)^2} \right) \tag{401}$$

and recall we want $x \ll 1$.

The relic abundance approximately given as,

$$n_0 = n_{freeze} \left(\frac{a_{freeze}}{a_0} \right)^3 \simeq n_{freeze} \left(\frac{T_{CMB}}{T_{freeze}} \right)^3 \tag{402}$$

using $a \sim 1/T$. We could do better by including the number of effective degrees of freedom today and at the temperature T, but it would make little difference. Hence,

$$\rho_0 = M n_{freeze} \left(\frac{T_{CMB}}{T_{freeze}} \right)^3 \quad , \quad \Omega_X = \frac{\rho_0}{\rho_{crit}} = \frac{8\pi G \rho_0}{3H_o^2} \tag{403}$$

We also have,

$$n_{freeze} \sim M^3 x^{\frac{3}{2}} e^{-\frac{1}{x}} \quad , \quad T_{freeze} = \frac{Mx}{k} \tag{404}$$

Then we want $\Omega_X \sim 1$ today (more precisely $\Omega_X \sim 0.26$).
Using the above (and dropping the $8\pi/3$ factor) we find,

$$x^{-\frac{3}{2}} e^{-\frac{1}{x}} \sim \frac{1}{M} \frac{H_0^2}{G\left(T_{CMB} k\right)^3} \sim 10^{-11} \left(\frac{100GeV}{M} \right) \tag{405}$$

Thus we have equation (401) from the freeze out condition $\Gamma/H \sim 1$, and equation (405) from obtaining the correct density today. Thus combining these we have;

$$x^{-1} \sim 10^6 \left(\langle \sigma v \rangle \times (100GeV)^2 \right) \quad , \quad x^{-\frac{3}{2}} e^{-\frac{1}{x}} \sim 10^{-11} \left(\frac{100GeV}{M} \right) \tag{406}$$

and we recall we require $x \ll 1$ for consistency. These are the constraints on our dark matter model. Hence we see we have a one parameter family of models.

Suppose we take the mass and cross section to be typical of that of the weak scale $M_W \sim 100GeV$ and weak interactions. Consider, $M \sim M_W$, then a typical weak cross section would give,

$$\langle \sigma v \rangle \sim \frac{\alpha_W^2}{16\pi M_W^2} \sim \frac{10^{-5}}{(100GeV)^2} \tag{407}$$

Now consider the conditions from our model;

$$x^{-\frac{3}{2}}e^{-\frac{1}{x}} \sim 10^{-10} \quad \Longrightarrow \quad x \simeq 0.038 \tag{408}$$

(consistent with $x \ll 1$) and then

$$\langle \sigma v \rangle \sim \frac{1}{x}\frac{10^{-6}}{(100GeV)^2} \sim \frac{10^{-5}}{(100GeV)^2} \tag{409}$$

We see this is precisely of the correct order. This is the so called 'WIMP miracle'. It is very suggestive that dark matter is simply another particle that interacts only via the weak interactions (like the neutrinos) with a mass around the weak scale. This could arise from supersymmetric models as the lightest stable supersymmetric partner - e.g. this can be a neutralino (mix of partners to the Z and photon and Higgs), a sneutrino (neutrino partner) or a gravitino (spin 3/2 partner to graviton).

3.3 Nucleosynthesis

The neutron/proton mass $\sim 100MeV$ corresponds to around $\sim 10^{12}K$. Above this temperature these and their antiparticles are abundant. However, below it, we only have the matter neutrons and protons, recalling that baryon number is conserved and there must be some process in the very early universe to create a small matter/anti matter asymmetry.

The mass difference between n and p is small,

$$Q = m_n - m_p = 1.29MeV \tag{410}$$

but creates a small potential favouring protons. Above this energy scale, for $T > 10^{10}K$, we expect the number of neutrons and protons to be equal, but around $10^{10}K$ it will play a role in determining the equilibrium number of the relic n and p. The relevant weak interactions between n and p are;

$$\begin{aligned} n + \nu &\rightarrow p + e \\ n + \bar{e} &\rightarrow p + \bar{\nu} \\ n &\rightarrow p + e + \bar{\nu} \end{aligned} \tag{411}$$

and their reverse reactions.

