2 Thermal physics

2.1 Relativistic Statistical Mechanics

Let us now consider relativistic stat mech in Minkowski spacetime, in canonical Minkowski coordinates \( x^\mu = (t, x^i) \). We will consider an ideal gas, formed by identical massless or massive point particles whose collisions are instantaneous.

Then for a gas made up of particles, each with 4-momentum \( p^{\mu}_{(n)} = (E_{(n)}, p^i_{(n)}) \), with trajectory \( x^{\mu}_{(n)}(t) \), so,

\[
p^{\mu}_{(n)} = \frac{dx^{\mu}_{(n)}}{d\lambda_{(n)}}
\]

for the particle’s affine parameter \( \lambda_{(n)} \). Note in the massive case, with mass \( m \), we would have,

\[
p^{\mu}_{(n)} = m \frac{dx^{\mu}_{(n)}}{d\tau_{(n)}}
\]

and hence the affine parameter is related to the particles proper time (also an affine parameter) as \( \lambda_{(n)} = \tau_{(n)}/m \).

We may express this momentum in terms of the Minkowski time \( t \),

\[
p^\mu_{(n)} = \frac{dt}{d\lambda_{(n)}} \frac{dx^\mu_{(n)}}{dt} = p^0_{(n)} \frac{dx^\mu_{(n)}}{dt} = E_{(n)} \frac{dx^\mu_{(n)}}{dt}
\]

The stress tensor for this set of massless or massive particles is,

\[
T^{\mu\nu} = \sum_n \delta^{(3)}(x^i_{(n)}(t) - x^i) \frac{1}{E_{(n)}} p^\mu_{(n)} p^\nu_{(n)}
\]

For example, the energy is,

\[
T^{tt} = \sum_n \delta^{(3)}(x^i_{(n)}(t) - x^i) \frac{1}{E_{(n)}} p^t_{(n)} p^t_{(n)} = \sum_n \delta^{(3)}(x^i_{(n)}(t) - x^i) E_{(n)}
\]
as one would expect.

**Brief aside:** Let us briefly motivate the form of this discrete stress tensor. This is easiest to understand in the massive case. Recall for dust the stress tensor is:

\[ T^{\mu \nu} = \rho u^\mu u^\nu \quad \text{(207)} \]

where \( \rho \) is the rest mass density. Then a Lorentz-invariant discrete version of this is,

\[
T^{\mu \nu}(x^\mu) = \sum_n \int d\tau(n) \delta^{(4)}(x^\alpha_{(n)}(\tau(n)) - x^\alpha) \ m \ u^\mu_{(n)} u^\nu_{(n)} \\
= \sum_n \int d\tau(n) \delta^{(4)}(x^\alpha_{(n)}(\tau(n)) - x^\alpha) \ \frac{1}{m} \ p^\mu_{(n)} p^\nu_{(n)} \quad \text{(208)}
\]

where \( u^\mu_{(n)} = dx^\mu_{(n)} / d\tau(n) \) is the 4-velocity of the particle, mass \( m \), so that \( p^\mu_{(n)} = mu^\mu_{(n)} \). Then we have \( dt / d\tau(n) = u^0_{(n)} = p^0_{(n)}/m = E_{(n)}/m \).

Note the important points; the continuous form \( \rho u^\mu u^\nu \) for dust gives us the term \( m \ u^\mu_{(n)} u^\nu_{(n)} \), and the delta function makes this discrete. Furthermore it is a spacetime delta function, and hence is Lorentz invariant.

However, we may simply integrate over the particle’s timelike world line, using,

\[
\int d\tau(n) \delta^{(4)}(x^\alpha_{(n)}(\tau(n)) - x^\alpha) = \int dt(n) \delta(t(n)(\tau(n)) - t) \ \delta^{(3)}(x^i_{(n)}(\tau(n)) - x^i) \\
= \int dt(n) \frac{d\tau(n)}{dt(n)} \delta(t(n)(\tau(n)) - t) \ \delta^{(3)}(x^i_{(n)}(\tau(n)) - x^i) \\
= \int dt(n) \delta(t(n) - t) \ m \ E_{(n)} \delta^{(3)}(x^i_{(n)}(t(n)) - x^i) \\
= \frac{m}{E_{(n)}} \delta^{(3)}(x^i_{(n)}(t) - x^i) \quad \text{(209)}
\]
Thus we find,

\[ T^{\mu\nu}(x^\mu) = \sum_n \int d\tau(n) \, \delta^{(4)} \left( x^\alpha_{(n)}(\tau(n)) - x^\alpha \right) \frac{1}{m} p^\mu_{(n)} p'^\nu_{(n)} \]

\[ = \sum_n \delta^{(3)} \left( x^i_{(n)}(t) - x^i \right) \frac{1}{E_{(n)}} p^\mu_{(n)} p'^\nu_{(n)} \]  

(210)

as given above. Note that now the delta function is only spatial. This would not be Lorentz invariant were it not for the inverse energy factor. This is analogous to the invariant measure \( d^3x/(2E) \) that we use in QFT. **End of aside.**

We introduce a phase space density \( n(t, x^i, p_j) \) where, \( n(t, x^i, p_j) d^3x d^3p \) gives the number of particles in phase space volume \( d^3x, d^3p \). Then we may write,

\[ \int_V d^3x \sum_n \delta^{(3)} \left( x^i_{(n)}(t) - x^i \right) \simeq \int_V d^3x \int_{-\infty}^{\infty} d^3p \, n(t, x^i, p_j) \]  

(211)

where we have,

\[ n(t, x^i, p_j) = \sum_n \delta^{(3)} \left( x^i_{(n)}(t) - x^i \right) \delta^{(3)} \left( p_{(n)j}(t) - p_j \right) \]  

(212)

Now consider a large number of particles. We then approximate the phase space density as a smooth function rather than a distribution. Then we specialise to an equilibrium setting. Now time independence implies \( n(t, x^i, p_j) = n(x^i, p_j) \) for our averaged phase space distribution (although not for the original distributional phase space density!).

We define the number density in real space \( n \) which is extracted from the phase space density as,

\[ n(x^i) \simeq \int_{-\infty}^{\infty} d^3p \, n(x^i, p_j) \]  

(213)

Now consider a homogeneous isotropic distribution which is parameterized by temperature \( T \), chemical potential \( \mu \) etc... In this case homogeneity
implies no \( x^i \) dependence in the real space density or phase space density, so that \( n(x^i, p_j) = n(p_j) \) only. Then,

\[
n(T, \mu, \ldots) \simeq \int_{-\infty}^{\infty} d^3 p \ n(p^i; T, \mu, \ldots) \quad (214)
\]

Isotropy further implies;

\[
n(T, \mu, \ldots) \simeq \int_{0}^{\infty} dp \ 4\pi p^2 \ n(p; T, \mu, \ldots) \quad (215)
\]

where \( p^2 = \delta_{ij} p^i p^j \) and \( 4\pi p^2 n(p) dp \) gives the number density in the momentum shell \( dp \).

Then in our averaged large particle approximation we have;

\[
T_{tt} = \sum_n \delta^{(3)} (x^i_{(n)}(t) - x^i) \ E_{(n)} \simeq \int_{0}^{\infty} dp \ 4\pi p^2 n(p) E_p \quad (216)
\]

where,

\[
E_p^2 = p^2 + m^2
\]

Isotropy implies \( T_{ti} = 0 \) and \( T_{ij} \propto \delta_{ij} \). Consider,

\[
\sum_n \delta^{(3)} (x^i_{(n)}(t) - x^i) \ p^i_{(n)} p^j_{(n)} \simeq \int d^3 p \ n(|p|) p^i p^j
\]

\[
= \delta_{ij} \int dp \ 4\pi p^2 n(p) f(p) \quad (218)
\]

where we must deduce \( f(p) \). Tracing we find;

\[
\sum_n \delta^{(3)} (x^i_{(n)}(t) - x^i) \ p^i_{(n)} p^j_{(n)} \delta_{ij} = \sum_n \delta^{(3)} (x^i_{(n)}(t) - x^i) |p_{(n)}|^2
\]

\[
= \int dp \ 4\pi p^2 n(p) p^2 \quad (219)
\]

Hence \( f(p) = p^2 / 3 \). Then,

\[
T^{ij} = \sum_n \delta^{(3)} (x^i_{(n)}(t) - x^i) \frac{1}{E_{(n)}} p^i_{(n)} p^j_{(n)} \simeq \delta^{ij} \int_{0}^{\infty} dp \ 4\pi p^2 n(p) \frac{p^2}{3E_p} \quad (220)
\]
Thus we find the energy density and pressure;

\[
T^{ti} = \rho = \int_0^\infty dp 4\pi p^2 n(p) E_p \\
T^{ti} = 0 \\
T^{ij} = P\delta^{ij} = \delta^{ij} \int_0^\infty dp 4\pi p^2 n(p) \frac{p^2}{3E_p}
\]

(221)

Massless case: \(E_p = p\) and hence

\[
P = \rho/3
\]

(222)

Note this is totally independent of the actual distribution \(n(p)\). Thus homogeneous isotropic massless particles behave as an ideal fluid with \(w = 1/3\), independent of whether they are hot or how they are interacting.

**Thermal equilibrium**

Consider the particles to have \(g\) internal degrees of freedom (eg. spins). Then at temperature \(T\) and chemical potential \(\mu\);

\[
n(p; T, \mu) = \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{\frac{E_p - \mu}{kT}} + 1}
\]

(223)

with + for fermions and − for bosons.

