An introduction to AdS-CFT

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1 Introduction

The AdS-CFT correspondence is a remarkable conjecture that certain conformal field theories (CFTs) in d-dimensions are dual to certain gravitational theories, now in at least $(d+1)$-dimensions, in the full quantum sense of duality. By a gravitational theory I mean a theory where spacetime is dynamical, rather than literally Einstein gravity. This spacetime always has particular asymptotics, those of Anti de Sitter spacetime.

In the best understood cases these gravitational theories are string theories, which in certain limits reduce to supergravity. There are other examples - 'higher spin theories' - where these theories have dynamic spacetime, but never a limit that looks like a more conventional Einstein gravity. In all known cases where the gravitational theory has a limit that appears to be a usual theory of gravity and matter (for example a supergravity) then the CFT is strongly coupled and has many degrees of freedom. Note that it is the many degrees of freedom that allow the higher dimensional 'bulk' physics to be packaged into a lower dimensional field theory. In all known cases where the CFT is weakly coupled (or even free) the dual gravitational theory is very exotic or strongly coupled. For this reason it seems that the AdS-CFT conjecture will remain a conjecture in the foreseeable future. However there is much evidence for it in particular examples (perhaps the best following from integrability in the CFT side).

The correspondence is very remarkable in two senses. Firstly in principle it gives a rigorous definition of quantum theories of spacetime, and in certain limits conventional Einstein gravity theories, albeit living in AdS spacetimes.

However this definition appears to be in terms of a strongly coupled field theory so extracting information about quantum gravity remains a challenge, although a well defined one. Certain very general lessons emerge though. For example the bulk spacetime is 'emergent' from the perspective of the CFT - strictly classical spacetime only emerges as the number of CFT degrees of freedom becomes very large. The fact that this quantum spacetime is emergent is probably a very important lesson for quantum gravity. Another example is that black hole information is not lost - the CFT has a perfectly unitary evolution. Again an important lesson.

The second remarkable sense follows from thinking in the reverse direction. Then we learn that in some instances strongly coupled field theories may be described in terms of simple Einstein gravity theories. This is a wonderful repackaging of the most relevant degrees of freedom in the field theory that allows remarkably simple gravity calculation to deduce results that appear to be impossible to obtain directly using usual QFT methods (due to the strong coupling). It is this second sense that has spawned many 'applications of AdS-CFT' to rather diverse areas of physics - for example AdS-CMT, using it to understand features of strongly coupled CFT relevant for condensed matter.

The best understood example in AdS/CFT is the case that the CFT is maximally supersymmetric $U(N)$ Yang-Mills theory in $(3+1)$-dimensions (so called $\mathcal{N}=4$ YM) where the dual description is in terms of IIB string theory in AdS spacetime. The correspondence was discovered by Maldacena building on previous work on D-branes in string theory. In this case, a large number of D3-branes have two descriptions. One is in terms of open strings that end on N D-branes, which yields the $U(N)$ gauge theory. The other is in terms of closed strings, for who the D-branes act as a source of gravitational and charge, and may be understood (in the decoupling limit) in terms of supergravity in AdS spacetime. The equivalence of these two descriptions was what lead to the conjecture, and is a convincing physical argument for it even though we lack a mathematical proof.

In this case the CFT has a simple(ish) Lagrangian and has a coupling g_{YM} that can be made small or large. When the gauge group size $N \to \infty$, and the so called 't Hooft coupling, $\lambda = N g_{YM}^2$ (which is the relevant coupling for large N perturbation theory) is taken to be large then the dual description reduces to supergravity. Then simple supergravity computations allow calculation in the dual strongly coupled gauge theory to be performed. This has had a huge impact in thinking about the physics of heavy ion colli-

sions, such as those in RHIC and ALICE (where the gauge theory governing it, QCD, is strongly coupled and has reasonable $N = 3$, and is not far in form from the $\mathcal{N} = 4$ theory). There are also instances where this CFT can be made to superconduct, and this strongly coupled superconducting behaviour is understood in terms of gravity. Since strongly coupled CFT superconductors are thought by many to underly certain high temperature superconducting materials this has created a huge amount of activity - an example of AdS-CMT. Properties such as transport and full non-perturbative dynamics of the theory is then accessible from simple gravity calculations, eg, using techniques such as numerical GR.

The correspondence is often known as the gauge/gravity duality since in the well known cases the CFT is a gauge theory. However more recently CFTs which are not gauge theories have arisen. It is also known as gauge/string duality, since really the dual is a string theory in the well known examples rather than a usual gravity theory. However, as discussed above, not all correspondences involve string theories (at least in their usual limits). AdS-CFT is actually a term that I don't like so much as there are correspondences that are well understood, but where the QFT is not conformal, and the gravity theory does not live in AdS - in fact there are cases where the QFT is just a quantum mechanics, and there has even been numerical work solving these theories and checking the duality holds. Probably the term 'holography' is the best general term, as it alludes to the fact that the 'bulk' gravity theory lives in more dimensions.

The purpose of these lectures is to give an introduction to the correspondence. While the known examples of AdS-CFT involve string theory or higher spin theories, it is reasonable to believe that any consistent theory of quantum gravity living in AdS must have a CFT description (although it may be a very complicated CFT). This is the thinking behind the 'bottom-up' approach to AdS-CFT, where one takes very simple models of gravity and matter in AdS and posits a dual CFT - for example, surely these are embeddable somewhere in the string landscape?

I will focus on this simple approach and describe the very basic properties of AdS-CFT. These properties all hold in the more complicated 'top-down' settings where actual dualities are known. However the additional physics in these cases often obscures the basics. There are many excellent reviews on AdS-CFT. The classic is that of MAGOO (hep-th/9905111), although it involves a lot of string theory. Maldacena has a nice more accessible review (arXiv:1106.6073). For a review on applications to superconductors

and condensed matter physics see Hartnoll's arXiv:0903.3246 and McGreevy arXiv:0909.0518. For a review on applications to heavy ion physics there is arXiv:1101.0618. I have a fairly recent review on application to curved spacetime QFT arXiv:1312.0612. For a very recent review, see that of Penedones, arXiv:1608.04948.

Plan:

It seems to me a sensible way to introduce AdS-CFT is to talk about AdS and CFTs and then show how they are related. Both AdS and CFTs are interesting in their own right so I hope everyone will find something interesting.

I will start by introducing some basics of CFTs. This is a huge and beautiful subject and I will only introduce the concepts I will use later to establish the correspondence. I will then discuss the notion of AdS spacetime and asymptotic AdS spacetimes. I will discuss the physics of a scalar field in AdS. Following from this we will have all the ideas necessary to state the basic AdS-CFT correspondence. If time permits we'll close by talking about simple thermal physics and the canonical example of $\mathcal{N} = 4$ SYM.

