AdS-CFT

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QFFF Special topics 2017

Solutions for problem sheet 1: CFT

1 a). Let us check; $x^\mu \to x^\mu - a^\mu$. Then $f \to f'(x^\mu) = f(x^\mu - a^\mu)$ so to linear order in a^μ, then $f' = f(x) + \delta f(x)$ where $\delta f = -a^\mu \partial_\mu f$. Now for $P_\mu = i\partial_\mu$,

$$f' = e^{ia^\mu P_\mu} f = f + ia^\mu P_\mu f + O(a^2) = f - a^\mu \partial_\mu f \tag{1}$$

to linear order.

Now take $x^\mu \to \lambda x^\mu = e^{-\epsilon} x^\mu$ where we are interested in infinitessimal $|\epsilon| \ll 1$ so,

$$x^\mu \to x^\mu - \epsilon x^\mu + O(\epsilon^2) \qquad \Longrightarrow \qquad \delta x^\mu = -\epsilon x^\mu \tag{2}$$

Then from above taking $a^\mu = \epsilon x^\mu$ then we have that to linear order,

$$f' = f + ia^\mu P_\mu f + O(a^2) = f + i\epsilon \left(ix^\mu \partial_\mu\right) f + O(\epsilon^2) = e^{i\epsilon D} f \tag{3}$$

and so we can take as the generator of dilatations,

$$D = ix^\mu \partial_\mu \tag{4}$$

1 b). Take,

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2x^\nu b_\nu + b^2 x^2} \tag{5}$$

and linearize in b^μ to obtain,

$$x^\mu \to \left(x^\mu - b^\mu x^2\right)\left(1 + 2x^\nu b_\nu\right) + O(b^2) = x^\mu - \left(b^\mu x^2 - 2x^\mu x^\nu b_\nu\right) + O(b^2) \tag{6}$$

Hence from above $a^\mu = (b^\mu x^2 - 2x^\mu x^\nu b_\nu)$, leading to,

$$\begin{aligned} f' = f + ia^\mu P_\mu f + O(a^2) &= f + i\left(i\left(b^\mu x^2 - 2x^\mu x^\nu b_\nu\right)\partial_\mu\right) f + O(\epsilon^2) \\ &= f + ib^\mu\left(i\left(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu\right)\right) f + O(\epsilon^2) \\ &= e^{ib^\mu K_\mu} f \end{aligned} \tag{7}$$

so that we see

$$K_\mu = i\left(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu\right) \tag{8}$$

1 c). Firstly,

$$\begin{aligned} [D, P_\mu]f = [ix^\nu\partial_\nu, i\partial_\mu]\, f &= -x^\nu\partial_\nu\partial_\mu f + \partial_\mu\left(x^\nu\partial_\nu f\right) \\ &= \left(\partial_\mu x^\nu\right)\partial_\nu f = \delta^\nu_\mu\partial_\nu f = \partial_\mu f \\ &= -i\left(i\partial_\mu f\right) = -iP_\mu f \end{aligned} \tag{9}$$

since partial's commute.

Similar manipulations should give the second commutator.

2. The only possible thing the one point function $< 0|\phi(x)|0 >$ can depend is x, so most generally it might be a function,

$$f(x) =< 0|\phi(x)|0 > \tag{10}$$

However the vacuum is Poincare invariant so that $P_\mu|0 >=< 0|P_\mu = 0$. Then we know,

$$0 =< 0|P_\mu\phi(x)|0 > \tag{11}$$

since $< 0|P_\mu = 0$. But we can commute P and ϕ,

$$0 =< 0|P_\mu\phi(x)|0 >=< 0|\phi(x)P_\mu|0 > + < 0|[P_\mu, \phi(x)]|0 > \tag{12}$$

The first term $< 0|\phi(x)P_\mu|0 >$ again vanishes as $P_\mu|0 >= 0$, leaving,

$$0 =< 0|[P_\mu, \phi(x)]|0 >=< 0|i\partial_\mu\phi(x)|0 >= i\partial_\mu < 0|\phi(x)|0 >= i\partial_\mu f \tag{13}$$

But if $\partial_\mu f = 0$ this implies the result we want, that $f(x)$ =constant.

3. Suppose ϕ is primary with scale dimension $\Delta > 0$. We already know the one-point function is a constant from the last question. Now suppose,

$$c =< 0|\phi(x)|0 > \tag{14}$$

for some constant c. For a CFT the vacuum is annhilated by D. Hence,

$$\begin{aligned} 0 &=< 0|D\phi(x)|0 >=< 0|\phi(x)D|0 > + < 0|[D, \phi(x)]|0 >=< 0|[D, \phi(x)]|0 > \\ &=< 0|i\left(\Delta + x^\mu \partial_\mu\right)\phi(x)|0 >= i\left(\Delta + x^\mu \partial_\mu\right) < 0|\phi(x)|0 >= i\Delta c \end{aligned} \tag{15}$$

since $\partial_\mu c = 0$. However as $\Delta \neq 0$ then c must vanish.