**Massless/relativitistic limit**

Suppose we have \(m = 0\) and \(\mu = 0\). Alternatively consider a massive field with potential, but at very high temperatures so that, \(m, \mu \ll kT\). Then we can approximate \(E_p \simeq p, \mu \simeq 0\). Then,

\[
n(T) = \int_0^\infty dp 4\pi p^2 n(p; T) = \frac{g}{(2\pi\hbar)^3} \int_0^\infty dp \frac{4\pi p^2}{e^{\frac{E_p}{kT}} + 1}
\]

\[
= \frac{g (kT)^3}{2\pi^2\hbar^3} \int_0^\infty dx \frac{x^2}{e^x + 1}
\]

(224)

Now we may evaluate;

\[
\int_0^\infty dx \frac{x^2}{e^x + 1} = \frac{7}{4} \zeta(3) \quad \text{ie.} \quad \frac{3}{2} \zeta(3), \ 2\zeta(3)
\]

(225)
where \(\zeta(3) = 1.202\ldots\) is the Riemann zeta function \((\zeta(s) = \sum_{n=1}^{\infty} n^{-s})\). Now we define the radiation constant,

\[
a = \frac{\pi^2 k^4}{15\hbar^3 c^3}
\]

and then,

\[
\begin{align*}
n_{\text{fermion}(-)}(T) &= \frac{3}{4} \times \frac{15\zeta(3)}{\pi^4} \frac{aT^3}{k} \\
n_{\text{boson}(+)}(T) &= \frac{15\zeta(3)}{\pi^4} \frac{aT^3}{k}
\end{align*}
\]

We may evaluate the energy density as;

\[
\rho(T) = \int_0^{\infty} dp 4\pi p^2 E_p \rho(p; T) = \frac{g}{(2\pi\hbar)^3} \int_0^{\infty} dp \frac{4\pi p^3}{e^{p/kT} + 1}
\]

\[
= \frac{g (kT)^4}{2\pi^2 \hbar^3} \int_0^{\infty} dx \frac{x^3}{e^x \pm 1}
\]

We may evaluate;

\[
\int_0^{\infty} dx \frac{x^3}{e^x \pm 1} = \frac{15 + 1}{240} \pi^4 \quad \text{ie.} \quad \frac{7}{8} \times \frac{1}{15} \pi^4, \frac{1}{15} \pi^4
\]

Then,

\[
\begin{align*}
\rho_{\text{fermion}(-)}(T) &= \frac{7}{8} \times \frac{1}{2} gaT^4 \\
\rho_{\text{boson}(+)}(T) &= \frac{1}{2} gaT^4
\end{align*}
\]

Recall that \(P = \rho/3\). This is the statistical mechanics description of our radiation/hot matter \(w = 1/3\).

Note a fermion species contributes 4/3 to the number density and 7/8 to the energy density and pressure compared to a boson.

**Examples:**
Consider photons so that \(g = 2\). Then,

\[
\rho_\gamma(T) = aT^4
\]
and hence the definition of the radiation constant $a$.

Consider electrons and antielectrons, so now $g = 2 \times 2$ (particle + antiparticle, each with two spins). Then.

$$\rho_{\text{el}}(T) = \frac{7}{4} a T^4 \quad (232)$$

**Non-relativistic limit**

Suppose $E_p - \mu \gg kT > 0$, hence one can only excite low momentum modes. Then,

$$n(p; T, \mu) = \frac{g}{(2\pi \hbar)^3} \frac{1}{e^{\frac{E_p-\mu}{kT}} \pm 1} \simeq \frac{g}{(2\pi \hbar)^3} e^{E_p-\mu} \quad (233)$$

Now since,

$$E_p = m + \frac{p^2}{2m} + \ldots \quad (234)$$

and hence our condition implies $m - \mu \gg kT$. Then,

$$n \simeq \frac{g}{(2\pi \hbar)^3} e^{\frac{\mu-m}{kT}} e^{-\frac{p^2}{2mkT}} \quad (235)$$

Now the number density may be evaluated;

$$n = \int_0^\infty dp 4\pi p^2 n(p) \simeq g \left(\frac{kmT}{2\pi \hbar^2}\right)^{\frac{3}{2}} e^{\frac{\mu-m}{kT}} \quad (236)$$

[Note that for a non-relativistic species rather than having a vanishing chemical potential, we would have a constant potential $\mu = m$. In this case we recover the usual expression we are used to from non-relativistic stat mech,

$$n_{\text{non-rel}} = g \left(\frac{kmT}{2\pi \hbar^2}\right)^{\frac{3}{2}} \quad (237)$$]
Continuing then,

\[ \rho = \int_0^\infty dp \, 4\pi p^2 n(p) E_p \simeq \left( m + \frac{3}{2}kT + \ldots \right) n \]

\[ P = \int_0^\infty dp \, 4\pi p^2 n(p) \frac{p^2}{3E_p} \simeq (kT + \ldots) n \]

and hence we see,

\[ P \ll \rho \quad (239) \]

This is our statistical mechanics description of cold matter \( w = 0 \). Thus by cold we really mean \( m - \mu \gg kT \).

### 2.2 Thermodynamics

We start with the first law;

\[ dE = TdS - pdV + \mu dN \quad (240) \]

where we have included a conserved total particle number \( N \). However in a closed system such as the universe or a box \( N \) =constant, and so \( dN = 0 \). Hence,

\[ dE = TdS - pdV \quad (241) \]

These quantities \( E, S \) depend on \( T, V \) and \( N \) in general. Note that if there is no conserved number or \( \mu = 0 \) then they depend only on \( T \) and \( V \).

We may define intensive quantities, the number density \( n \) using the conserved particle number, and energy and entropy density \( \rho, s \) using volume, as,

\[ N = nV, \quad E = \rho V \quad (242) \]

Then,

\[ d(\rho V) = Td(sV) - pdV \quad (243) \]
and so,

\[ d\rho - Td_s = \frac{dV}{V} (Ts - \rho - p) = \frac{dn}{n} (\rho + p - Ts) \]  

(244)

In general since \( \rho, P, s \) are intensive they are functions only of \( T \) and \( n = N/V \). (ie. not \( T, V, N \) separately).

**Special case:** if either \( \mu = 0 \) or there is no conserved particle number, then we have \( \rho, P, s \) being only functions of temperature \( T \) (not on \( n \) or equivalently \( V \)). Then their independence on \( n \) from above implies the coefficient of \( dn/n \) must vanish so,

\[ Ts = \rho + p \]  

(245)

**Massless case:** Consider the massless case (eg. photons or a real scalar). Recall \( \rho = \rho(T) \) and \( P = \rho/3 \). Hence there is no chemical potential - the quantities only depend on \( T \). Then,

\[ s = \frac{\rho + p}{T} = \frac{4}{3} \frac{\rho}{T} \]  

(246)

Hence,

\[ s_{\text{boson}} = \frac{2}{3} \frac{g a T^3}{\rho}, \quad s_{\text{fermion}} = \frac{7}{12} \frac{g a T^3}{\rho} \]  

(247)

[ Note: one can check \( d\rho = Tds \).]

**Non-relativistic case:** Consider the non-relativistic case with conserved particle number and chemical potential. Recall;

\[ \rho(T, n) = \left( m + \frac{3}{2} kT \right) n, \quad P(T, n) = kTn \]  

(248)

Thus we see there is a chemical potential since these quantities are functions of \( T \) and \( n \). Then,

\[ d\rho = \left( m + \frac{3}{2} kT + \ldots \right) dn + \frac{3}{2} kndT \]  

(249)
So,

\[ d\rho - Tds = \frac{dn}{n} (Ts - \rho - p) \]
\[ (m + \frac{3}{2}kT)dn + \frac{3}{2}knTds - Tds = dn \left( (m + \frac{3}{2}kT) + kT + \frac{Ts}{n} \right) \]

so dividing by \(-nT\),

\[ \frac{ds}{n} - \frac{s}{n^2}dn = \frac{3k}{2} \frac{dT}{T} - k \frac{dn}{n} \implies d \left( \frac{s}{n} \right) = d \left( k \log \left( \frac{T^{3/2}}{n} \right) \right) \]

which implies,

\[ \frac{s}{n} = C + k \log \left( \frac{T^{3/2}}{n} \right) \]

for a constant \(C\). Recall that,

\[ n = g \left( \frac{kmT}{2\pi\hbar^2} \right) \frac{3}{2} e^{\frac{-m}{kT}} \]

and hence,

\[ \frac{s}{n} = \text{const} + \frac{m - \mu(T)}{T} \]

Note that as \(T \to 0, \mu \to m\). In the case of constant potential \(\mu = m\) then we have,

\[ \frac{s}{n} = \text{const} \implies n \sim T^{3/2} \]

This implies that (on average) the entropy per particle is constant.

**Adiabatic expansion**

Consider an FRW universe or box with side length \(a(t)\), which contains a homogeneous isotropic ideal gas. Then suppose the universe or box expands adiabatically, ie. sufficiently slowly so that thermal equilibrium is approximately maintained and no entropy is produced. For a box or closed universe
this implies the total entropy $S =$constant. Now since $S = sV$ and $V \sim a^3$ this implies,

$$sa^3 = \text{const} \tag{256}$$

For an open or flat universe one might be concerned that the total volume is infinite. However adiabaticity in this case implies the entropy in a comoving volume is constant. Again this implies $sa^3 =$constant.

**Massless case:** Consider the massless case with no chemical potential. Then we had $s \sim T^3$. Thus we deduce,

$$T \sim \frac{1}{a} \tag{257}$$

Naively this may be understood as $kT \sim p$, and momentum redshifts as $p \sim 1/a$ so $T \sim 1/a$. Note that this implies;

$$\rho \sim T^4 \sim \frac{1}{a^4} \tag{258}$$

which precisely agrees with our perfect fluid with $w = 1/3$.

**Non-relativistic case:** Consider the non-relativistic limit with conserved particle number $N$. Now we have $na^3 =$constant (again either viewed as $N =$constant, or the number in the comoving volume is constant. Hence we have,

$$\text{const} = \frac{s}{n} = C + k \log \left( \frac{T^{3/2}}{n} \right) \quad \Longrightarrow \quad T \sim n^{2/3} \sim \frac{1}{a^2} \tag{259}$$

and also,

$$\text{const} = \frac{s}{n} = \text{const} + \frac{m - \mu(T)}{T} \quad \Longrightarrow \quad \mu = m = \text{const} \tag{260}$$

Naively we might think $kT \sim \frac{p^2}{2m}$ and we know momentum redshifts as $p \sim 1/a$ so $T \sim 1/a^2$.