2 What is a CFT?

Consider a Poincare (Translations + Lorentz) invariant QFT on d-dimensional Minkowski spacetime, $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. The action of the translations and Lorentz symmetry on Minkowski is given by,

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu - a^\mu \\ x^\mu &\to x'^\mu = \Lambda^\mu{}_\nu x^\nu \end{aligned} \tag{1}$$

for $\Lambda \in SO(1, d-1)$ and the algebra is,

$$\begin{aligned} [M_{\alpha\beta}, M_{\mu\nu}] &= -i\eta_{\alpha\mu} M_{\beta\nu} + \dots \\ [M_{\mu\nu}, P_\alpha] &= -i\left(\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu\right) \end{aligned} \tag{2}$$

where P_α is the generator of translations and $M^{\mu\nu}$ is the generator of Lorentz transformations. The action of these on a function $f(x)$ is,

$$\begin{aligned} P_\mu f(x) &= i\partial_\mu f \\ M_{\mu\nu} f(x) &= i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) f \end{aligned} \tag{3}$$

If we have a local relativistic QFT it is invariant under these transformations, and built from fields $\Phi(x)$ which are representations of the Poincare group (where we suppress internal indices, such as spin/flavour). These fields have specific transformation properties,

$$\begin{aligned} [P_\mu, \Phi] &= i\partial_\mu \Phi \\ [M_{\mu\nu}, \Phi] &= \left[i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + \Sigma_{\mu\nu}\right] \Phi \end{aligned} \tag{4}$$

where Σ generate the instrinsic spin of the field (a finite representation of Lorentz). The Casimir operator of the Poincare group is the mass operator $m^2 = -P^\mu P_\mu$ which we use to construct representations. For example we might consider Φ to be a free massive or massless scalar field.

Now suppose that in addition we assume the theory is invariant under a scale symmetry,

$$x^\mu \to \lambda x^\mu \tag{5}$$

These transformations are generated by D, the *dilation* or *dilatation* or *scaling* operator, which acts on a function as,

$$Df(x) = ix^\mu \partial_\mu f \tag{6}$$

and hence obeys,

$$\begin{aligned}[M_{\alpha\beta}, D] &= 0 \\ [P_\mu, D] &= iP_\mu \end{aligned} \tag{7}$$

Consider the transformation of a field which forms a representation of Poincare and dilations. It may be more general than that of a function. In addition to the change of coordinate, the field itself may also scale, so,

$$\Phi(x) \to \Phi'(x) = \lambda^\Delta \Phi(\lambda x) \tag{8}$$

where Δ is called the *scaling dimension* of the field. Then the generator acts on the field as,

$$[D, \Phi(x)] = i\left(\Delta + x^\mu \partial_\mu\right)\Phi(x) \tag{9}$$

which we note preserves the brackets of D with $M_{\mu\nu}$ and P_μ above. Note that the mass operator $m^2 = -P^\mu P_\mu$ can no longer be a Casimir operator of Poincare and dilations, as it does not commute with D. Hence a representation of the symmetry group cannot have a definite mass unless it vanishes. A consequence of this is that we should not expect there to be 'particles' in the usual sense that we can scatter, and hence an S-matrix. The interactions in a QFT with scale invariance will necessarily have a very long range, and generically mean such an S-matrix description is problematic.

A simple example of such a theory is a free scalar field in 2-dimensions,

$$S = \int dtdx(\partial_\mu \Phi)^2 \tag{10}$$

where we see if,

$$\Phi \to \lambda^0 \Phi \tag{11}$$

so that $\Delta = 0$ then the theory is scale invariant. An interacting theory must have only dimensionless couplings as otherwise the coupling constant will break scale symmetry. This is analogous to the fact that a field cannot have a well defined mass (since it may be thought of as a dimenionful coupling). Examples of classically scale invariant theories are Yang-Mills theory in 4-dimensions and massless Φ^4 theory in 4-dimensions, so,

$$S = \int dtd^3x\left[(\partial_\mu \Phi)^2 - \alpha\Phi^4\right] \tag{12}$$

where α is a dimensionless coupling and now we have invariance if,

$$\Phi(x) \to \Phi' = \lambda^1 \Phi(\lambda x) \tag{13}$$

so that $\Delta = 1$. Transformations are always confusing. Let us check this taking $\Phi \to \Phi'$!

$$\begin{aligned} S \to S' &= \int d^4x \left[(\partial_\mu \Phi')^2 - \alpha \Phi'^4\right] = \int d^4x \left[(\lambda^2 \partial_\mu \Phi(\lambda x))^2 - \alpha \lambda^4 \Phi^4(\lambda x)\right] \\ &= \int (\lambda^{-4} d^4x') \left[(\lambda^4 \partial'_\mu \Phi(x'))^2 - \alpha \lambda^4 \Phi^4(x')\right] \end{aligned} \tag{14}$$

with $x' = \lambda x$, noting $\partial/\partial x^\mu = \lambda \partial/\partial x'^\mu$. Classically we see that the scaling dimension Δ is just the naive 'engineering' dimension of the field.

An important point is that classical scaling symmetry is generally anomalous and broken in the quantum theory. This is perhaps not surprising thinking about the most physical regulator which is an energy cut-off which totally destroys scaling. For example both these interacting examples are not conformal in the quantum theory - although the free 2-dimensional example is still conformal.

Suppose we do have a theory that is scale invariant quantum mechanically. The crucial point is that under weak assumptions, scale invariance leads to the enhancement of symmetry to the *conformal symmetry group*. For $d > 2$ in addition to Poincare and dilations one has invariance under an additional *special conformal transformation*,

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x^\mu a_\mu + a^2 x^2} \tag{15}$$

whose generator K^μ has the action on a function,

$$K_\mu f(x) = i\left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu\right) f \tag{16}$$

and forms a closed algebra with $M_{\mu\nu}, P_\mu, D$ - the *conformal algebra*;

$$\begin{aligned} [M_{\alpha\beta}, K_\mu] &= -i\left(\eta_{\alpha\mu} K_\beta - \eta_{\beta\mu} K_\alpha\right) \\ [P_\alpha, K_\beta] &= 2iM_{\alpha\beta} - 2i\eta_{\alpha\beta} D \\ [D, K_\mu] &= iK_\mu \end{aligned} \tag{17}$$

If we take the action of K_μ on the field Φ to be,

$$[K_\mu, \Phi(x)] = \left[i\left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2x_\mu \Delta\right) - 2x^\nu \Sigma_{\mu\nu}\right] \Phi(x) \tag{18}$$

then one can check this gives a representation of the conformal algebra above.

A *conformal field theory* is a theory that is invariant under the conformal group above, generated by $M_{\mu\nu}, P_\mu, D, K_\mu$ which act on the operators in the quantum field theory to give a representation of the conformal group. Note that in $d = 2$ the conformal group is in fact infinite dimensional and hence conformal symmetry becomes a very powerful constraint on a QFT - however here that will not play an important role.

The conformal algebra is in fact the algebra $SO(2, d)$ (leaving the metric $- - + + \ldots +$ invariant in analogy with Lorentz $SO(1, d-1)$ which leaves $- + + \ldots +$ invariant). This is seen by taking generators J_{ab} with $a, b = 0, 1, \ldots, d+1$ with,

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu} \\ J_{\mu d} &= \frac{1}{2}\left(K_\mu - P_\mu\right) \\ J_{\mu(d+1)} &= \frac{1}{2}\left(K_\mu + P_\mu\right) \\ J_{(d+1)d} &= D \end{aligned} \tag{19}$$

so that,

$$[J_{ab}, J_{cd}] = i\left(G_{bc}J_{ad} - G_{ac}J_{bd} - G_b J_{ac} + G_{ad}J_{bc}\right) \tag{20}$$

where $G_{ab} = \mathrm{diag}(- - + + \ldots +)$.

2.1 Primary fields

From the algebra we see that P_μ and K_μ may be thought of as raising and lowering operators with respect to D. We can build representations of the conformal group by considering operators which are eigenfunctions of D and spin with eigenvalue $-i\Delta$ under D, and the usual $(l, m, \ldots)$ under spin - we call these *operators with dimension* Δ. Then P_μ and K_μ act to increase or decrease Δ by one. We are interested in representations which have a lower bound on Δ (this is constrained by unitarity). Hence there must be an operator in the representation which is annhilated by the lowering operator K_μ. This is called a *primary operator*. The other operators in the representation are built by acting with P_μ and the spin raising/lowering operators.