So below $T \ll 10^{12}K$ the n and p are non-relativistic particles, with distributions (units $\hbar = c = 1$),

$$n_i = g_i \left(\frac{m_i kT}{2\pi} \right)^{\frac{3}{2}} e^{\frac{\mu_i - m_i}{kT}} \quad , \qquad i = n, p \tag{412}$$

Now both particles have $g_i = 2$ as they are spin 1/2.

Consider a comoving volume with numbers $N_n = a^3 n_n$ and $N_p = a^3 n_p$, baryon conservation implies $N_n + N_p$ =constant.

Consider these particles in equilibrium, and consider a change only in the relative number of n to p (keeping everything else fixed - S, V). For this 'chemical equilibrium' we have,

$$dE = \mu_n dN_n + \mu_p dN_p \tag{413}$$

but we must have $dE = 0$ for an equilibrium processes. Now since for every weak process we have $\Delta N_n = -\Delta N_p$, then this implies the chemical potentials of n and p are equal;

$$\mu_n = \mu_p \equiv \mu \tag{414}$$

Note in principle the charge of the proton could affect this, but such a correction would be irrelevent.

Now we may eliminate the unknown μ as,

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p} \right)^{\frac{3}{2}} e^{\frac{m_p - m_n}{kT}} \simeq e^{-\frac{Q}{kT}} \tag{415}$$

recalling that we are at temperatures $Q \ll m_n, m_p$.

Now define the total nucleon density,

$$n_N = n_n + n_p \tag{416}$$

so that $a^3 n_N$ =const, and the fractions,

$$X_n = \frac{n_n}{n_N} \ , \quad X_p = \frac{n_p}{n_N} \tag{417}$$

and then we arrive at the 'Saha' equation of chemical equilibrium;

$$\frac{X_n}{X_p} = \frac{X_n}{1 - X_n} = e^{-\frac{Q}{kT}} \tag{418}$$

which simply implies;

$$X_n = \frac{1}{1 + e^{\frac{Q}{kT}}} \tag{419}$$

Thus for $T \gg 10^{11}K$ this is very close to $X_n \sim 0.5$, and for $T \ll 10^8 K$ $X_n \to 0$.

However, the equilibrium is maintained by the 2 body weak processes $n + \nu \to p + e$ and $n + \bar{e} \to p + \bar{\nu}$. Similarly to our discussion of other weak cross sections, this goes as,

$$\Gamma \simeq G_W^2 \, (kT)^5 \sim sec^{-1} \left(\frac{T}{10^{10}K} \right)^5 \tag{420}$$

One might be confused as to why the neutron-proton mass is not important. The point is that the most relevant scale controlling the process is that of the W and Z particles that mediate the process - and these are already incorporated in the Fermi constant $G_W \sim 10^{-5}/(GeV)^2$.

As discussed before, we have,

$$H \sim sec^{-1} \left(\frac{T}{10^{10}K} \right)^2 \tag{421}$$

at this point in the radiation era. Hence we find that the interactions become weaker with decreasing temperature, with $\Gamma/H \sim 1$ at $T_{freeze} \sim 10^{10}K$. After this point the n and p no longer interact, and n_n and n_p are separately conserved. Thus,

$$X_n(T) = X_n(T_{freeze}) \quad , \qquad T < T_{freeze} \tag{422}$$

An accurate calculation gives $T_{freeze} \simeq 0.3 \times 10^{10}K$, leading to,

$$X_n(T) = 0.17 \quad , \qquad T < 0.3 \times 10^{10}K \tag{423}$$

below freezout.

