AdS-CFT

Toby Wiseman (Imperial College)
QFFF Special topics 2017

Problem sheet 3: AdS-CFT

1. Consider the $(d+1)$-dimensional planar Schwarzschild-AdS spacetime;

$$ds^2 = \frac{\ell^2}{z^2}\left(-f dt^2 + dx^a dx^a + \frac{1}{f}dz^2\right) \ , \quad f = 1 - \left(\frac{z}{z_0}\right)^d \tag{1}$$

with coordinates $x^A = (t, x^a, z)$. Let us assume the spatial coordinates x^a, $a = 1, \ldots, d-1$, are compactiifed to form a torus with period L, so $x^a \sim x^a + L$.

Firstly show that the horizon at $z = z_0$ has spatial area A equal to,

$$A = \left(\frac{\ell}{z_0}\right)^{d-1} L^{d-1} \tag{2}$$

Recall that the entropy of the black hole is $S = A/4G$.
Secondly consider the Killing vector $K = \partial/\partial t$. The surface gravity κ is given from,

$$\kappa^2 = -\left(\partial^a |K|\right)\left(\partial_a |K|\right)\big|_{horizon} \tag{3}$$

where $|K| = \sqrt{K^2}$. Given the surface gravity, the temperature T (wrt K) is $T = \kappa/2\pi$. Confirm that,

$$T = \frac{d}{4\pi z_0} \tag{4}$$

Hence show that the thermodynamics of a holographic CFT at finite temperature T with effective central charge C, on the space $\mathbb{R}_t \times T^{d-1}$ with the torus periods size L, is given by,

$$S = 4\pi C L^{d-1} \left(\frac{4\pi T}{d} \right)^{d-1} \tag{5}$$

where $C = \ell^{d-1}/(16\pi G)$.

For $\mathcal{N} = 4$ super Yang-Mills we have $C = N^2/(8\pi^2)$. Show that when the gravity dual is valid we should have,

$$S = \frac{\pi^2}{2} N^2 T^3 V \tag{6}$$

where $V = L^3$.

2. Consider a holographic CFT now defined on $\mathbb{R}_\tau \times S^{d-1}$, so on the Einstein static universe. Choose units so that the sphere has unit radius, so,

$$ds^2_{(d)} = -d\tau^2 + d\Omega^2_{(d-1)} \tag{7}$$

A bulk solution is global Schwarzschild-AdS (simply global AdS if $r_0 = 0$);

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2_{(d-1)} \ , \quad f = \frac{r^2}{\ell^2} + 1 - \left(\frac{r_0^2}{\ell^2} + 1\right)\left(\frac{r_0}{r}\right)^{d-2} \tag{8}$$

where r_0 is the parameter determining the size of the black hole, and $t = \ell\tau$. Firstly find a 'defining function' that shows that this metric has the conformal boundary with geometry $ds^2_{(d)}$ above. Then compute the thermodynamics of the black hole, and in particular the free energy $F = E - TS$ as a function of r_0. (*Hint:* Use the 1st law $dE = TdS$ to determine the energy from the temperature and entropy).

Now show that below a certain temperature, T_{min}, no black hole solution exists. Then the only possible bulk solution is global AdS itself.

Further show that there is a critical temperature $T_c > T_{min}$ where the black holes have a lower free energy than global AdS, and hence are favoured over global AdS. This first order phase transition is the 'Hawking-Page' transition.

[*Aside:* When the CFT is a gauge theory this is an example of a (large N) confinement/deconfinement phase transition. The low temperature phase has energy $\sim O(1)$, and the high temperature phase has energy $\sim O(N^2)$, with a first order phase transition between them.]