Consider our example of the field $\Phi(x)$ above. Then the operator given by $\Phi(0)$, ie. $\Phi(x)$ at the origin in Minkowski (about which we defined our

conformal generators) is a primary operator since $[D, \Phi(0)] = -i\Delta\Phi(0)$ and $[K, \Phi(0)] = 0$. One may then check that acting with P_μ to give the operator $O = A^{\mu_1 \cdots \mu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} \Phi(0)$ gives a dimension $(\Delta + n)$ operator, for some A which is taken to be a representation of the Lorentz group determining the spin of O; for example, $\eta^{\mu\nu} \partial_\mu \partial_\nu \Phi(0)$ is a scalar operator with dimension $(\Delta + 2)$.

We describe our field $\Phi(x)$ as a *primary field*. Its value at a point gives a primary operator with some spin with respect to the conformal generators centered at that point. Terms in the Taylor expansion of the field about the point gives the other operators in the representation generated by this primary. In the example of the free 2-dimensional scalar the basic field itself is a primary scalar field.

A field may have definite scaling dimension but not be primary. For such a field we will still have the relation in equation (9), but not the action in equation (18) for K_μ.

For a CFT the vacuum state $|0>$ is a conformally invariant state. In particular it is annhilated by the generators of the algebra; eg. $P_\mu|0>= D|0>= 0$, likewise $<0|D = 0$ etc...

2.2 Correlators

A crucial point in CFT is that the correlators of fields that have scale dimension, or are primary, are constrained by conformal invariance of the theory.

A basic property of fields with scaling dimension is that,

$$\langle 0|\phi_1(x_1)\phi_2(x_2)\ldots|0\rangle \to \lambda^{\Delta_1+\Delta_2+\cdots} \langle 0|\phi_1(\lambda x_1)\phi_2(\lambda x_2)\ldots|0\rangle \tag{21}$$

under a dilation.

Perhaps surprisingly the spatial form of the 2 point correlators is completely fixed. Consider a scalar field ϕ with scaling dimension Δ (not necessarily primary). Then,

$$< \phi(x)\phi(y) >= \frac{c}{(x-y)^{2\Delta}} \tag{22}$$

for some constant c. Suppose we have scalar primary fields ϕ_1 and ϕ_2 with different scaling dimensions. Then the two point function $< \phi_1(0)\phi_2(x) >= 0$ vanishes. Suppose all the scalar primary fields in the theory with the same

dimension Δ are labelled ϕ_i.Then the two point function of these must take the form,

$$< \phi_i(x)\phi_j(y) >= \frac{d_{ij}}{((x-y)^2)^\Delta} \tag{23}$$

for constant real symmetric matrix d_{ij}. Usually we then take linear combinations of fields to diagonalize d_{ij} so that, $< \phi_i(x)\phi_j(y) >= \delta_{ij}/(x-y)^{2\Delta}$. Similarly the 3 point function is constrained. Consider now all scalar primaries ϕ_a with dimension Δ_a, then,

$$< \phi_a(x_1)\phi_b(x_2)\phi_c(x_3) >= \frac{c_{abc}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}} \tag{24}$$

for constants c_{abc}. We have similar relations for primary operators with spin too. 4 point functions are less constrained as with 4 x's it is possible to write down two conformally invariant expressions. The correlation function is then an undetermined function of these. Still, reducing all two and three point functions of primaries to some constants is pretty powerful.

2.3 Stress tensor

An important operator in all CFTs is the stress tensor $T_{\mu\nu}$ (which is conserved $\partial^\mu T_{\mu\nu} = 0$). There may possibly also be conserved currents j_μ associated to global symmetries (also conserved). These operators have scaling dimension $\Delta_T = d$ and $\Delta_J = d-1$ which are equal to their engineering dimension. This implies there is no quantum correction to the dimension. The lack of quantum correction is due to conservation, and would otherwise not be expected. Note that while these operators have definite scaling dimension - they are not primary operators (ie. not annhilated by K_μ).

The stress tensor exists for any CFT. Conformal symmetry implies that it must be traceless. The stress tensor is defined by the variation of the action S under a perturbation to the metric $\delta g_{\mu\nu}$ as,

$$\delta S = \int d^d x T_{\mu\nu}\delta g^{\mu\nu} \tag{25}$$

From this we see it must have scaling dimension $\Delta = d$, since the action is scale invariant (as the theory is conformal). Then for the dilation transformation to Minkowski we have,

$$x^\mu \to \lambda x^\mu \quad \Longrightarrow \quad ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \to \lambda^2\eta_{\mu\nu}dx^\mu dx^\nu \tag{26}$$

so that,

$$\delta g_{\mu\nu} = 2\epsilon\eta_{\mu\nu} \tag{27}$$

where we write $\lambda = e^{\epsilon}$ and consider $|\epsilon| \ll 1$, so a small dilation. Then we see,

$$\delta S = \int d^d x T^{\mu}_{\mu}\epsilon \tag{28}$$

but if dilation is a symmetry we must have $\delta S = 0$, hence $T^{\mu}_{\mu} = 0$. [Note more carefully we could have $T^{\mu}_{\mu} = \partial_\mu V^\mu$ of a vector operator with dimension $\Delta_V = d - 1$, but if V^μ has its engineering dimension we would expect it to be conserved, so $\partial_\mu V^\mu = 0$.]

Further since the stress tensor has dimension $\Delta = d$ then its two point function is constrained to be;

$$< T_{\mu\nu}(0)T_{\sigma\rho}(x) >= \frac{C}{(x^2)^d} I_{\mu\nu,\sigma\rho} \tag{29}$$

where $I_{\mu\ldots}$ is a tensor determined as,

$$\begin{aligned} I_{\mu\nu,\sigma\rho} &= \frac{1}{2}\left(A_{\mu\sigma}A_{\nu\rho} + A_{\mu\rho}A_{\nu\sigma}\right) - \frac{1}{d}\eta_{\mu\nu}\eta_{\sigma\rho} \\ A_{\mu\nu} &= \eta_{\mu\nu} - \frac{2}{x^2}x_\mu x_\nu \end{aligned} \tag{30}$$

The constant C can roughly be thought of as characterizing the number of physical degrees of freedom in the theory. For example in a CFT built from N free fields then $C \propto N$. It is referred to as the effective central charge (in 2d it is the central charge).

2.4 Sources

One way to think about getting interesting behaviour from a QFT is to put it into a state which is not the vacuum. However another way to think about it is to take it in its vacuum, but deform the Lagrangian of the theory by sources for local operators. Consider the path integral for a theory with a field $\Phi(x)$, so,

$$Z = \int D\Phi\, e^{i\int d^d x L[\Phi]} \tag{31}$$

(setting $\hbar = 1$). Recall that for an operator O we compute the vev in vacuum by taking,

$$\langle O \rangle = \frac{\int D\Phi \, O e^{i \int d^d x L[\Phi]}}{Z} \tag{32}$$

We may deform the theory by a source $J(x)$ for the field Φ as,

$$Z[J] = \int D\Phi \, e^{i \int d^d x (L[\Phi] + J\Phi)} \tag{33}$$

Now Z is a functional of the source called the generating function. We may now use this source to generate correlation functions, by differentiating and setting the source to zero,

$$\frac{-i}{Z} \frac{\partial}{\partial J(x_1)} Z \bigg|_{J=0} = \frac{\int D\Phi \, \phi(x_1) e^{i \int d^d x (L[\Phi])}}{\int D\Phi \, e^{i \int d^d x (L[\Phi])}} = \langle \phi(x_1) \rangle \tag{34}$$

Continuing to differentiate again gives the two point function;

$$\frac{1}{Z} \left(-i \frac{\partial}{\partial J(x_1)} \right) \left(-i \frac{\partial}{\partial J(x_2)} \right) Z \bigg|_{J=0} = \langle \phi(x_1) \phi(x_2) \rangle \tag{35}$$

and so on. More generally we may think of the source as a physical 'external field' as we are familiar with in condensed matter. For example the external magnetic field is a source for the spins in a ferromagnet. In a gauge theory the gauge field A_μ couples to a current j^μ (which is conserved due to gauge invariance). We can think of this current as a source for the gauge field - in Maxwell's equations it literally gives the source term $\nabla^\mu F_{\mu\nu} = j_\nu$.