In fact this isn't quite accurate, as along with the 2 body interactions there is also the 1 body decay process $n \to p + e + \bar{\nu}$. The rate for this is given by the neutron lifetime, $\Gamma_n \sim (1000sec)^{-1}$.

Recall that the temperature $T \sim 0.3 \times 10^{10}K$ is reached after around 1 second. If neutrons simply persisted for the next ~ 1000 seconds then they would all be gone. However, in fact the process of nucleosynthesis occurs shortly after this time, before further neutrons can decay. This is the process by which the unstable neutrons are assembled into stable nuclei.

The details of the process are complicated, involving various nuclear reactions (see Weinberg). However, 2 body processes are the only relevant ones, as densities are low. Hence the first step in the reaction involves deuterium $d = H^2$, as,

$$p + n \to d + \gamma \tag{424}$$

which has a large Γ/H and thus maintains an equilibrium distribution for deuterium. However, the abundance of deuterium is highly suppressed until a lower temperature $T \sim 10^9 K$, as determined by a Saha type equation.

Setting the total baryon density $n_B = n_n + n_p + 2n_d$, then this is conserved. These are all non-relativistic species so,

$$n_i = g_i \left(\frac{m_i k T}{2\pi} \right)^{\frac{3}{2}} e^{\frac{\mu_i - m_i}{kT}} \quad , \qquad i = n, p, d \tag{425}$$

where now $g_n = g_p = 2$ as before, and for deuterium $g_d = 3$ (spin-1). Chemical equilibrium implies that,

$$\mu_n + \mu_p = \mu_d \tag{426}$$

and the binding energy liberated by the forward process is,

$$B = m_n + m_p - m_d = 2.2 MeV \tag{427}$$

Define the fractions,

$$X_i = \frac{n_i}{n_B} \ , \quad i = n, p, d \tag{428}$$

then one finds the Saha equation (see ex sheet 3, qu 3),

$$\frac{X_d}{X_n X_p} = \frac{n_d}{n_n n_p} n_B = \frac{3}{4} \left(\frac{m_d}{m_n m_p} \right)^{\frac{3}{2}} \left(\frac{2\pi}{kT} \right)^{\frac{3}{2}} e^{\frac{B}{kT}} n_B \tag{429}$$

and using $m_p \simeq m_n \simeq m_d/2$, and,

$$n_B \sim n_{B,0} \left(\frac{T}{T_{CMB}} \right)^3 \tag{430}$$

with $n_{B,0} \sim 1m^{-3}$, then,

$$X_d \simeq \frac{3}{4}\left(\frac{4\pi T}{kT_{CMB}^2 m_p}\right)^{\frac{3}{2}} e^{\frac{B}{kT}} n_{B,0} X_n X_p \tag{431}$$

So for $T \gg B, m_p$ then $X_d \sim T^{3/2} \ll 1$. However, the coefficient is very small since the number of baryons is very small. Hence even at the scale $T \sim 10^{12}K$ one finds $X_d \sim 10^{-10}$. However, as soon as kT decreases to the scale B then X_d starts increasing rapidly with further cooling.

One finds for $X_n, X_p \sim O(1)$, then $X_d \ll 1$ until T drops to $T \simeq 0.8 \times 10^9 K$.

Repeating the same Saha calculation for the other heavier nuclei, one finds that in fact they would be abundant at higher temperatures. However, since they can't form except through suppressed multi particle interactions, they have to wait until d becomes abundant and then they are assembles from other quick 2 body processes; Firstly, the relatively unstable H^3 and He^3 form (both $B \sim 8MeV$),

$$d + d \to H^3 + p\ , \quad d + d \to He^3 + n \tag{432}$$

but then assemble into the preferred stable He^4 (with $B \sim 30MeV$) via,

$$d + H^3 \to He^4 + n\ , \quad d + He^3 \to He^4 + p \tag{433}$$

For all these processes the rates will go as $\Gamma \propto n_d$, and so above $T > 10^9 K$ when $X_d \simeq 0$ these processes are very slow. Below $10^9 K$ when d can form then these reactions occur rapidly to convert the d into the more relatively stable He^4.