4. The 2-point function of two scalars, $< \phi_1(x)\phi_2(y) >$ can only be a function of x^μ, y^μ. Furthermore by Lorentz invariance it can only be a scalar function of these. By translation invariance it can only depend on the difference $x^\mu - y^\mu$. Hence it must simply be a function of $(x-y)^2$. So we must have,

$$f((x-y)^2) =< 0|\phi_1(x)\phi_2(y)|0 > \tag{16}$$

for some function f. Due to translation invariance we can w.l.o.g. set one of the arguments to be at the origin, so,

$$f(x^2) =< 0|\phi_1(0)\phi_2(x)|0 > \tag{17}$$

As the fields have definite scale dimensions Δ_i, then D has the action on them as,

$$[D, \phi_i(x)] = i\left(\Delta_i + x^\mu \partial_\mu\right)\phi_i(x) \tag{18}$$

Now D annhilates the vacuum so that,

$$\begin{aligned} 0 &=< 0|D\phi_1(0)\phi_2(x)|0 >=< 0|\phi_1(0)D\phi_2(x)|0 > + < 0|[D, \phi_1(0)]\phi_2(x)|0 > \\ &=< 0|\phi_1(0)D\phi_2(x)|0 > + < 0|i\Delta_1\phi_1(0)\phi_2(x)|0 > \\ &=< 0|\phi_1(0)\phi_2(x)D|0 > + < 0|\phi_1(0)[D, \phi_2(x)]|0 > +i\Delta_1 f(x^2) \\ &=< 0|\phi_1(0)i\left(\Delta_2 + x^\mu \partial_\mu\right)\phi_2(x)|0 > +i\Delta_1 f(x^2) \\ &= i\left(\Delta_2 + x^\mu \partial_\mu\right) < 0|\phi_1(0)\phi_2(x)|0 > +i\Delta_1 f(x^2) \\ &= i\left(\Delta_1 + \Delta_2 + x^\mu \partial_\mu\right) f(x^2) \end{aligned} \tag{19}$$

Now,

$$x^\mu \partial_\mu f(x^2) = f'(x^2) x^\mu \partial_\mu \left(x^2\right) = 2x^2 f'(x^2) \tag{20}$$

Hence we see,

$$(\Delta_1 + \Delta_2)\, f(x^2) + 2x^2 f'(x^2) = 0 \tag{21}$$

Writing $\mu = x^2$,

$$\mu f'(\mu) = -\frac{\Delta_1 + \Delta_2}{2} f(\mu) \qquad \Longrightarrow \qquad f(\mu) = c\mu^{-\frac{\Delta_1+\Delta_2}{2}} \tag{22}$$

for a constant of integration c. Hence we see,

$$< 0|\phi_1(0)\phi_2(x)|0 >= \frac{c}{(x^2)^{\frac{\Delta_1+\Delta_2}{2}}} \tag{23}$$

and by translation,

$$< 0|\phi_1(x)\phi_2(y)|0 >= \frac{c}{((x-y)^2)^{\frac{\Delta_1+\Delta_2}{2}}} \tag{24}$$

5. Now assume they are primary fields so that K_μ has an action of them. As above we use translations and Lorentz invariance to consider the 2-point function, $< 0|\phi_1(0)\phi_2(x)|0 >$. Then by definition, K_μ will annhilate $\phi_1(0)$ (as it is primary) and also the vacuum. Then,

$$\begin{aligned}
0 =& < 0|K_\mu \phi_1(0)\phi_2(x)|0 >=< 0|\phi_1(0)K_\mu\phi_2(y)|0 > \\
=& < 0|\phi_1(0)\phi_2(x)K_\mu|0 > + < 0|\phi_1(0)[K_\mu, \phi_2(x)]|0 > \\
=& < 0|\phi_1(0)[K_\mu, \phi_2(x)]|0 >=< 0|\phi_1(0) i\left(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu - 2x_\mu \Delta_2\right)\phi_2(x)|0 > \\
=& \, ic\left(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu - 2x_\mu\Delta_2\right)(x^2)^{-\frac{\Delta_1+\Delta_2}{2}}
\end{aligned} \tag{25}$$

Now,

$$\begin{aligned}
\partial_\mu (x^2)^{-\frac{\Delta_1+\Delta_2}{2}} &= -\frac{\Delta_1+\Delta_2}{2}(x^2)^{-1-\frac{\Delta_1+\Delta_2}{2}}\partial_\mu(x^2) \\
&= -(\Delta_1+\Delta_2)(x^2)^{-1-\frac{\Delta_1+\Delta_2}{2}} x_\mu
\end{aligned} \tag{26}$$

so that,

$$\begin{aligned}
x^2\partial_\mu (x^2)^{-\frac{\Delta_1+\Delta_2}{2}} &= -(\Delta_1+\Delta_2)(x^2)^{-\frac{\Delta_1+\Delta_2}{2}} x_\mu \\
x_\mu x^\nu \partial_\nu (x^2)^{-\frac{\Delta_1+\Delta_2}{2}} &= -(\Delta_1+\Delta_2)(x^2)^{-\frac{\Delta_1+\Delta_2}{2}} x_\mu
\end{aligned} \tag{27}$$

Thus we see,

$$
\begin{aligned}
0 &= c\left(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu - 2x_\mu\Delta_2\right)(x^2)^{-\frac{\Delta_1+\Delta_2}{2}} \\
&= c\left((1-2)\left(-(\Delta_1+\Delta_2)(x^2)^{-\frac{\Delta_1+\Delta_2}{2}}x_\mu\right) - 2x_\mu\Delta_2(x^2)^{-\frac{\Delta_1+\Delta_2}{2}}\right) \\
&= c\left((\Delta_1+\Delta_2)\left((x^2)^{-\frac{\Delta_1+\Delta_2}{2}}x_\mu\right) - 2x_\mu\Delta_2(x^2)^{-\frac{\Delta_1+\Delta_2}{2}}\right) \\
&= c(\Delta_1-\Delta_2)\left(x_\mu(x^2)^{-\frac{\Delta_1+\Delta_2}{2}}x_\mu\right) \qquad (28)
\end{aligned}
$$

If $\Delta_1 \neq \Delta_2$ then we must have $c = 0$ and the 2-point function vanishes. If $\Delta_1 = \Delta_2$ then the two point function can be non-trivial, with no constraint on c.