We may deform a CFT by sources in exactly the same manner. In fact this is a rather natural way to think about a CFT since there are not generally asymptotically free particles to scatter. In this case we may add a source term J for an operator O with dimension Δ, so that,

$$Z_{CFT}[J] = \int DX \, e^{i \int d^d x (L_{CFT} + JO)} \tag{36}$$

where DX is the measure integrating over the fields entering the Lagrangian. Since the Lagrangian must have scaling dimension d so that the action is invariant, then the source J must scale with dimension $(d - \Delta)$.

3 What is AdS spacetime?

A nice way to think of $(d+1)$-dimensional AdS spacetime is as a hypersurface - a hyperboloid radius ℓ,

$$U^2 + V^2 - X^i X^i = \ell^2 \tag{37}$$

with $i = 1, \ldots d$ embedded in $(d+2)$-dimensional Minkowski spacetime,

$$ds^2 = -dU^2 - dV^2 + dX^i dX^i \tag{38}$$

This surface has closed time like curves - the circles at X^i =const are timelike. However if we unwrap these circles, one obtains the 'universal cover' which is AdS spacetime.

Taking,

$$\begin{aligned} U &= \ell \cosh\rho \cos\tau \\ V &= \ell \cosh\rho \sin\tau \\ X_i &= \ell \sinh\rho \Omega_i \end{aligned} \tag{39}$$

where Ω_i parameterize a $(d-1)$-sphere, $\sum_i \Omega_i^2 = 1$, then one can check,

$$ds^2 = \ell^2 \left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{(d-1)} \right) \tag{40}$$

Then τ is the time coordinate, and in the universal cover $\tau \in \mathbb{R}$ rather than being an angle. This is known as 'global AdS'. We often write this as,

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2_{(d-1)} \ , \quad f = \frac{r^2}{\ell^2} + 1 \tag{41}$$

using $r = \ell \sinh\rho$ and $t = \ell\tau$.

The isometries of the spacetime are easily seen in the embedding to be $SO(2,d)$, although in the actual $(d+1)$-coordinates it is less obvious, with only the (maximal compact) subgroup $SO(2) \times SO(d)$ being manifest.

From the first form we see that as $\rho \to \infty$ there is an asymptotic region where,

$$ds^2 \to \ell^2 \left(e^{2\rho} \left(-d\tau^2 + d\Omega^2 \right) + d\rho^2 \right) \tag{42}$$

so time and the sphere 'blow up'. In fact a space which can be written as,

$$ds^2 = \frac{1}{Z^2(x)} g_{\mu\nu}(x) dx^\mu dx^\nu \tag{43}$$

where $g_{\mu\nu}$ is a regular metric and Z is called the 'defining function' and goes to zero 'linearly' on some hypersurface so that $dZ \neq 0$ there is called conformally compact. It has an asymptotic region that we call the conformal boundary. Its geometry is conformal to the induced geometry of the slice of $g_{\mu\nu}$ where $Z = 0$. By taking $e^{\rho} = \frac{1}{z}$ we see our metric is of this form;

$$ds^2 \to \frac{\ell^2}{z^2}\left(-d\tau^2 + d\Omega^2 + dz^2\right) \tag{44}$$

where $Z = \frac{1}{\ell}z$ and so the metric has a conformal boundary at the hypersurface $Z = z = 0$. The conformal boundary has a geometry given by (the conformal class of) the metric $g_{\mu\nu}|_{Z=0}$ at the boundary. Note that this conformal boundary is only defined up to a conformal factors - ie. up to the choice of Z which may be any linearly vanishing function. So we see here that AdS has a conformal boundary that is (conformal to),

$$ds_b^2 \to -d\tau^2 + d\Omega^2 \tag{45}$$

so $\mathbb{R} \times S^{d-1}$, time cross a unit sphere (the Einstein static universe).

An important point about a conformal boundary is that a spacelike curve that extends to the boundary must have infinite distance. However, a null geodesic reaches the boundary at finite time τ. This is because null geodesics are not affected by a conformal transformation of the metric - for them they see the space $g_{\mu\nu}$ which we can think of as having a 'real' boundary at $Z = 0$.

We will focus here on a chart of AdS which does not cover the whole space. This is called the Poincare patch or Poincare-AdS and is found from taking,

$$\begin{aligned} U &= \frac{z}{2}\left(1 + \frac{1}{z^2}\left(\ell^2 + (x^a)^2 - t^2\right)\right) \\ V &= \frac{\ell}{z}t \\ X^a &= \frac{\ell}{z}x^a \\ X^d &= \frac{z}{2}\left(1 - \frac{1}{z^2}\left(\ell^2 - (x^a)^2 + t^2\right)\right) \end{aligned} \tag{46}$$

for $a = 1, \ldots, d-1$. Then this covers half the hyperboloid and induces the

metric,

$$ds^2 = g_{AB}dx^A dx^B = \frac{\ell^2}{z^2}\left(-dt^2 + dx^a dx^a + dz^2\right) = \frac{\ell^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \tag{47}$$

writing $x^\mu = (t, x^a)$, $x^A = (x^\mu, z)$. We can immediately read off that the conformal boundary is d-dimensional Minkowski, identifying $Z = z/\ell$ again, with the boundary at $z = 0$.

In fact Poincare-AdS can be 'derived' from global AdS by zooming in on a portion of global AdS near the boundary. Then one looses sight of the fact the boundary is spatially a sphere, and it looks flat. So $z = 0$ is a conformal boundary - what about $z \to \infty$ - what happens there? We'll return to that in a minute.

How does the AdS symmetry look in these coordinates? One can show that its action on the Minkowski boundary precisely reproduces the conformal symmetry group action we saw earlier. This provides a precise link between AdS spaces and conformal symmetry. Importantly for us, we easily see an isometry,

$$z \to \lambda z \ , \quad x^\mu \to \lambda x^\mu \tag{48}$$

which clearly leaves the Poincare-AdS metric invariant. Restricted to the boundary this has the action $x^\mu \to \lambda x^\mu$ which is precisely the dilaton operation on Minkowski.

The AdS space is an Einstein space meaning that writing,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{49}$$

it solves the equation,

$$R_{\mu\nu} = \Lambda g_{\mu\nu} = -\frac{d-1}{\ell^2}g_{\mu\nu} \tag{50}$$

so the Ricci tensor is proportional to the metric. We may think of this as the Einstein equation where the matter is a negative cosmological constant, $\Lambda \sim -\frac{1}{\ell^2}$. Hence the 'Anti' in AdS. In fact the AdS spacetime is so symmetric that its Riemann tensor is simply given in terms of the metric;

$$R_{\mu\nu\alpha\beta} \sim \Lambda g_{\mu\alpha}g_{\nu\beta} + \ldots \tag{51}$$

so that its Weyl tensor vanishes.