However the process largely stops there. Whilst there are many stable heavier nuclei, it is difficult to proceed further since only 2-body processes work, and there are no stable nuclei at atomic number 5 ($H + He^4$) or 8 ($He^4 + He^4$).

There are some small amounts of heavier elements produced, and in the end the nuclear rates become too weak, leaving a relic 'ash'. In the end to a good approximation the n and p are bound up into just H^1 and He^4. Since we have seen initially $X_n = n_n/n_B = 0.17$ and $X_p = n_p/n_B = 1 - 0.17 = 0.83$ at around $T \sim 10^9 K$. From these we have,

$$n_n = 2n_{He}\ , \quad n_p = n_H + 2n_{He} \tag{434}$$

and so in the end we expect,

$$n_{He} = \frac{1}{2} \times 0.17 n_B \ , \quad n_H = (1 - 2 \times 0.17)\, n_B \tag{435}$$

so that,

$$\frac{n_H}{n_B} \sim 0.66 \ , \quad \frac{n_{He}}{n_B} \sim 0.08 \tag{436}$$

For the ash one finds very small quantities,

$$\begin{aligned} \frac{n_d}{n_B} &\sim 10^{-3} \\ \frac{n_{He^3}}{n_B} &\sim 10^{-4} \\ \frac{n_{Li^7}}{n_B} &\sim 10^{-9} \\ &\ldots \end{aligned} \tag{437}$$

(Note that any H^3 produced later beta decays to He^3). Whilst these are small, they can be accurately predicted using a full simulation of the nuclear processes, and these can be compared to the observed abundances in primordial gas clouds. Detailed calculations show the results agree very well. Since nucleosynthesis is very sensitive to the initial fraction of X_n and X_p, which depends on when the $n \to p$ weak interaction becomes weak, it is a sensitive probe of H around the temperature $T \sim 10^9 K$, which in turn depends on the number of relativistic degrees of freedom.

Adding a single extra light particle affects the results of nucleosynthesis so that they no longer agree with the data. Likewise, modifying gravity in some way so that H behaves differently around $T \sim 10^9 K$ also ruins the agreement of nucleosynthesis. Hence it is a very powerful constraint on any modifications of the standard model and big bang picture.

3.4 Recombination

Consider now $T < 10^8 K$ and the equilibrium process,

$$p + e \to H \tag{438}$$

forming neutral atoms. Now charge neutrality and baryon conservation implies,

$$n_p = n_e \quad , \qquad a^3 n_B = \text{const} \tag{439}$$

and from nucleosynthesis we have,

$$(1-2\times 0.17)\, n_B = 0.66 n_B = n_H + n_p \tag{440}$$

The binding energy of the neutral H atom is,

$$Q = m_p + m_e - m_H = 13.6 eV \tag{441}$$

Now naively $kT \sim Q$ implies $T \sim 10^5 K$, and so one might expect neutral atoms to form at this temperature. However, in fact the process is suppressed until lower temperatures.

Consider the Saha equation. Chemical equilibrium implies,

$$\mu_p + \mu_e = \mu_H \tag{442}$$

and hence,

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e}\left(\frac{m_H}{m_p m_e}\right)^{\frac{3}{2}}\left(\frac{2\pi}{kT}\right)^{\frac{3}{2}} e^{\frac{Q}{kT}} \tag{443}$$

Now $g_p = g_e = 2$ as they are spin-1/2. The hyperfine splitting between the spin-0 and spin-1 H_{1s} ground states can be ignored so $g_H = 1 + 3 = 4$. Now since $m_p \sim m_H$ then,

$$\frac{n_H}{(n_p)^2} \simeq \left(\frac{2\pi}{m_e kT}\right)^{\frac{3}{2}} e^{\frac{Q}{kT}} \tag{444}$$