**Application:** photon-baryon mix Consider a mixture of photons and baryons in a adiabatically expanding universe. (This partly models our universe at temperatures $T \ll 10^{13}K$, although omits the leptons and neutrinos.)
The total entropy density \( s_{\text{tot}} = s_\gamma + s_B \) and \( s_{\text{tot}}a^3 \) = constant. Then since baryon number is conserved we also have \( n_B a^3 \) = constant. This implies that the total entropy per baryon is constant. We define the dimensionless quantity,

\[
\sigma = \frac{s_{\text{tot}}}{k n_B} = \text{const}
\]  

(261)

(261)
to measure this.

Now we use,

\[
s_\gamma = \frac{4}{3} aT^3, \quad \frac{s_B}{n_B} = \text{const} + k \log \left( \frac{T^{3/2}}{C n_B} \right)
\]

(262)

for a constant \( C \), and hence,

\[
\sigma = \frac{4}{3} \frac{aT^3}{k n_B} + \log \left( \frac{T^{3/2}}{C n_B} \right)
\]

(263)

Recall that, \( n_\gamma = \frac{30\zeta(3)aT^3}{\pi^4 k} \), then,

\[
\text{const} = \sigma = \frac{4\pi^4}{3 \times 30\zeta(3)} \left( \frac{n_\gamma}{n_B} \right) + \log \left( \frac{T^{3/2}}{C n_B} \right)
\]

(264)

This equation governs the dependence of \( n_B \) on \( n_\gamma \) and \( T \).

Note that there are two interesting limits. Firstly \( n_\gamma \gg n_B \) as in our universe. In this case, due to the logarithm, this implies,

\[
\frac{n_\gamma}{n_B} \simeq \text{const}
\]

(265)

and hence (since \( n_B a^3 \) = const and \( n_\gamma \sim T^3 \)),

\[
a^3T^3 \simeq \text{const}
\]

(266)

so we recover the behaviour as if there were only photons, namely \( T \sim 1/a \).

Conversely if \( n_\gamma \ll n_B \) (not the case in our universe) then the argument of the log, \( T^{3/2}/n_B \), must be approximately constant, and we recover \( T \sim 1/a^2 \),
the behaviour of baryons.

**Phonon radiation in our universe:** Consider $\Omega_m \simeq 0.32$, $\Omega_B \simeq 0.05$, $\Omega_\gamma \ll 1$ and $h \simeq 0.67$. Then consider the relic CMB photons. These have a thermal distribution (although they are not in equilibrium) with a temperature today $T_{CMB} = 2.725K$. Thus we have,

$$\rho_\gamma = \frac{3H_0^2}{8\pi G} \Omega_\gamma = aT^4 \implies \Omega_{\text{gamma}} \simeq 5.5 \times 10^{-5} \quad (267)$$

Note that the total radiation today is due to these photons together with relic neutrinos, which we will see later have a similar fraction $\Omega_{\nu\bar{\nu}}$. In fact as we will show later,

$$\Omega_r = \Omega_\gamma + \Omega_{\nu\bar{\nu}} = \left(1 + 3 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{\frac{4}{3}}\right)\Omega_\gamma \simeq 1.68 \Omega_\gamma \quad (268)$$

Hence the total radiation fraction $\Omega_r \ll 1$ as we assumed earlier.

Note that while a negligible fraction of the energy density is in radiation today, in fact the photons vastly outnumber the baryons. We have,

$$n_\gamma \simeq \frac{30\zeta(3) aT^3}{\pi^4} \frac{k}{m} \sim 4 \times 10^8 m^{-3}$$

$$n_B \simeq \frac{\rho_B}{m_{\text{proton}}} = \frac{3H_0^2}{8\pi G m_{\text{proton}}} \Omega_B \sim 0.25 m^{-3} \quad (269)$$

and hence $n_\gamma \gg n_B$.

An important epoch was **radiation-matter equality**, the time when the energy density in radiation and matter were equal. Consider our $\Lambda$CDM model. Then,

$$\rho_{\text{tot}} = \rho_{\text{crit}} \left(\Omega_\Lambda + \Omega_m (1 + Z)^3 + \Omega_r (1 + Z)^4\right) \quad (270)$$

Then radiation-matter equality occurred at a redshift $Z_{eq}$ when,

$$\Omega_m \simeq \Omega_r (1 + Z_{eq}) \implies Z_{eq} \simeq 3500 \quad (271)$$
Note that since the redshift is large we can consistently ignore the cosmological constant contribution which will be totally subdominant.

As we discuss later in more detail, the photon temperature redshifts as $T \sim 1/a$. Hence the temperature at radiation-matter equality was,

$$T_{eq} \simeq T_{CMB} (1 + Z_{eq}) \sim 10^4 K$$ \hfill (272)

### 2.3 Boltzmann equation in general

We may always find coordinates so that we can write the metric as,

$$ds^2 = -dt^2 + g_{ij}(t, x) dx^i dx^j$$ \hfill (273)

where $g_{ij}$ is the metric on a (spatial - ie. Riemannian) manifold $S$. As before we assume we can describe a large number of identical particles by a phase space density $n(t, x^i, p_j)$. Formally the phase space is $T^*S$. Note that the phase space momentum is a covector field, ie. has $p_j$ with its index down stairs. The interpretation is the $n$ gives the number of particles in a phase space volume $d^3x^i d^3p_j$. ie. the total number of particles $N_V$ in a region of phase space $V$ is,

$$N_V(t) = \int_V d^3x^i d^3p_j n(t, x^k, p_l)$$ \hfill (274)

The importance of momentum being a covector is that the measure is trivial. However it is more convenient to work with a vector momentum, where $p_i = g_{ij} p^j$. The phase space density is simply a function on phase space, so we may use any coordinates we like to describe it. However, we must take case to use the correct measure when we integrate it.

The dynamics of the phase space density is determined by the Liouville theorem. This famously states that the the volume of a region of phase space is preserved under time evolution. In terms of the density it simply states;

$$L[n] \equiv \frac{d}{dt} n = 0$$ \hfill (275)

where $L = d/dt$ is the Liouville operator, and must be understood as ‘flow’
derivative,

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x^i} + \dot{p} \frac{\partial}{\partial p^i} + \ddot{p} \frac{\partial}{\partial \dot{p}^i}
\]  

(276)

where \(\dot{\cdot} = d/dt\).

We begin by treating the particles as non-interacting, and hence freely propagating along geodesics. We now proceed to compute the Liouville operator by determining \(\dot{x} \) and \(\dot{p} \) or \(\ddot{p} \).

Each particle has a parameter \(\lambda\) so that \(p^\mu = dx^\mu/d\lambda\) [ note for a massive particle \(\lambda = \tau/m\) ]. Since they are free they obey,

\[
0 = \frac{d^2 x^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \frac{dp^\mu}{d\lambda} + \tilde{\Gamma}^\mu_{\alpha\beta} p^\alpha p^\beta
\]  

(277)

where we should emphasise that \(\tilde{\Gamma}^\mu_{\alpha\beta}\) is the Christoffel symbol of the full spacetime, not just the spatial metric \(g_{ij}\). Now,

\[
\frac{d}{dt} = \frac{1}{p^i} \frac{d}{d\lambda}
\]  

(278)

and hence,

\[
\frac{dx^i}{dt} = \frac{1}{p^i} p^i, \quad \frac{dp^\mu}{dt} + \frac{1}{p^i} \tilde{\Gamma}^\mu_{\alpha\beta} p^\alpha p^\beta = 0
\]  

(279)

**Approach 1: using a covector momentum**

The most natural way to proceed from the perspective of phase space is to compute \(\dot{p}_i\). We may use,

\[
p_i = g_{ij} p^j \implies \frac{dp_i}{dt} = \frac{dg_{ij}}{dt} p^j + g_{ij} \frac{dp^j}{dt}
\]

\[
= \frac{dx_k}{dt} \partial_k g_{ij} p^j + g_{ij} \frac{dp^j}{dt}
\]

\[
= \frac{1}{p^i} \delta^k_i \partial_k g_{ij} + g_{ij} \frac{dp^j}{dt}
\]  

(280)
and then (after some calculation - Ex) the geodesic equations tells us;

\[
\frac{dp_i}{dt} = \frac{1}{2p^j p^k \partial_i g_{jk}}
\]  
(281)

Thus we arrive at the equation Liouville equation;

\[
L[n] = 0 \quad L = \frac{\partial}{\partial t} + \frac{p^j}{p^i} \frac{\partial}{\partial x^i} + \frac{1}{2p^i p^j \partial_i g_{jk}} \frac{\partial}{\partial p_i}
\]  
(282)

Note we must be careful with the partial derivatives that in \(\partial / \partial x^i\) we fix \(p_i\) (not \(p^i\)).