3.1 Asymptotically AdS spacetimes; a bottom-up view

More generally we say an asymptotically AdS space is one that has a conformal boundary and near that it shares the same geometry as AdS. More precisely we say it is a spacetime that has a conformal boundary and locally approaches an Einstein space with vanishing Weyl tensor near that boundary.

This spacetime will be governed by dual gravitational theory and generally in AdS-CFT would really be quantum (whatever that means). In bottom-up approaches to AdS-CFT we usually imagine a setting where the CFT is described by a dual gravity theory comprising semiclassical Einstein gravity, perhaps coupled to some matter fields.

Something that is generally true is that in any realization of AdS-CFT where the bulk theory is described by some conventional gravity theory (eg supergravity in dimensions greater than $(d+1)$) we may always turn off matter fields and truncate to simple pure gravity with a negative cosmological constant in $(d+1)$-dimensions, so,

$$R_{\mu\nu} = -\frac{d-1}{\ell^2} g_{\mu\nu} \tag{52}$$

Let us discuss an important example of an asymptotically AdS spacetime that solves these equations, in the knowledge that it will embed into more complicated theories.

The AdS-Schwarzschild spacetime is a simple generalization of AdS in global coordinates which solves these Einstein equations,

$$f = \frac{r^2}{\ell^2} + 1 - \frac{\alpha}{r^{d-2}} \tag{53}$$

Then writing $\alpha = \left(\frac{r_0^2}{\ell^2} + 1\right) r_0^{d-2}$ we see there is a horizon at $r = r_0$ which is much like the usual Schwarzschild horizon and in particular its spatial section is a (d-1)-sphere. For large r this spacetime just looks like global AdS, and hence is asymptotically AdS.

In our case of interest, Poincare-AdS, the black hole metric solving the Einstein equation above is simply,

$$ds^2 = \frac{\ell^2}{z^2}\left(-f dt^2 + dx^a dx^a + \frac{1}{f} dz^2\right) \ , \quad f = 1 - \left(\frac{z}{z_0}\right)^d \tag{54}$$

and there is a horizon at $z = z_0$. Now unlike usual Schwarzschild we see the spatial geometry of the horizon is simply flat space $\mathbb{R}^{d-1}$ which is perhaps

surprising. A standard calculation (ie. computing the surface gravity) shows the Hawking temperature of the horizon (defined with respect to $\partial/\partial t$) is,

$$T_{Hawking} = \frac{d}{4\pi z_0} \tag{55}$$

This temperature is the temperature the space has as observed by someone at the conformal boundary - the asymptotic region - whose time translation is generated by $\partial/\partial t$. For large z we see the same behaviour as for Poincare AdS, and hence this is an example of a spacetime which is asymptotic to AdS, but not AdS. We can also compute the Bekenstein-Hawking entropy, given in terms of the area A of the horizon,

$$S_{BH} = \frac{A}{4G} \tag{56}$$

This is divergent since horizon is planar - but the entropy density is finite.

Let us return to what $z \to \infty$ is for Poincare AdS. Taking the black hole above to zero temperature $z_0 \to \infty$ we recover Poincare-AdS. We also see that $z \to \infty$ is a zero temperature horizon - more precisely what is known as an extremal horizon. This is a null hypersurface with zero surface gravity.

Geometrically the g_{tt} component of the metric is very important. It determines the relative redshift for an observer at constant spatial position in the bulk. We see that since $g_{tt} \to \infty$ near the boundary, an observer sitting near the boundary sees events further away in the bulk as being redshifted. So in this spacetime we might say physics happening near the boundary has high energy, physics far from the boundary is nearer to the extremal horizon, and has low energy. Both $z \to 0$ and $z \to \infty$ are an infinite distance away from any point in the interior for a spatial curve.

Another important point is that to sit at constant spatial position you must accelerate - it is not geodesic. A particle initially at rest will follow a geodesic that 'falls' to increasing z - 'to lower energy'.

3.2 A bulk scalar field

Let us consider the behaviour of a massive scalar field $\phi(x^\mu, z)$ in Poincare-AdS. We will take a bottom-up perspective imagining the theory governing the bulk is semiclassical Einstein coupled to the scalar. We will consider its fluctuations, taking them small so we can ignore backreaction. Then we

simply have to solve the equation,

$$\nabla^2\phi = m^2\phi \tag{57}$$

which derives from the bulk action,

$$S = -\frac{1}{2}\int \sqrt{g}\left(g^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2\right) \tag{58}$$

for g_{AB} the Poincare AdS metric before. Let us proceed by Fourier decomposing in the $x^\mu = (t, x^a)$ directions, so we consider a mode,

$$\phi(x^A) = e^{ik_\mu x^\mu} f(z) \tag{59}$$

This reduces to the ode,

$$f''(z) - \frac{d-1}{z}f'(z) - \left(k^2 + \frac{m^2\ell^2}{z^2}\right) f = 0 \tag{60}$$

and then $k = \sqrt{k^2}$ and writing $f(z) = z^{\frac{d}{2}}h(kz)$ and defining $\mu = kz$ this reduces to the modified Bessel equation,

$$\mu^2 h''(\mu) + \mu h'(\mu) - \left(\mu^2 + \nu^2\right) h = 0 \ , \quad \nu^2 = m^2 + \frac{d^2}{4} \tag{61}$$

with general solution,

$$h(\mu) = aK_\nu(\mu) + bI_\nu(\mu) \tag{62}$$

Recall that,

$$I_\nu(\mu) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)}\left(\frac{\mu}{2}\right)^{2n+\nu} \ , \quad K_\nu(\mu) = \frac{\pi}{2}\frac{I_{-\nu}(\mu) - I_\nu(\mu)}{\sin(\nu\pi)} \tag{63}$$

Hence for generic ν then $I_\nu \sim \mu^\nu$ and $K_\nu \sim \mu^{-\nu} + \mu^{+\nu}$ as $\mu \to 0$. Recall also that $I_\nu \sim \frac{1}{\sqrt{\mu}}e^\mu$ and $K_\nu \sim \frac{1}{\sqrt{\mu}}e^{-\mu}$ as $|\mu| \to \infty$. Thus we see near the boundary, $z \sim 0$, the two solutions for f have expansions,

$$f(z) = az^{\frac{d}{2}-\sqrt{\ell^2m^2+\frac{d^2}{4}}}\left(1+\ldots\right) + c(a,b)z^{\frac{d}{2}+\sqrt{\ell^2m^2+\frac{d^2}{4}}}\left(1+\ldots\right) \tag{64}$$

where the $\ldots$ are a power series expansion in z^2 and the constant $c(a,b)$ is a particular linear combination of a and b. We write these as,

$$f(z) = az^{d-\Delta}\left(1+\ldots\right) + cz^{\Delta}\left(1+\ldots\right) \tag{65}$$

where $\Delta = \frac{d}{2} + \sqrt{\ell^2 m^2 + \frac{d^2}{4}}$ is the (greater) root of the quadratic equation;

$$\Delta\left(\Delta - d\right) = \ell^2 m^2 \tag{66}$$

Note that we have called this Δ but at the moment have not linked it to our earlier CFT discussion of scaling dimension.

A careful analysis of this system shows that to have good dynamics we require Δ to be real (to avoid terribly oscillatory behaviour near $z = 0$, and in fact also $\Delta \geq \frac{d-2}{2}$ (cf unitarity!). Drawing a graph of m^2 against Δ we see that in fact m^2 may be negative, with $m^2 > -\frac{d^2}{4}$ (this is the 'BF' bound). It seems odd that mass squared can become negative. In fact this is because in many respects AdS behaves like a box, whose size is $\sim \ell$. The box acts to cut off long wavelength behaviour. A tachyonic field may become stable if it is placed in a small enough box - thus AdS allows a range of tachyonic masses to give stable behaviour.