Now defining the fractions;

$$X_p = \frac{n_p}{n_p + n_H}\ , \quad X_H = \frac{n_H}{n_p + n_H}\ , \quad n_n + n_H \simeq 0.66 n_B \tag{445}$$

one obtains,

$$\frac{X_p^2}{1-X_p} \simeq \frac{1}{0.66 n_B}\left(\frac{m_e kT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{Q}{kT}} \tag{446}$$

Using $n_B = n_{B,0}\left(T/T_{CMB}\right)^3$, thus we find;

$$\begin{aligned} X_p &\simeq 1\ , \quad T > 4000K \\ X_p &\simeq 0\ , \quad T < 3000K \end{aligned} \tag{447}$$

An important point; this temperature is below the scale at which the photons are in equilibrium with the charges - recall Thomson scattering is inefficient below $\sim 10^5 K$ - there is much scattering, but it is elastic. Thus the naive process,

$$p + e \to H_{1s} + \gamma \tag{448}$$

is not efficient at these temperatures. The resulting photon, whilst rapidly scattering, only looses energy slowly (due to inefficient thermalization and then redshifting).

For small fractions of H_{1s} equilibrium is initially maintained. However, as the universe cools and the fraction of H_{1s} quickly starts to increase, the number density of the emitted photons produced is sufficiently high that they reionize any more H_{1s} produced before their energy is diffused or redshifted away. The process $p + e \to H_{1s} + \gamma$ can then no longer maintain equilibrium effectively.

In addition to the ground state, H_{1s}, excited states of neutral hydrogen such as $H_{2s}, H_{2p} \ldots$ also form. Since these excited states have a binding energy $Q < Q_{1s} = 13.6eV$ the Saha analysis shows that in equilibrium their abundance is suppressed relative to H_{1s}. Around temperature scales $> 2000K$ their abundance is sufficiently small that they can maintain equilibrium via the process $p + e \to H^\star + \gamma$, as the resulting γ's are not too numerous to reionize the small number of excited H.

So then the question is how does appreciable neutral hydrogen form? In fact the process of recombination proceeds by a somewhat complicated non-equilibrium route once any appreciable H_{1s} has formed. The most effective channel becomes the decay,

$$H_{2s} \to H_{1s} + \gamma + \gamma \tag{449}$$

where the small equilibrium abundance of excited H_{2s} decays to H_{1s}. The key points are that;

- when the spin of the H_{2s} and H_{1s} states are the same then a single photon (spin-1) cannot mediate a decay to the H_{1s} ground state, hence such transitions can only occur using 2 photons.

- each photon then separately carries less energy than is required to excite H_{1s} to any excited state.

This more complicated process is rather inefficient, as it involves the slower 2 photon decay. Also the direct production of H_{2s} via $p + e \rightarrow H_{2s} + \gamma$ also shuts off after too many reactions since there are too many γ's around to immediately reionize any H_{2s} produced. Instead the H_{2s} may form from other excited states, which then loose energy through a cascade of photons, finally arriving at H_{2s}.

Thus the process of H_{1s} does proceed, but mainly via H_{2s} and too slowly to maintain the Saha equilibrium value above. A full out-of-equilibrium Boltzmann analysis is required. Such analysis shows X_p decreases rather more slowly than the equilibrium Saha prediction past $T \sim 3500K$, but still becomes very small by $T \sim 2500K$.

A Saha analysis for He^4 shows that it becomes neutral at a slightly higher temperature - see example sheet 3. Therefore the universe becomes locally charge neutral at $T \sim 3000K$, corresponding to a redshift $Z \sim 1100$. The 2-sphere in the universe at redshift $Z \sim 1100$ which produces photons that just reach us today - the CMB photons we detect - is called the *surface of last scattering*. However in a more sophisticated analysis, this is not treated as a sharp surface, but rather has a thickness in redshift due to the finite time it takes to go from ionised to neutral H.