**Approach 2: using a vector momentum**

A more natural approach from the perspective of the spacetime is to determine \(\dot{p}^i\) which is obtained directly from the geodesic equation. In fact in this case we may drop any assumption about coordinates and take a general metric,

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu
\]  
(283)

Then,

\[
\frac{dx^i}{dt} = \frac{1}{p^j} \dot{p}^j , \quad \frac{dp^i}{dt} + \frac{1}{p^j} \Gamma^{i}_{\alpha\beta} p^\alpha p^\beta = 0
\]  
(284)

where now \(\Gamma\) (rather than \(\dot{\Gamma}\)) is the full spacetime connection. Then the Liouville theorem tells us,

\[
L[n] = 0 \quad L = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + \dot{p}^i \frac{\partial}{\partial p^i}
\]

\[
= \frac{\partial}{\partial t} + \frac{p^j}{p^i} \frac{\partial}{\partial x^i} + \frac{1}{p^j} \frac{dp^j}{dx} \frac{\partial}{\partial p^i}
\]

\[
= \frac{\partial}{\partial t} + \frac{p^j}{p^i} \frac{\partial}{\partial x^i} - \frac{1}{p^j} \Gamma^{i}_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^i}
\]

\[
= \frac{1}{p^j} \left[ p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^{i}_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^i} \right]
\]  
(285)

Thus we arrive at quite a simple form for the Liouville equation;

\[
\left[ p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^{i}_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^i} \right] n = 0
\]  
(286)
Note we must be careful with the partial derivatives that in \( \partial/\partial x^\mu \) we fix \( p^i \) (not \( p_i \)).

Again we must be careful to use the proper phase space measure \( d^3x^\mu d^3p^i n \) when integrating \( n \).

**Approach 3: covariant form**

The last expression is not manifestly covariant as the summed index on the connection is only spatial. However we can do better by thinking of the density \( n \) not as a function of just \( p^i \), but also \( p^t \), but then imposing the mass shell constraint \( p^2 = m^2 \) after evaluating the derivatives in the Liouville equation. Then the density looks more covariant \( n = n(x^\mu, p^\nu) \). Now,

\[
\begin{align*}
\left. \frac{\partial}{\partial x^\mu} \right|_{p^t} &= \left. \frac{\partial}{\partial x^\mu} \right|_{p^i} + \left. \frac{\partial p^i}{\partial x^\mu} \right|_{p^t} \left. \frac{\partial}{\partial p^i} \right|_{x^\nu, p^t} \\
\left. \frac{\partial}{\partial p^t} \right|_{x^\mu} &= \left. \frac{\partial}{\partial p^t} \right|_{x^\nu, p^t} + \left. \frac{\partial p^i}{\partial p^t} \right|_{x^\mu} \left. \frac{\partial}{\partial p^i} \right|_{x^\nu, p^t}
\end{align*}
\]

But, since \( g_{\mu\nu}p^\mu p^\nu = m^2 \) then,

\[
0 = \left. \frac{\partial}{\partial p^k} \right|_{x^\mu} \left( g_{tt}p^t p^t + 2g_{ti}p^t p^i + g_{ij}p^i p^j \right) \\
= 2g_{tt} \left. \frac{\partial p^t}{\partial p^k} \right|_{x^\mu} + 2g_{ti} \left. \frac{\partial p^i}{\partial p^k} \right|_{x^\mu} p^i + 2g_{kj} p^j \\
= 4 \left( p_1 \left. \frac{\partial p^t}{\partial p^k} \right|_{x^\mu} + p_k \right)
\]

and hence,

\[
\frac{\partial p^t}{\partial p^k} = -\frac{p_k}{p_t}
\]
Likewise,

\[
0 = \frac{\partial}{\partial \mu} \left( g_{\alpha} p^\alpha + 2 g_{\nu} p^\nu + g_{ij} p^i p^j \right)
\]

\[
= 2 g_{\alpha} p^\alpha \frac{\partial p^\alpha}{\partial \mu} + p^\mu p^i \frac{\partial g_{\mu i}}{\partial \mu} + 2 g_{\nu} p^\nu \frac{\partial p^\nu}{\partial \mu} + 2 p^\nu p^i \frac{\partial g_{\nu i}}{\partial \mu} + p^i p^j \frac{\partial g_{ij}}{\partial \mu}
\]

\[
= 2 \left( g_{\alpha} p^\alpha + g_{\nu} p^\nu \right) \frac{\partial p^\nu}{\partial \mu} + p^\mu p^i \frac{\partial g_{\mu i}}{\partial \mu} + 2 p^\nu p^i \frac{\partial g_{\nu i}}{\partial \mu} + p^i p^j \frac{\partial g_{ij}}{\partial \mu}
\]

\[
= 2 p_i \frac{\partial p^i}{\partial \mu} + p^\alpha p_\beta \frac{\partial g_{\alpha \beta}}{\partial \mu}
\]  
(290)

Recalling,

\[
\frac{\partial g_{\alpha \beta}}{\partial \mu} = g_{\alpha \nu} \Gamma^\nu_{\mu \beta} + g_{\beta \nu} \Gamma^\nu_{\mu \alpha}
\]  
(291)

so that,

\[
p^\alpha p_\beta \frac{\partial g_{\alpha \beta}}{\partial \mu} = 2 p_\nu p^\nu \Gamma^\nu_{\mu \alpha}
\]  
(292)

then we see,

\[
0 = 2 p_i \frac{\partial p^i}{\partial \mu} + 2 p_\nu p^\nu \Gamma^\nu_{\mu \alpha}
\]  
(293)

Hence we arrive at,

\[
\frac{\partial p^i}{\partial \mu} \bigg|_{p_\nu} = -\frac{1}{p^\nu} p_\nu p^\nu \Gamma^\nu_{\mu \alpha}
\]

\[
\frac{\partial p^i}{\partial \mu} \bigg|_{p_\mu} = -\frac{p_k}{p_\mu}
\]  
(294)
Now we may substitute these into our Boltzmann equation above;

\[
0 = \left[ p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^i_{\alpha \beta} p^\alpha p^\beta \frac{\partial}{\partial p^i} \right] n(t, x^i, p^j)
\]

\[
= \left[ p^\mu \left( \frac{\partial}{\partial x^\mu} p^\nu \right) + \frac{\partial p^\mu}{\partial x^\mu} \frac{\partial}{\partial p^\nu} \right] - \Gamma^i_{\alpha \beta} p^\alpha p^\beta \left( \frac{\partial}{\partial p^i} \right. \left. \frac{\partial}{\partial x^\mu} \right) n(x^\alpha, p^\beta)
\]

\[
= \left[ p^\mu \left( \frac{\partial}{\partial x^\mu} \Big|_{p^\nu} - \frac{1}{p^i} p^\nu p^\alpha \frac{\partial}{\partial p^\nu} \Big|_{x^\alpha, p^\beta} \right) - \Gamma^i_{\alpha \beta} p^\alpha p^\beta \left( \frac{\partial}{\partial p^i} \Big|_{x^\alpha, p^\beta} \right) n(x^\alpha, p^\beta)
\]

\[
= \left[ p^\mu \left( \frac{\partial}{\partial x^\mu} \Big|_{p^\nu} - \Gamma^i_{\alpha \beta} p^\alpha p^\beta \frac{\partial}{\partial p^i} \Big|_{x^\alpha, p^\beta} \right) - \frac{1}{p^i} p^\nu p^\alpha \frac{\partial}{\partial p^\nu} \left( p_i \Gamma^i_{\alpha \beta} \frac{\partial}{\partial p^i} \right) \right] n(x^\alpha, p^\beta)
\]

\[
= \left[ p^\mu \left. \frac{\partial}{\partial x^\mu} \right|_{p^\nu} - \Gamma^\mu_{\alpha \beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} \right] n(x^\alpha, p^\beta)
\]

Thus we have an elegant covariant form for the Boltzmann equation;

\[
\left[ p^\mu \left. \frac{\partial}{\partial x^\mu} \right|_{p^\nu} - \Gamma^\mu_{\alpha \beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} \right] n(x^\alpha, p^\beta) = 0
\]

(295)

where derivatives are evaluated taking \( n \) to depend on the full 4-vector \( p^\mu \), but then the mass shell condition \( p^2 = m^2 \) is imposed.

### 2.4 Boltzmann equation for FRW

Consider approach 1, so \( n = n(t, x^i, p^j) \). Now consider the case of flat FRW so,

\[
ds^2 = -dt^2 + g_{ij}dx^i dx^j, \quad g_{ij}(t, x) = a(t)^2 \delta_{ij} \quad \Rightarrow \quad \partial_k g_{ij} = 0
\]

(297)

Then the Boltzmann equation is simply;

\[
\left[ \frac{\partial}{\partial t} + \frac{p^i}{p^\mu} \frac{\partial}{\partial x^i} \right] n = 0 \quad \Rightarrow \quad p^\mu \left. \frac{\partial}{\partial x^\mu} n \right|_{p^i} = 0
\]

(298)
as the momentum derivatives have vanishing coefficient. If we further assume
the phase space distribution itself is homogeneous, so \( n(t, p_i) \), then we
simply have,

\[
L[n] = \left. \frac{\partial}{\partial t} n \right|_{x',p_j} = 0 \tag{299}
\]

Further more let us assume the distribution is also isotropic. Then we may
parameterize \( n \) as,

\[
n = n(t, p) , \quad p \equiv \sqrt{g_{ij} p_i p_j} \tag{300}
\]

A very important point is that \( p \) is the magnitude of the spatial momentum
a comoving observer sees for a 4-momentum \( p^\mu \). Recall for an observer with
4-velocity \( v^\mu \) they observe a particle with 4-momentum \( p^\mu \) to have energy \( E \)
and spatial momentum in the direction \( n^\mu \),

\[
E = -v^\mu p_\mu , \quad p(\nu) = n^\mu p_\mu \tag{301}
\]

where \( n^\mu \) is a unit spatial vector orthogonal to \( v^\mu \), so \( n^2 = -1 \) and \( v^\mu n_\mu = 0 \).
Now consider a comoving observer so \( v^\mu = (1,0,0,0) \). Then for the total
spatial momentum \( p \), i.e. the momentum in the direction of the particle, we
require,

\[
n^t = 0 ; \quad n^i = \frac{1}{\sqrt{g_{jk} p^j p^k}} p^i \tag{302}
\]

and then,

\[
p = n^\mu p_\mu = \frac{1}{\sqrt{g_{jk} p^j p^k}} p^i p_i = \sqrt{g_{jk} p^j p^k} = \sqrt{g^{jk} p_j p_k} \tag{303}
\]

as claimed. For thermal physics it is important that \( p \) is the physical 3-
momentum magnitude. It is this quantity that enters the Bose/Fermi distri-
bution. So in flat FRW in thermal equilibrium we would have,

\[
n(t, p) = \frac{1}{(2\pi \hbar)^3} \frac{1}{e^{\frac{E_p - m}{kT}} \pm 1} , \quad E_p^2 = p^2 + m^2 \tag{304}
\]

as usual, but with the understanding that \( p^2 = g_{ij} p^i p^j = a^2(t) \delta_{ij} p^i p^j \).
Now let us derive the phase space measure. Suppose we parameterize the
covector momentum as,

\[ p_1 = \mu \cos \theta \cos \phi, \quad p_2 = \mu \cos \theta \sin \phi, \quad p_3 = \mu \sin \theta \]  

(305)

so that \(\mu^2 = \delta^{ij} p_i p_j\). Isotropy implies that \(\mu = \mu(t, p)\) and there is no \(\theta\) or \(\phi\) dependence to \(n\). \(n\) is a function of \(t, p\) or equivalently \(t, \mu\).