Now we need some boundary conditions to fix up the constants of integration. We may think of the two constants as being data for the Fourier mode associated to there being two important asymptotic regions. Firstly the conformal boundary at $z = 0$, and secondly the past (extremal) horizon of Poincare AdS, at $z \to \infty$ and $t \to -\infty$.

The fact that the conformal boundary requires boundary conditions is associated to the fact that null rays can reach it in finite affine parameter. Likewise high energy modes of the scalar field will get near the boundary in a finite time, and we have to say how the scalar behaves there. Again it is a bit like for a scalar field in the box, we must fix the value of the scalar field on the walls of the box.

Let us for simplicity assume that $\Delta > d/2$ (this includes negative m^2). One can handle more negative Δ but it is a little more subtle. Then the term $z^{d-\Delta}$ dominates that going as z^{Δ} as $z \to 0$. We then want to fix this leading behaviour of the scalar field near the boundary $z = 0$.

In real space we see,

$$\phi(x, z) = \phi_0(x) z^{d-\Delta} + \phi_2(x) z^{d+2-\Delta} + \ldots + \phi_d(x) z^{\Delta} + \ldots \tag{67}$$

with the $\ldots$ being ascending powers of z (as we assumed $\Delta > d/2$), and so we wish to fix the function $\phi_0(x)$ over the boundary spacetime. For $\Delta = d$ this is simply the value of the scalar at $z = 0$, but for other Δ it is just the leading behaviour (and can even diverge for $\Delta > d$!). Note that all the ϕ_n

for $n < d$ are determined by ϕ_0. Then ϕ_d isn't, being the other data, and after that all higher terms are determined by the pair $\phi_{0,d}$.

Having fixed ϕ_0 we then may solve the wave equation to see the response of the scalar field, which is characterized by ϕ_d, the coefficient of z^Δ. So it is a little different to a scalar in flat space which is thought of only in terms of initial data. Here we have boundary data to fix too. We also have initial data as we discuss shortly associated to the past horizon.

Before we proceed it is interesting to note that under the bulk isometry $z \to \lambda z$, $x^\mu \to \lambda x^\mu$ which leaves the metric invariant the scalar field must just transform as,

$$\phi(x, z) \to \phi(\lambda x, \lambda z) \tag{68}$$

We may think of the functions $\phi_0(x), \phi_d(x)$ as living on the boundary spacetime where this bulk isometry acts as a dilation. How do the terms $\phi_0(x)$ and $\phi_d(x)$ change if we read them off from a solution after performing such a bulk isometry? We see that under this dilation in fact a, b do not transform as functions but have a scaling dimension,

$$\phi_0(x) \to \lambda^{d-\Delta}\phi_0(\lambda x)\ , \quad \phi_d(x) \to \lambda^\Delta \phi_d(\lambda x) \tag{69}$$

Fixing ϕ_0 we think of as a boundary condition, and then we see ϕ_d as the response to it. Given our discussion above it might be natural to think of $\phi_d(x)$ as a scalar field on the boundary with dimension Δ, and then $\phi_0(x)$ can be interpreted as its source field, with dimension $(d - \Delta)$, which pushes it to have some dynamics.

Now we must fix the initial condition for the field on the past extremal horizon at $z \to \infty$, $t \to -\infty$. We have seen that as $z \to \infty$,

$$\phi \sim \frac{az^{\frac{d}{2}}}{\sqrt{kz}} e^{ik^\mu x_\mu - kz}\,(1 + \ldots) + \frac{cz^{\frac{d}{2}}}{\sqrt{kz}} e^{ik^\mu x_\mu + kz}\,(1 + \ldots) \tag{70}$$

where $k = \sqrt{k^\mu k_\mu}$. For wavenumbers k^μ which are spacelike, so k is real in the source we expect the field to be regular, and hence exponentially decay near the horizon rather than blow up. For k^μ timelike, so $k = \sqrt{k^2}$ is pure imaginary we want the perturbation to be ingoing at the future horizon, with no outgoing past horizon component. Both these require that $b = 0$. Hence physical initial data and regularity requires us to take the solution,

$$f(z) = az^{\frac{d}{2}} K_\nu(kz) \tag{71}$$

for a constant a. Expanded in a power series in z,

$$f(z) = a_0 z^{d-\Delta} + \ldots + a_1 z^{\Delta} + \ldots \tag{72}$$

and the constants a_0 and a_1 are just functions of a and k which may be read off. Again $\ldots$ represent ascending power of z. In fact,

$$a_1 = \frac{\Gamma\left(-\frac{2\Delta-d}{2}\right)}{\Gamma\left(+\frac{2\Delta-d}{2}\right)} \left(\frac{k}{2}\right)^{2\Delta-d} a_0 \tag{73}$$

The k dependence is important as now returning to our real space form, it implies that,

$$\phi_d(x) \propto \int d^d y \frac{1}{|x-y|^{2\Delta}} \phi_0(y) \tag{74}$$

[Why?: if

$$\phi_0(x) = \int d^d k a_k e^{ik\cdot x} \tag{75}$$

then,

$$\begin{aligned} \int d^d y \frac{\phi_0(y)}{|x-y|^{2\Delta}} &= \int d^d k \int d^d y \frac{1}{|x-y|^{2\Delta}} a_k e^{ik\cdot y} \\ &= \int d^d k a_k e^{ik\cdot x} \int d^d y \frac{1}{|x-y|^{2\Delta}} e^{ik\cdot(y-x)} \\ &= \int d^d k a_k e^{ik\cdot x} \int d^d z \frac{1}{|z|^{2\Delta}} e^{ik\cdot z} \end{aligned} \tag{76}$$

but on general grounds we expect $\int d^d z \frac{1}{|z|^{2\Delta}} e^{ik\cdot z} \propto |k|^{2\Delta-d}$ up to a k independent constant so,

$$\int d^d y \frac{\phi_0(y)}{|x-y|^{2\Delta}} \propto \int d^d k \, a_k k^{2\Delta-d} e^{ik\cdot x} \sim \phi_d \tag{77}$$

as claimed. End of aside.]

3.3 The scalar action

Consider the action for the scalar evaluated on this solution (the 'on-shell' action). If we integrate by parts, the action becomes,

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d x dz \sqrt{g}\phi\left(\nabla^2\phi - m^2\phi\right) - \frac{1}{2}\int_{\partial\mathcal{M}} dS^A \phi\partial_A\phi \tag{78}$$

but for a solution the bulk integral vanishes. Hence the 'on-shell' action is simply given by the boundary term

$$S_{on-shell} = -\frac{1}{2}\int_{\partial\mathcal{M}} dS^A \phi\partial_A\phi \tag{79}$$

where dS^μ is the outward directed area element of the boundary.