An important point is that recall $Z_{eq} \sim 3600$ for matter-radiation equality. Hence the last scattering actually occurred in the early stages of the matter era, not in the radiation era. However, since the photons had decoupled, $T_\gamma \sim 1/a$.

3.5 Baryogenesis

Recall today the number density of photons is $n_\gamma \sim 4 \times 10^8 m^{-3}$ and baryons $\sim 0.25 m^{-3}$, but there are no antibaryons. Hence for every baryon we have $\sim 2 \times 10^9$ photons. However in the very early universe in equilibrium we would expect equal numbers of baryons and anti-baryons. Likewise we observe charge neutrality, with the baryon charge being balances by an excess of electrons.

One of the big questions is what created this very small matter/anti matter asymmetry in the very early universe, hence leading to small non-zero B and L, at a level of approximately 1 part in 10^9, but maintaining charge neutrality.

The simplest idea is that at early times we have equality of matter/anti matter, but baryon/lepton number was not conserved by some high energy process, whilst charge was, allowing the conversion $\bar{p} \to e$.

Note that since B and L are global charges, there is no obstruction to having these broken by some high energy processes. In fact anomalies may even do this. However, charge is associated to gauge symmetry, so one would not expect charge asymmetry to be allowed (compatible with observation).

Example: In SM baryon and lepton number are conserved in perturbation theory, but non-perturbative effects (spharlerons) violate this. In fact only $B - L$ (and obviously charge) are conserved. However, these SM processes are not believed to be strong enough to produce the desired asymmetry.

Sakharov conditions:

- $\Delta B \neq 0$ reactions must exist.
- C and CP violating reactions.
- Must be out of equilibrium

The first condition is obvious. For the second, observe that a state with equal numbers of particles and antiparticles in each momentum and spin state is invariant under both C and CP. Hence to have an asymmetry both must be broken. Note, only breaking C one could have an asymmetry between left handed baryons and anti baryons, but if one doesn't break CP this would be compensated by a reverse asymmetry between the right handed versions.

This last condition is less obvious. If baryon number B is not conserved, there is no chemical potential for it, and the number density can only depend on mass. However, CPT is not broken and implies the mass of particles and antiparticles are equal. Hence in equilibrium the number of particles and antiparticles must be equal.

Example: GUT baryogenesis $U(1) \times SU(2) \times SU(3) \subset SU(5)$; heavy $\sim 10^{16} GeV$ leptoquark X, charge +4/3, $B - L = +2/3$. Decays to SM via;

$$\begin{aligned} X &\to \bar{e} + \bar{d} && \text{branching ratio } = r \\ X &\to u + u && \text{branching ratio } = (1 - r) \end{aligned} \tag{450}$$

Note these preserve $B-L$ and charge. CPT requires also the processes to exist,

$$\begin{aligned} &\bar{X} \rightarrow e + d && \text{branching ratio } = \bar{r} \\ &\bar{X} \rightarrow \bar{u} + \bar{u} && \text{branching ratio } = (1-\bar{r}) \end{aligned} \tag{451}$$

but the branching ratio $\bar{r}$ is not constrained by CPT. In fact $\bar{r} \neq r$ implies only C and CP are broken.

At early times there is equilibrium with equal numbers of X and $\bar{X}$, hence $B = L = 0$. Then the process freezes out. This must happen for $kT \ll m_X$ so that there is no reverse reaction. The relic X and $\bar{X}$ then later decay, and depending on the branching ratios $r, \bar{r}$, then produce $B, L \neq 0$ matter excess, but with net $B-L$ and charge being zero.

Let the initial densities of $X, \bar{X}$ at freeze out be $n_X|_{freeze}$. Then after decay one would obtain a matter excesses;

$$n_e = n_d = (\bar{r} - r)\, n_X|_{freeze} \ , \quad n_u = n_e + n_d \tag{452}$$

so we would require $\bar{r} > r$.

Note the weak processes then allow interchange $u + e \rightarrow d + \nu$, so that today $n_e \neq n_d$.