Consider the phase space measure \(d^3 p_i\). We may perform the \(\theta, \phi\) integrals in the usual way,

\[
\int d^3 p_j = \int_0^\infty d\mu \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\pi} d\phi \mu^2 \cos \theta = \int_0^\infty 4\pi \mu^2 d\mu
\]

(306)

In terms of the physical 3-momentum \(p\), we have,

\[
\mu^2 = \delta^{ij} p_i p_j = a^2(t) g^{ij} p_i p_j = a^2(t)p^2
\]

(307)

Thus the phase space measure given by,

\[
dN = n d^3 x^i d^3 p_j = n d^3 x^i (4\pi \mu^2 d\mu) = n \left(a^3(t) d^3 x^i\right) (4\pi p^2 dp)
\]

(308)

but now we recognise \(a^3 d^3 x^i\) as the usual spatial measure. Hence we see the real space number density is derived from the our homogeneous isotropic phase space density in FRW as,

\[
n(t) = \int_0^\infty dp \ 4\pi p^2 n(t, p)
\]

(309)

Now we have to treat the Boltzmann equation carefully. We note for a function \(f(t, p)\) then,

\[
\frac{\partial}{\partial t} \left|_{x^i, p_j} f(t, p) \right| = \left( \frac{\partial}{\partial t} \left|_{x^i, p_j} \right. \right) + \left( \frac{\partial}{\partial t} \left|_{x^i, p_j} \right. \right) \left( \frac{\partial}{\partial p} \right) f(t, p)
\]

\[
= \left( \frac{\partial}{\partial t} \left|_p \right. \right) + \left( \frac{\partial}{\partial t} \left|_{x^i, p_j} \right. \right) \left( \frac{\partial}{\partial p} \right) f(t, p)
\]

(310)
but recall \( p^2 = \frac{1}{a^2(t)} \delta_{ij} p_i p_j \) so,

\[
\frac{\partial p}{\partial t} \bigg|_{x^i, p_j} = \frac{\partial}{\partial t} \bigg|_{x^i, p_j} \left( \frac{1}{a(t)} \sqrt{\delta_{ij} p_i p_j} \right) = \frac{1}{a(t)} \sqrt{\delta_{ij} p_i p_j} \frac{\partial}{\partial t} \left( \frac{1}{a(t)} \right) = -\frac{1}{a^2(t)} \sqrt{\delta_{ij} p_i p_j} a'(t) = -p \frac{a'(t)}{a(t)}
\]

(311)

Then the Boltzmann equation is;

\[
\left( \frac{\partial}{\partial t} - p \frac{a'(t)}{a(t)} \frac{\partial}{\partial p} \right) n(t, p) = 0
\]

(312)

[ A quick way to obtain this result is to start from a homogeneous isotopic distribution \( n(t, p) \) and apply the Liouville theorem directly;

\[
0 = \frac{d}{dt} n(t, p) = \left( \frac{\partial}{\partial t} \bigg|_p + \frac{\partial p}{\partial t} \frac{\partial}{\partial p} \bigg|_t \right) n(t, p)
\]

(313)

and then use the fact above, \( \frac{\partial p}{\partial t} = -p \frac{a'(t)}{a(t)} \). However, of course one cannot obtain the measure without thinking in the full phase space. ]

In this collisionless case we may solve the Boltzmann equation trivially. Changing variable from \( t \) to \( a = a(t) \), then it implies,

\[
0 = \left( \frac{\partial}{\partial t} \bigg|_p - p \frac{a'(t)}{a(t)} \frac{\partial}{\partial p} \bigg|_p \right) n(t, p) = a'(t) \left( \frac{\partial}{\partial t} \bigg|_p - p \frac{1}{a} \frac{\partial}{\partial p} \bigg|_a \right) n(t, p)
\]

(314)

which implies;

\[
n = n(a(t) p)
\]

(315)
Recall that for a particle \( a(t)p \) = constant as a result of homogeneity. Hence this result is simply the statement that \( dn/dt = 0 \).

**Example:** Consider massless particles in FRW with a thermal distribution. Then consider the particle interactions immediately switch off at a time \( t = t_i \), when the gas has temperature \( T = T_i \). Thus,

\[
n(t_i, p) = \frac{g}{(2\pi \hbar)^3} \frac{1}{e^{\frac{\mu(t_i) - m}{kT_i}} + 1}
\]

Subsequently the evolution is governed by the Boltzmann equation, so, \( n = n(a(t)p) \). Thus for \( t > t_i \) the solution is,

\[
n(t, p) = \frac{g}{(2\pi \hbar)^3} \frac{1}{e^{\frac{\mu(t) - m}{kT_i}} + 1}
\]

which obviously agrees with our boundary condition at \( t = t_i \). We may write this as,

\[
n(t, p) = \frac{g}{(2\pi \hbar)^3} \frac{1}{e^{\frac{\mu(t) - m}{kT_{eff}}}} \quad , \quad T_{eff}(Z) = \frac{a(t_i)}{a(t)} T_i = \frac{1}{1 + Z(t)} T_i
\]

where \( 1 + Z(t) = a(t)/a(t_i) \) is the redshift of the past time \( t_i \) measured at the later time \( t \).

An important application of this is that photons and neutrinos maintain an equilibrium Bose/Fermi distribution if they are initially in thermal equilibrium and then suddenly start **free streaming** - i.e. stop interacting. The effective temperature governing their distribution goes as \( T_{eff} \sim 1/a \).

**Example:** Consider a similar scenario with non-relativistic particles in FRW with a thermal distribution. Then,

\[
n(t_i, p) = \frac{g}{(2\pi \hbar)^3} e^{-\frac{\mu(t_i) - m}{kT_i}}
\]

Then for \( t > t_i \) the solution is,

\[
n(t, p) = \frac{g}{(2\pi \hbar)^3} e^{-\frac{(a(t)p)^2}{2m \alpha(t_i)^2 T_i}} e^{-\frac{\mu(t_i) - m}{kT_i}}
\]
so,

\[ n(t, p) = \frac{g}{(2\pi \hbar)^3} e^{-\frac{2mE_{eff}(Z)}{\hbar^2}} e^{\frac{\mu_{eff}(Z)}{kT_{eff}(Z)}} \]  

(321)

with

\[ T_{eff}(Z) = \frac{1}{(1 + Z(t))^2} T_i, \quad \mu_{eff}(Z) = \frac{1}{(1 + Z(t))^2} (\mu(t_i) - m) \]  

(322)

### 2.5 Boltzmann equation with collision term

Now we consider an interacting gas. We describe this using our free Boltzmann equation and a ‘collision term’. This collision term may have a simple form for local interactions, such as in perturbative QFT. In general we write,

\[ L_{free}[n] = C[n] \]  

(323)

In our FRW case we have,

\[ \left( \frac{\partial}{\partial t} \bigg|_p - \frac{\dot{a}}{a} \frac{\partial}{\partial p} \bigg|_t \right) n(t, p) = C \left[ n(t, p) \right] \]  

(324)

Consider the local density in real space;

\[ n(t) = \int_0^\infty dp \frac{4\pi p^2 n(t, p)}{} \]  

(325)

Then, integrating the Boltzmann equation;

\[ \int_0^\infty dp 4\pi p^2 C[n] = \int_0^\infty dp 4\pi p^2 \left( \frac{\partial}{\partial t} \bigg|_p - \frac{\dot{a}}{a} \frac{\partial}{\partial p} \bigg|_t \right) n(t, p) \]

\[ = \frac{\partial}{\partial t} \left( \int_0^\infty dp 4\pi p^2 n \right) \bigg|_p - 4\pi \frac{\dot{a}}{a} \int_0^\infty dp p^3 \frac{\partial n}{\partial p} \bigg|_t \]

\[ = \frac{dn(t)}{dt} + 4\pi \frac{\dot{a}}{a} \int_0^\infty dp n \frac{\partial (p^3)}{\partial p} \bigg|_t \]  

(326)

where we have integrated by parts and thrown away the boundary terms,

\[ \left[ p^3 n(t, p) \right]_0^\infty \]  

(327)
which is justified by smoothness of \( n(t,p) \) in \( p \) at \( p \to 0 \), and good high energy behaviour for \( p \to \infty \). Then,
\[
\int_0^\infty dp 4\pi p^2 C[n] = \frac{dn(t)}{dt} + 3\frac{\dot{a}}{a} \int_0^\infty dp 4\pi p^2 n
\]
\[
= \frac{dn(t)}{dt} + 3\frac{\dot{a}}{a} n(t)
\] (328)

Hence we obtain,
\[
\frac{1}{a^3(t)} \frac{d}{dt} (a^3(t)n(t)) = \int_0^\infty dp 4\pi p^2 C[n]
\] (329)

Note that \( a^3(t)n(t) \) is proportional to the number of particles comoving volume. Defining the comoving number density as
\[
N(t) = a^3(t)n(t)
\] (330)
then we have,
\[
\frac{dN}{dt} = a(t)^3 \int_0^\infty dp 4\pi p^2 C[N]
\] (331)
so that in the absence of collisions we have the free Boltzmann equation and consequently \( N = \text{constant} \).