We have two potential surfaces that may give contributions - $z = 0$ and $z \to \infty$. Let us evaluate the boundary term on our solution for a boundary $\partial\mathcal{M}$ that is at a fixed z. Then given the Poincare-AdS metric,

$$I(z) \equiv \int_{\partial\mathcal{M}} dS^A J_A = \int d^d x \left(\frac{\ell}{z}\right)^d n^A J_A = -\int d^d x \left(\frac{\ell}{z}\right)^{d-1} J_z \tag{80}$$

where n^μ is the unit outer (for the $z = 0$ boundary) normal, $n^\mu = 0$, $n^z = -\frac{z}{\ell}$ so that $n^2 = 1$, and $\left(\frac{\ell}{z}\right)^d$ is the volume element $\sqrt{g|_z}$ on the metric induced on a constant z surface. For our solution we have $J_z = \phi\partial_z\phi$ so find,

$$I = -\int d^d x \left(\frac{\ell}{z}\right)^{d-1} (\phi\partial_z\phi) \tag{81}$$

From our discussion of the z behaviour of solutions as $z \to \infty$ we see there will be no contribution there as at worst $\phi \to z^{\frac{d-1}{2}}$. However there may be a contribution at $z = 0$ where the field behaves as in (67). Then,

$$\begin{aligned}\phi\partial_z\phi &= \left(\phi_0(x)z^{d-\Delta} + \ldots + \phi_d(x)z^\Delta + \ldots\right)\left((d-\Delta)\phi_0(x)z^{d-\Delta-1} + \ldots + \Delta\phi_d(x)z^{\Delta-1} + \ldots\right) \\ &= (d-\Delta)\phi_0(x)^2 z^{2d-2\Delta-1} + \ldots + d\phi_0(x)\phi_d(x)z^{d-1} + \ldots \end{aligned} \tag{82}$$

with the $\ldots$ being ascending power of z. Putting this into the surface term we find,

$$I(z) = -\int d^d x\, \ell^{d-1}\left((d-\Delta)\phi_0(x)^2 z^{d-2\Delta} + \ldots + d\phi_0(x)\phi_d(x)z^0 + \ldots\right) \tag{83}$$

The on-shell action is then given by,

$$S_{os} = -\frac{1}{2}\lim_{z\to 0} I(z) \to \infty \tag{84}$$

But this is simply distastrous?! For $\Delta > d/2$ then all the leading terms up to z^0 are horribly divergent! Only this term is finite near $z = 0$. So we conclude the action evaluated on our solution is divergent, due to the boundary.

Is this surprising? Not really, since the space near the conformal boundary has infinite volume. If fields do not fall off quickly enough they will naturally give rise to an infinite action.

However it turns out that all the divergences can be cured by adding additional local surface terms to the action. Since these are surface terms, or total divergences from the bulk point of view they don't change the equation of motion or our solution. Consider instead the new action,

$$S_{reg} = \frac{1}{2}\int_{\mathcal{M}} d^d x dz \sqrt{g}\phi\left(\nabla^2\phi - m^2\phi\right) - \frac{1}{2}\int_{\partial\mathcal{M}} dS^A \phi\partial_A\phi + \int_{\partial\mathcal{M}} dS\left(k\phi^2 + \ldots\right) \tag{85}$$

for constant k, and the $\ldots$ represent terms involving derivatives of ϕ, and dS is the volume element of the surface. So on-shell it is,

$$S_{reg,os} = -\frac{1}{2}\int_{\partial\mathcal{M}} dS^A \phi\partial_A\phi + \int_{\partial\mathcal{M}} dS\left(k\phi^2 + \ldots\right) \tag{86}$$

We'll ignore the $\ldots$ terms and focus on the leading new term on a constant z surface,

$$I_{ct}(z) = \int d^d x \left(\frac{\ell}{z}\right)^{d-1}\left(\phi^2\right) \tag{87}$$

so that,

$$S_{reg,os} = \lim_{z\to 0}\left(-\frac{1}{2}I(z) + I_{ct}(z)\right) \tag{88}$$

Using our expansion near $z = 0$ gives,

$$I_{ct} = \int_{\partial\mathcal{M}} dSk\phi^2 = -\int d^d x\, \ell^{d-1}\left(\phi_0(x)^2 z^{d-2\Delta} + \ldots + 2\phi_0(x)\phi_d(x)z^0 + \ldots\right) \tag{89}$$

Using this and I we see the on-shell action is,

$$S_{reg,os} = \lim_{z\to 0}\left[\frac{1}{2}\int d^dx\,\ell^{d-1}\left((d-\Delta)\phi_0(x)^2 z^{d-2\Delta} + \ldots + d\phi_0(x)\phi_d(x)z^0 + \ldots\right)\right.$$
$$\left. +k\int d^dx\,\ell^{d-1}\left(\phi_0(x)^2 z^{d-2\Delta} + \ldots + 2\phi_0(x)\phi_d(x)z^0 + \ldots\right) + \ldots\right] \tag{90}$$

Now we see that if we choose $k = -\frac{d-\Delta}{2}$ the (leading) divergence cancels leaving the term z^0 which taking the limit gives,

$$S_{reg,os} = \frac{2\Delta - d}{2}\int d^dx\,\ell^{d-1}\phi_0(x)\phi_d(x) \tag{91}$$

Note that this extra surface term that removes the divergence does contribute to the coefficient of the finite term - so its form is important.

Actually we have been a bit naughty and dropped the other possible divergences. One can show that a finite number of counter terms are required to cancel these divergences and these don't affect the finite term near $z = 0$.

Let us return to our solution. Then we see,

$$S_{reg,os} = c\int d^dx\int d^dy\,\frac{1}{|x-y|^{2\Delta}}\phi_0(x)\phi_0(y) \tag{92}$$

for a constant c depending only on ℓ and dimension. This sort of thing looks familiar!

4 The AdS-CFT correspondence

Certain d-dimensional CFTs (non-gravitational) are thought to be physically equivalent to theories of gravity coupled to matter in $(d+1)$-dimensions that are asymptotic to AdS. The CFT is said to live at the 'boundary' of the asymptotic AdS space. The gravity theory must have matter that supports this AdS behaviour, and is the 'bulk'. This is thought to be a fully quantum duality.

The correspondence states that each for every bulk matter field - the scalars, vectors etc... - there is a dual primary operator that corresponds to it. The metric itself corresponds to the CFT stress tensor (which is a scaling operator, but not a primary). For simplicity here let us focus on a massive bulk scalar ϕ which is dual to a scalar operator O in the CFT with scaling dimension Δ. Then we may deform the CFT by a source $J(x)$, so its generating function is,

$$Z_{CFT}[J] = \int DX e^{i\int d^d x L_{CFT} + JO} \tag{93}$$

In the bulk we have seen the scalar field in asymptotic AdS requires boundary conditions - fixing $\phi_0(x)$ as discussed above. Then the path integral of this (quantum) gravity theory in the bulk will depend implicitly on these boundary conditions, and we have,

$$Z_{grav}[J] = \int D[g\ldots]e^{i\int d^d x dz \frac{R}{16\pi G} + (\partial\phi)^2 - m^2\phi^2 \ldots}\Big|_{\phi_0(x)=J(x)} \tag{94}$$

so the path integral is taken over configurations with the imposed boundary conditions. It is easy to write this, but the quantum path integral for gravity is totally ill posed. Ignoring this important point the AdS-CFT correspondence then states that,

$$Z_{CFT}[J] = Z_{grav}[J] \tag{95}$$

There are two ways to think about this. Firstly if this is true it provides a rigorous definitely of the quantum gravity path integral in terms of a well defined QFT one. This is remarkable.

However the second way to think about this is when the gravity bulk solution looks classical. Then the path integral is dominated one or more

classical saddle points. Typically one of these will dominate the others, so,

$$Z_{CFT}[J] = Z_{grav}[J] \sim e^{iS_{reg,os}[\phi_0]}\Big|_{\phi_0(x)=J(x)} = e^{iS_{reg,os}[J]} \tag{96}$$

where we are writing the action of the (dominant) classical solution with boundary conditions given by ϕ_0 as $S_{reg,os}[\phi_0]$ and note that in order for this to make sense it must be regulated as above.