### 2.6 Application 1: Decay into thermal products

Suppose we have a species \( X \) that decays into products \( Z_i \) which are in thermal equilibrium at a temperature \( T \).
\[
X \rightarrow Z_1 + Z_2 + \ldots
\] (332)

We may use the Boltzmann equation to study the out of equilibrium dynamics of \( X \), in the approximation that the products \( Z_i \) are held in equilibrium by other interactions.

Let us denote the rate of decay of a single \( X \) particle as \( \Gamma \), and denote the number density of \( X \) as \( n(t) \) and the comoving number density as \( N(t) = a^3n \). Note that since the decay is into thermal products, this rate will be a
function of temperature. We assume that the temperature of the products $Z_i$ is $T = T(t)$. For example, typically $T \sim 1/a(t)$ for the expanding universe.

Then the Boltzmann equation is,

$$\frac{1}{a^3(t)} \frac{d}{dt} (a^3(t)n(t)) = \frac{\text{#created/vol}}{\text{#decay/vol}}$$

$$= \frac{\text{#created/vol}}{\text{#decay/vol}} - \Gamma(T) \times n(t)$$

(333)

Now detailed balance tells us that in equilibrium the left hand side must vanish. Suppose at temperature $T$ the equilibrium number density for $X$ is $n_{eq}(T)$, and correspondingly the comoving density is $N_{eq}(T)$. Then,

$$\frac{1}{a^3(t)} \frac{d}{dt} (a^3(t)n(t)) = \Gamma(T) (n_{eq}(T) - n(t))$$

(334)

and hence,

$$\frac{dN(t)}{dt} = \Gamma(T) (N_{eq}(T) - N(t))$$

(335)

determines the time evolution of $N(t)$. For cosmology it is convenient to write the evolution in terms of $a$ rather than $t$, using;

$$\frac{d}{dt} = \frac{da}{dt} \frac{d}{da} = H(a) \frac{d}{d\ln a}$$

(336)

Then we find,

$$\frac{d\ln N(a)}{d\ln a} = -\frac{\Gamma(T)}{H(a)} \left( 1 - \frac{N_{eq}(T)}{N(a)} \right)$$

(337)

If we further assume that $a = 1/T$ (where say $T$ is the photon temperature), we may write this as,

$$\frac{d\ln N}{d\ln T} = \frac{\Gamma}{H} \left( 1 - \frac{N_{eq}}{N} \right)$$

(338)

where all quantities are functions of $T$.

Let us consider the behaviour of this equation. Suppose at high temperatures the particle $X$ is close to thermal equilibrium. Let us take,

$$N = N_{eq} (1 + f)$$

(339)
where initially $|f| \ll 1$. Then,

$$\ln N = \ln N_{eq} + f$$

(340)

so,

$$\frac{df}{d\ln T} + \frac{d \ln N_{eq}}{d \ln T} = \frac{\Gamma}{H} \left(1 - \frac{1}{1 + f}\right) \approx \frac{\Gamma}{H} f$$

(341)

and hence,

$$\frac{df}{d\ln T} = \frac{\Gamma}{H} f - \frac{d \ln N_{eq}}{d \ln T}$$

(342)

For decreasing temperature we may regard the first term on the right as a restoring term, and the second as a driving term. Suppose that $d \ln N_{eq}/d \ln T \sim O(1)$ and that $1 \ll \Gamma / H$. Then as $T$ decreases, we will have,

$$f \approx \frac{1}{\Gamma / H} \frac{d \ln N_{eq}}{d \ln T} \ll 1$$

(343)

Consider examples of this driving term;

- Suppose $X$ is highly relativistic - then $n_{eq} \sim T^3$ so $N_{eq} \sim$constant. Thus $d \ln N_{eq}/d \ln T \approx 0$.

- Suppose $X$ is non relativistic with a potential so $n_{eq} \sim T^{3/2}$ so $N_{eq} \sim T^{-3/2}$. Thus $d \ln N_{eq}/d \ln T \approx -3/2$.

Thus usually we have the case that $d \ln N_{eq}/d \ln T \sim O(1)$. (Note this is not always the case - eg. non-relativistic species with no potential $n_{eq} \sim T^{-3/2} e^{-m/kT}$.) Then the condition that equilibrium is preserved as the temperature drops is;

$$1 \ll \frac{\Gamma}{H}$$

(344)

This is a crucially important condition. Intuitively it says that if the decay (and hence creation) rate is much quicker than the expansion rate then the species is held in equilibrium. However, if the universe expands too fast, equilibrium cannot hold.
Suppose the converse is true, namely that $\Gamma/H \ll 1$. In this case we simply have our free Boltzmann equation,
\[
\frac{d \ln N}{d \ln T} = 0 \implies N = \text{const} \tag{345}
\]
and hence $n \sim T^3 \sim 1/a^3$.

Suppose for some reason for $T > T_{\text{freeze}}$ we have $1 \ll \Gamma/H$, but then quickly changes for $T < T_{\text{freeze}}$ to $\Gamma/H \ll 1$. Then $T_{\text{freeze}}$ is called the \textbf{freeze out} temperature. The the approximate solution to the boltzmann equation is (as we have seen before in our photon example above);
\[
N \simeq N_{eq}(T) \quad T > T_{\text{freeze}}
\]
\[
N \simeq N_{eq}(T_{\text{freeze}}) \quad T < T_{\text{freeze}} \tag{346}
\]
For quick transitions this is a good approximation to a full solution of the Boltzmann equation. Note that rather than $N$ decreasing as $N_{eq}(T)$, instead a \textbf{relic density} of $X$ particles is left over.

The relic density of the matter today, $t_0$, is simply calculated. Suppose it is a massive particle, mass $m$. Then,
\[
\rho_{\text{relic}} = m n(t_0) = m \left( \frac{a^3(T_{\text{freeze}})}{a_0^3} \right) n_{eq}(T_{\text{freeze}})
\]
\[
= m \left( \frac{T_{\text{CMB}}}{T_{\text{freeze}}} \right)^3 n_{eq}(T_{\text{freeze}}) \tag{347}
\]
assuming that $a \sim 1/T_{\text{photon}}$ which is to a good approximation true. Then,
\[
\Omega_{\text{relic}} = \frac{\rho_{\text{relic}}}{\rho_{\text{crit}}} = \frac{8\pi G}{3H_0^2} \rho_{\text{relic}} \tag{348}
\]
Now recall that in in the radiation and matter eras $H \sim 1/t$. Hence for single particle decay, we might not expect that $\Gamma/H$ decreases with temperature. However, let us consider decays involving more than one particle in the initial state.
2.7 Application 2: Decay via interaction with thermal species

A similar equation arises if we consider a process where $X$ decays via interaction with a thermal species $Y$ into thermal products $Z_i$, so,

$$X + Y \rightarrow Z_1 + Z_2 + \ldots$$ (349)

Now the rate of decay of an $X$ particle, $\Gamma$, depends on the cross section of the process $\sigma$, the relative velocity $v$, and the number density of $Y$, $n_Y(T)$ as,

$$\Gamma = \langle \sigma v \rangle n_Y(T)$$ (350)

where the $\langle \rangle$ signify the expectation value in the thermal state, and given the distribution for $X$ - thus it is a complicated expression. We may think of $\langle \sigma v \rangle$ as the 'interaction volume'.

Thus we have,

$$\frac{d \ln N}{d \ln T} = \frac{\Gamma}{H} \left( 1 - \frac{N_{eq}}{N} \right) = \frac{\langle \sigma v \rangle n_Y}{H} \left( 1 - \frac{N_{eq}}{N} \right)$$ (351)

Again the species will be driven to equilibrium for falling temperatures if $1 \ll \Gamma/H$. For $\Gamma/H \ll 1$ it will behave as a free 'relic'.

**Example:** If $X$ is a non-relativistic particle, and $Y$ is relativistic, then one often finds this quantity is roughly constant, so $\langle \sigma v \rangle \sim \text{const}$ eg. think of a cannon ball.

Suppose particle $Y$ has mass $m_Y$. Consider early times when $m_Y \ll kT$ so it behaves as a relativistic particle and then $n_Y \sim T^3$. Assume $X$ is non-relativistic, with $\langle \sigma v \rangle \sim \text{constant}$.

Then in the radiation era we have $H \sim 1/a^2 \sim T^2$. Hence we see at early times we expect $\Gamma/H \sim T^3/T^2 \sim T \gg 1$.

Assume this holds true until the temperature $kT \sim m_Y$ is reached. Then below this $Y$ is non-relativistic. Suppose it has no potential so $n_Y \sim T^{3/2}e^{-m_Y/kT}$. 

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Then very quickly we will transition to, \( \Gamma / H \sim e^{-m/kT}/T^{3/2} \ll 1 \).

**Example:** neutron decay,

\[
    n + \bar{e} \rightarrow p + \bar{\nu}
\]  

(352)

We will discuss this later. In fact there \( \Gamma / H \sim T^3 \) early on so it is slightly more complicated.

### 2.8 Application 3: Decay of particle antiparticle

Consider a decay of a particle \( X \) and antiparticle \( \bar{X} \) with conserved particle number \( X = \bar{X} \). Consider decay into thermal products \( Z_i \),

\[
    X + \bar{X} \rightarrow Z_1 + Z_2 + \ldots
\]  

(353)

Now assume a thermal distribution for \( X \) initially and there is no potential so \( n_X = n_{\bar{X}} \) initially and for all times after. Thus we may write \( n = n_X = n_{\bar{X}} \). Then we have,

\[
    \frac{1}{a^3(t)} \frac{d}{dt} \left( a^3(t)n_X(t) \right) = (\#created/vol) - (\#decay/vol)
\]

\[
    = (\#created/vol) - \langle \sigma v \rangle n_X(t)n_{\bar{X}}(t)
\]

\[
    = (\#created/vol) - \langle \sigma v \rangle n(t)^2
\]  

(354)

and by detailed balance then,

\[
    \frac{1}{a^3(t)} \frac{d}{dt} \left( a^3(t)n_X(t) \right) = \langle \sigma v \rangle \left( n_{eq}^2 - n(t)^2 \right)
\]  

(355)

where \( n_{eq}(T) \) is the equilibrium distribution for \( X \) and \( \bar{X} \). Then,

\[
    \frac{d \ln N}{da} = \langle \sigma v \rangle N \left( 1 - \frac{N_{eq}^2}{N^2} \right) = \frac{\Gamma}{H} \left( 1 - \frac{N_{eq}^2}{N^2} \right)
\]  

(356)

with \( \Gamma = \langle \sigma v \rangle n \).
2.9 Boltzmann equations and QFT

Our discussion of the behaviour of the real space number density has shown us that the quantity $\Gamma/H$ controls when in a cooling universe a species breaks from equilibrium, and its approximate behaviour after. However we need to understand how to compute $\Gamma$ directly from a QFT model. Let us return to the Boltzmann equation for the phase space density,

$$\frac{dn(t, p)}{dt} = C[n]$$

(357)

An interesting point to note is that elastic collisions (eg. Thompson scattering $e + \gamma \rightarrow e + \gamma$ with $E_\gamma \ll m_e$) has $C[n] = 0$. While there is scattering, the number of particles in a momentum shell $dp$ is constant.

Let us reconsider a one body, decay as considered above, but with a specific QFT interaction. Consider again $X \rightarrow Z_i$ where the products are thermally distributed. Recall we can consider $C = C_{\text{decay}} + C_{\text{creation}}$ thinking of the forward and reverse processes separately. Then (see eg. Peskin and Schroder)

$$C_{\text{decay}} = \frac{n(t, p)}{2E} \prod_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)} \sum_{\text{av spins and } p_X} |M|^2 (2\pi)^4 \delta^4(p_X - \sum_f \vec{p}_f)$$

The factor $1/2E$ is required for Lorentz invariance, and then,

$$d\Gamma = C_{\text{decay}} 4\pi p^2 dp$$

(359)

gives $d\Gamma$ the rate of decay/volume of $X$ due to a momentum shell $p$. Here $M$ is the matrix element for the process $X \rightarrow Z_i$.

Comment on conventions: $\langle p|q \rangle = 2E_p (2\pi)^3 \delta^3(p - q)$. And $\text{out} \langle f|i \rangle_m = \langle f|S|i \rangle$. Then $S = 1 + iT$ and $\langle f|iT|i \rangle = (2\pi)^4 \delta^{(4)}(\text{mom}) iM$. Also $\hbar = c = 1$.

Example: Consider a scalar particle $X$, mass $M$, decaying into 2 identical scalar particles, mass $m$, in the final state. Suppose this is given in perturbation theory by a a tree level process with coupling $\lambda$. Suppose firstly the decay occurs in the zero temperature vacuum. In this case we simply have,

$$|M|^2 = \lambda^2$$

(360)
with no momentum dependence. Furthermore,

\[
\int \frac{d^3p_1}{(2E_1)} \int \frac{d^3p_2}{(2E_2)} \delta^4(p_X - p_1 - p_2) = \int \frac{d^3p_1}{4E_1E_2} \delta(E_X - E_1 - E_2) |p_X = p_1 + p_2|
\]

and using \( E_1 = \sqrt{m^2 + |p_1|^2} \), and,

\[
\int f(x)\delta(g(x))dx = \sum_{\{x|g(x)=0\}} \frac{f(x)}{|g'(x)|}
\]

then,

\[
\int \frac{d^3p_1}{2E_1} \delta(\bar{E} - \sqrt{m^2 + |p_1|^2}) (\ldots) = \int_0^\infty \frac{|p_1|^2dp_1}{2E_1} \frac{d\Omega_1}{E_1} \delta(\bar{E} - \sqrt{m^2 + |p_1|^2}) (\ldots)
\]

\[
= \left\{ \begin{array}{ll}
\int d\Omega_1 \frac{|p_1|^2}{2E_1} (|p_1|E_1 (\ldots)) |\bar{E} = E_1 & E > \bar{E} \\
0 & E < \bar{E}
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\int d\Omega_1 \left( \frac{|p_1|^2}{2} (\ldots) \right) |\bar{E} = E_1 & E > \bar{E} \\
0 & E < \bar{E}
\end{array} \right.
\]

Thus we can compute,

\[
C_{\text{decay}} = \frac{n(t,p)}{2E} \sum_{av_{pX}} \frac{\lambda^2}{(2\pi)^2} \int d\Omega_1 \left( \frac{|p_1|}{4E_2} \right) |E_X = E_1 + E_2, p_X = p_1 + p_2|
\]

This is still not trivial to compute. However, let us now assume that the particle \( X \) is very heavy and highly non-relativistic, and the products are much lighter, \( m \ll M \). Then \( E_X \simeq M \), and \( p_X \simeq 0 \), and the products will be highly relativistic so \( E_i \simeq |p_i| \). In this case, the kinematics simply so that,

\[
p_1' \simeq -p_2', \quad E_1 \simeq E_2 \simeq |p_1| \simeq |p_2| \simeq \frac{M}{2}
\]

Hence one obtains for \( p \ll M \),

\[
C_{\text{decay}} = \frac{n(t,p) \lambda^2}{2M} \frac{1}{4\pi}
\]
Thus we see the decay rate in this limit has no dependence on $p$ except trivially through the factor of $n(t,p)$. Integrating over the species $X$ momentum we obtain;

$$\frac{1}{a^3(t)} \frac{d(a^4 n(t))}{dt} = - \int dp_X 4\pi p_X^2 C_{\text{decay}} \sim -\frac{n(t)}{2M} \frac{\lambda^2}{4\pi}$$

(367)

using $\int dp_X 4\pi p_X^2 n(t,p) = n(t)$ and noting there will be no particle creation spontaneously from vacuum. Hence we simply obtain,

$$\frac{dN}{dt} = -\frac{\lambda^2}{8\pi M} N$$

(368)

**Thermal end state:**

However, in cosmology we are interested in the situation that the final state particles are in thermal equilibrium. Then the matrix element is not evaluated in vacuum, but in this thermal state. For a tree level process one finds,

$$\prod_f \int \frac{d^3 p_f}{(2\pi)^3(2E_f)} |M|^2_{\text{thermal}} = \prod_f \int \frac{d^3 p_f}{(2\pi)^3(2E_f)} |M|^2_{\text{vacuum}} (1 \pm F)_f$$

(369)

The factors $F$ are the filling fraction of momentum states of the final state particles. In vacuum these are zero, but in a thermal product state we have the filling fraction for a species $f$ which is a boson $(\pm)$ or fermion $(-)$ are;

$$\frac{1}{(2\pi)^3} (1 \pm F)_f = \frac{1}{(2\pi)^3} \left( 1 \pm \frac{1}{e^{\frac{E-f}{kT}} + 1} \right) = \frac{1}{(2\pi)^3} \frac{1}{1 \mp e^{\frac{E-f}{kT}}}$$

(370)

giving stimulated boson emission, and reduced fermion emission.

Returning to our previous example, this modifies the result. Now, using our same approximation $m \ll M \sim E_X$

$$C_{\text{decay}} = \frac{n(t,p)}{2E} \sum_{avpX} \frac{\lambda^2}{(2\pi)^2} \int d\Omega_1 \left( \frac{|p_1|}{4E_2} \right) \left( \frac{1}{1-e^{-\frac{E_1}{kT}}} \right) \left( \frac{1}{1-e^{-\frac{E_2}{kT}}} \right)$$

(371)
Now assuming our theory is T invariant (= CP invariant) then the backward process is simply given by detailed balance, so,

\[
\frac{dN}{dt} = \frac{\lambda^2}{8\pi M} \left( \frac{1}{1 - e^{-\frac{M}{kT}}} \right)^2 (N_{eq}(T) - N) \tag{372}
\]

Typically if we are treating \(X\) non-relativistically we would be considering the situation \(kT \ll M\), and then,

\[
\frac{dN}{dt} \simeq \frac{\lambda^2}{8\pi M} (N_{eq}(T) - N) = \Gamma (N_{eq}(T) - N) \tag{373}
\]

exactly as we had discussed before. Hence in this simple case we have derived \(\Gamma = \frac{\lambda^2}{8\pi M}\) and in fact is constant.

Let us briefly check our units. \(\Gamma\) is a rate and hence should have mass dimension \([-1]\). The Lagangian must contain the interaction term \(XZ_1Z_2\) and should be mass dimension \([-4]\). Since \(X, Z_i\) are scalars with dimension 1, then \([\lambda] = 1\). Thus indeed we have \([\Gamma] = [\lambda^2] - [M] = +2 - 1 = +1\) as required.

**Example 2:** Now consider 2 - 2 scattering, where \(X + Y \rightarrow W + Z\) and both \(Y, W\) and \(Z\) are thermally distributed. Now we have for the forward process,

\[
C_{\text{decay}} = \frac{n(t, p)}{2E} \int \frac{d^3p_Y}{(2E_Y)^2} \frac{d^3p_f}{(2\pi)^3(2E_f)} \sum_{\text{av spins and } p_X} |M|_{\text{vac}}^2 (1 \pm F)_f (2\pi)^4 \delta^4(p_X - \sum_f p_f) \tag{374}
\]

where \(n_Y\) would be a thermal distribution,

\[
n_Y = \frac{1}{(2\pi)^3 e^{\frac{E_Y - \mu_Y}{kT}} \mp 1} \tag{375}
\]

Assume the process is dominated by a tree level scattering with coupling \(\lambda\). Let us consider the situation where \(X\) has mass \(M_X\), and \(Z\) has mass \(M_Z\) with \(\Delta M = M_X - M_Z\). Consider \(kT \ll M_X, M_Z\) and these to behave non-relativistically. Consider the other species, \(Y\) and \(W\) to be highly relativistic.
at the thermal scales of interest. 