Why the regulation? We require surface terms associated to the conformal boundary. But this is thought of as the high energy part of the space time. Thus we are subtracting divergences associated to UV behaviour, which is exactly what is requires in the CFT. It was implicit in our discussion that Z_{CFT} has been renormalized, and we see there is a mirror procedure in the bulk.

We obtain two point functions by differentiating twice with respect to J, as we know in the CFT,

$$\langle O(x)O(y)\rangle = -\frac{1}{Z_{CFT}}\frac{\partial^2}{\partial J(x)\partial J(y)}Z_{CFT}[J]\Big|_{J=0} = \frac{1}{|x-y|^{2\Delta}} \tag{97}$$

from before. However, since we have computed,

$$S_{reg,os}[\phi_0] = c\int d^dx \int d^dy\, \frac{1}{|x-y|^{2\Delta}}\phi_0(x)\phi_0(y) \tag{98}$$

then we get precisely the same result in the bulk,

$$\begin{aligned} -\frac{1}{Z_{grav}}\frac{\partial^2}{\partial J(x)\partial J(y)}Z_{grav}[J]\Big|_{J=0} &= e^{-iS_{reg,os}}\frac{\partial^2}{\partial J(x)\partial J(y)}e^{iS_{reg,os}}\Big|_{J=0} \\ &= e^{-iS_{reg,os}}\frac{\partial^2}{\partial J(x)\partial J(y)}e^{c\int d^dx\int d^dy\,\frac{1}{|x-y|^{2\Delta}}J(x)J(y)}\Big|_{J=0} \\ &= \frac{1}{|x-y|^{2\Delta}} \end{aligned} \tag{99}$$

We can compute higher point correlators with additional functional differentiation of the on-shell gravity action. In our scalar example, we will have to include the interaction of the scalar with the bulk graviton going beyond the two point function - this leads to 'Witten' diagrams.

Without a source one point functions must vanish. However with a source they will not. Then,

$$\langle O(x)\rangle_J = i\frac{\partial}{\partial J(x)}\ln Z_{CFT} \tag{100}$$

so then we can compute using the gravity,

$$\langle O(x)\rangle_J = i\frac{\partial}{\partial J(x)}\ln Z_{grav} \simeq -\frac{\partial}{\partial J(x)}S_{reg,os}[J] \tag{101}$$

A careful gravity analysis shows that,

$$\delta S_{reg,os} = (2\Delta - d)\int d^dx\, \ell^{d-1}\delta\phi_0(x)\phi_d(x) \tag{102}$$

and so taking $J = \phi_0$ we find,

$$\langle O(x)\rangle_J = (2\Delta - d)\ell^{d-1}\phi_d(x) \tag{103}$$

So for our scalar we interpret the leading fall off $\phi_0(x)$ as a boundary condition corresponding to a source in the CFT. And the subleading fall off ϕ_d is the response of the field and determines the one-point function.

4.1 Stress tensor

Every CFT has a stress tensor and this is understood to be dual to the bulk metric itself. The bulk metric for an asymptotically AdS spacetime has an expansion near $z = 0$ and one can find coordinates (Feffermann-Graham) where,

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + h_{\mu\nu}(z,x)dx^\mu dx^\nu\right) \tag{104}$$

and,

$$h_{\mu\nu}(z,x) = h^{(0)}_{\mu\nu}(x) + \ldots + z^d h^{(d)}_{\mu\nu}(x) + \ldots \tag{105}$$

The term $h^{(0)}$ is interpreted as the source for the stress tensor which is simply the boundary spacetime - ie. the spacetime the CFT lives on. By deforming this away from Minkowski we can get interesting dynamics. We think of this leading term as going as $z^{d-\Delta}$ where $\Delta = d$ for the stress tensor operator. The term $h^{(d)}$ then determines the one point function of the stress tensor. This term going as z^d we think of as z^Δ.

There is a sublety in odd bulk dimensions, but in even dimensions the CFT stress tensor is related to $h^{(d)}$ as,

$$\langle T_{\mu\nu}\rangle = \frac{d\ell^{d-1}}{16\pi G}h^{(d)}_{\mu\nu} \tag{106}$$

where the CFT lives on the metric $h^{(0)}_{\mu\nu}$. In odd dimensions there is an extra term, although it only involves $h^{(0)}$. One can compute the two point function of the stress tensor by considering gravity perturbations, analogous to scalar example. There is a well defined way to regulate the gravity action. One finds,

$$\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)\rangle = \frac{\ell^{d-1}}{16\pi G}\frac{1}{|x-y|^{2d}}I_{\mu\nu,\alpha\beta} \tag{107}$$

This allows us to identity the CFT constant,

$$C = \frac{\ell^{d-1}}{16\pi G} \tag{108}$$

Thus the ratio of ℓ to the Planck length determines the effective number of degrees of freedom in the CFT. In order to have classical gravity we therefore require $C \gg 1$, so the CFT must have many degrees of freedom.

For example in the case of $\mathcal{N} = 4$, one finds,

$$C = \frac{N^2}{8\pi^2} \tag{109}$$

showing one requires $N \gg 1$.

4.2 Finite temperature

Consider the CFT on Minkowski where we compactify the space into a torus, so the CFT lives on $\mathbb{R}_t \times T^{d-1}$. Now put the CFT at finite temperature T. Semi-classical gravity then tells us this should be described by a black hole, with Hawking temperature T (wrt $\partial/\partial t$). The metric will be the planar-AdS Schwarschild given earlier, although we compactify the coordinates $x^a \sim x^a + L$ to give a torus volume $V = L^{d-1}$. Now recall,

$$T_{Hawking} = \frac{d}{4\pi z_0} \tag{110}$$

The black hole entropy is dual to the entropy of the CFT at this finite temperature. Therefore the black hole is describing the thermal physics of the CFT! A calculation then predicts that,

$$S_{CFT} = \frac{dCV}{T}\left(\frac{4\pi T}{d}\right)^d \tag{111}$$

This is universal for any CFT with a gravity dual. Note the entropy is proportional to C the effective number of degrees of freedom, and to volume as it should be.

It is an incredibly tricky calculation in strongly coupled field theory to compute entropy from first principles. The fact we can do so here highlights the remarkable types of computation that holography allows. In particular for $\mathcal{N} = 4$ at large 't Hooft coupling it predicts,

$$S_{CFT} = \frac{\pi^2}{2} N^2 V T^3 \tag{112}$$

4.3 Top down

In top down approaches the dual gravity theory typically is D-dimensional with $D > d + 1$ and involves an additional compact space. So the vacuum geometry is,

$$g_{(D)} = AdS_{(d+1)} \times X \tag{113}$$

for a compact $(D-d-1)$-dimensional X. Since X has constant size, it shrinks relative to the AdS factor near the boundary and hence does not change the structure of the conformal boundary which remains $(d+1)$-dimensional. We may always Kaluza-Klein reduce these theories on X to obtain $(d+1)$-dimensional theories with infinite towers of modes. Note that in this case, the effective central charge is given in terms of the volume of X and the higher dimensional Newton constant,

$$C = \frac{\ell^{d-1}}{16\pi G_D} \mathrm{Vol}(X) \tag{114}$$

In the canonical $\mathcal{N} = 4$ example the vacuum is $AdS_5 \times S_5$. Isometries of the space X then give rise to additional global symmetries in the boundary. For example $\mathcal{N} = 4$ has an $SO(6)$ R-symmetry associated to the sphere.