Consider the initial and final momenta; \( p_X \simeq (M_X, 0, 0, 0) \), \( p_Y \simeq (p, p, 0, 0) \), \( p_Z \simeq (E_Z, p + q \cos \theta, q \sin \theta \cos \phi, q \sin \theta \sin \phi) \), \( p_W \simeq (p - q \cos \theta, -q \sin \theta \cos \phi, -q \sin \theta \sin \phi) \). Then 3-momentum is conserved. Since \( kT \ll M_X \), then \( p \sim kT \ll M_X \).

Consider energy;

\[
M_X + p = E_Z + q \simeq M_Z + q \quad \Rightarrow \quad q \simeq \Delta M + p \quad (376)
\]

Check;

\[
E_Z = M_Z + \frac{(p + q \cos \theta)^2 + q^2 \sin^2 \theta}{2M_Z} + \ldots \quad (377)
\]

But since \( p^2 \ll M_X^2 \), \( M_Z^2 \) and \( q^2 = (\Delta M + p)^2 \ll M_Z^2 \) also, then indeed \( E_Z \simeq M_Z \). The kinematics is thus much simplified so there is no angular dependence.

\[
C_{\text{decay}} \simeq \frac{n_X}{2E_X} \int \frac{d^3p_Y}{2E_Y (2\pi)^2} \int d\Omega_W \lambda^2 \left( \frac{1}{1 - e^{-\frac{p_W}{4E_Z}}} \right) \left( \frac{1}{1 - e^{-\frac{p_W}{4E_Z}}} \right) \frac{|p_W|}{4E_Z} \quad \text{on-shell}
\]

\[
\simeq \frac{n_X}{2M_X} \int_0^\infty \frac{4\pi p^2 dp}{2p^2} \frac{1}{2(2\pi)^3} \left( \frac{1}{e^{\frac{p}{4E_Z}} - 1} \right) 4\pi \lambda^2 \left( \frac{1}{1 - e^{-\frac{\Delta M + p}{4M_Z}}} \right) \frac{|p(\Delta M + p)|}{4M_Z}
\]

\[
\simeq \frac{n_X}{M_X M_Z} \frac{\lambda^2}{4(2\pi)^3} \int_0^\infty dp \frac{p (\Delta M + p)}{(e^{\frac{p}{4E_X}} - 1) (1 - e^{-\frac{\Delta M + p}{4M_Z}})} \quad (378)
\]

In fact we have to be careful with the lower limit if \( \Delta M < 0 \). Then \( \int_{-\Delta M}^\infty \) are the limits due to the on-shell constraint. Let \( x = p/kT \), and \( y = \Delta M/kT \), then,

\[
C_{\text{decay}} \simeq \frac{n_X}{M_X M_Z} \frac{\lambda^2}{4(2\pi)^3} (kT)^3 \int_0^\infty dx \frac{x (y + x)}{(e^x - 1) (1 - e^{-y - x})}
\]

\[
\simeq \frac{n_X}{M_X M_Z} \frac{\lambda^2}{4(2\pi)^3} (kT)^3 f\left( \frac{\Delta M}{kT} \right) \quad (379)
\]

where one can evaluate \( f(y) \). For example, if \( |\Delta M| \ll kT \) then \( f(y) \simeq f(0) = \pi^2/3 \), and \( \Delta M \ll M_X, M_Z \) so,

\[
C_{\text{decay}} \sim \frac{n_X \lambda^2}{M_X^3} (kT)^3 \quad |\Delta M| \ll kT \quad (380)
\]
On the other hand, for \( kT \ll |\Delta M| \) then \( f(y) \simeq \pi^2 y/6 \), so,

\[
C_{\text{decay}} \sim \frac{n_X \lambda^2 \Delta M}{M_X M_Z} (kT)^2 \quad kT \ll |\Delta M|
\] (381)

In addition to this process we must also include the 1-body decay process, \( X \rightarrow Y + W + Z \). Suppose that \( \Delta M < 0 \). In this case the 1-body decay cannot occur. Suppose also that \( |\Delta M| \ll kT \). Then, integrating over \( d^3p_X \), would lead to a Boltzmann equation for the real space density of \( X, n \),

\[
\frac{1}{a^3} \frac{dt}{dt} (a^3n) = \frac{\lambda^2}{96\pi} \frac{(kT)^3}{M_X^2} (n_{eq}(T) - n(t)) = \Gamma (n_{eq}(T) - n(t))
\] (382)

Again we check our units. \( \Gamma \) has mass dimension \([\Gamma] = +1\). The Lagrangian must contain the interaction term \( \lambda XYZW \) and should be mass dimension -4. Since the particles are all scalars with dimension 1, then \([\lambda] = 0\). Temperature has mass dimension \([kT] = +1\). Thus indeed we have \([\Gamma] = [\lambda^2] + [(kT)^3] - [M^2] = 0 + 3 - 2 = +1 \) as required.

An important comment is that the form of this expressions is quite constrained on dimensional grounds. Since for a tree level process \( C_{\text{decay}} \propto \lambda^2 \), and in this case also \( C_{\text{decay}} \propto n_Y \sim (kT)^3 \), then at low temperatures, but not so low that \( \Delta M \) is important, then the only possibility for the most relevant mass scale is \( M_X \) so,

\[
\Gamma \sim \frac{\lambda^2 (kT)^3}{M_X^2}
\] (383)

Suppose instead that the process was occurring at very high temperatures so \( M_X \ll kT \), and all the species were essentially relativistic. In this case then the only relevant mass scale is given by \( kT \). We could precisely work out the expression, but it must go as,

\[
\Gamma \sim \lambda^2 (kT)
\] (384)
simply on dimensional grounds.

An important example of such a process is the 2-2 scattering \( n + e \rightarrow p + \nu \) mediated by the weak interaction. Note now the particles are fermions, not bosons, and this is a more complicated matrix element. For \( kT \ll M_W \) then this vertex is described by the weak Fermi interaction vertex, coupling \( G_W \sim 10^{-5}/(GeV^2) \). Suppose we consider temperatures \( M_{n,p} \ll kT \ll M_W \), then it is clear that the decay rate of neutrons,

\[
\Gamma \sim G_W^2 (kT)^5
\]

and since \([G_W] = -2\) and \([\Gamma] = +1\) then,

\[
\Gamma \sim G_W^2 (kT)^5
\]

Such a process will be important in nucleosynthesis later.

**Example 3:** Again consider 2 - 2 scattering but now of a complex scalar particle \( X \) and antiparticle \( \bar{X} \), mass \( M \), into 2 scalar particles mass \( m \). Assume particle number conservation is preserved by the interactions and so \( n(t, p) = n_X(t, p) = n_{\bar{X}}(t, p) \). Assume a tree level vertex with coupling \( \lambda \). Then,

\[
C_{\text{decay}} = \frac{n(t, p)}{2E} \int \frac{d^3p'}{2E_p'} \frac{n(t, p')}{(2\pi)^2} \sum_{\text{av spins and } p_X} d\Omega_i |M_{\text{vac}}|^2 (1 + F)_1 (1 + F)_2 \left| \frac{p_i}{4E_2} \right|_{\text{on-shell}}
\]

so that integrating over \( p \),

\[
\frac{1}{a^3(t)} \frac{d(a^3 n(t))}{dt} = \text{creation} - \int \frac{d^3p}{2E} n(t, p) \int \frac{d^3p'}{2E_p'} n(t, p') \sum_{av} \ldots
\]

Again suppose our \( X \) particles are highly non-relativistic so that \( p \ll E \sim M \). Also again suppose that \( M \gg m \) so the products are highly relativistic. Then, \( p_1' \simeq -p_2' \) and so \( E_1 \simeq E_2 \simeq M \) by momentum and energy conservation. The filling factors,

\[
(1 + F)_1 (1 + F)_2 = \left( \frac{1}{1 - e^{-E_1/kT}} \right) \left( \frac{1}{1 - e^{-E_2/kT}} \right)
\]

\[
\simeq \left( \frac{1}{1 - e^{-M/kT}} \right)^2 \sim 1
\]

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again assuming that $kT \ll M$. Then we find,

$$
\frac{1}{a^3(t)} \frac{d}{dt} (a^3 n(t)) \approx \text{creation} - \frac{n^2(t)}{4M^2} \frac{1}{(2\pi)^2} \frac{4\pi \lambda^2 M}{4M}
$$

$$
= \frac{\lambda^2}{16\pi M^2} \left( n_{eq}^2(T) - n^2(t) \right)
$$

$$
= \langle \sigma v \rangle \left( n_{eq}^2(T) - n(t)^2 \right)
$$

(390)

In this example we have derived that,

$$
\langle \sigma v \rangle = \frac{\lambda^2}{16\pi M^2}
$$

(391)

and is therefore constant in time.

Let us again check our units. $\sigma v$ is a rate per volume, and hence should have mass dimension $[\sigma v] = -2$. The Lagrangian must contain the interaction term $\lambda X \bar{X} Z_1 Z_2$ and since $X, \bar{X}, Z_i$ are scalars with dimension 1, then $[\lambda] = 0$ i.e. is dimensionless. This gives, $[\sigma v] = [\lambda^2] - [M^2] = 0 - 2 = -2$ as required.

Suppose instead we were interested in a high temperature limit, where all particles were relativistic, so $M_X \ll kT$. In this case, simply on dimensional grounds we would expect,

$$
\langle \sigma v \rangle = \frac{\lambda^2}{(kT)^2}
$$

(392)