AdS-CFT

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QFFF Special topics 2017

Solutions for problem sheet 2: AdS

1. Consider,

$$U^2 + V^2 - X^i X^i = U^2 + V^2 - X^a X^a - X^d X^d = \ell^2 \tag{1}$$

Then we must check the parameterization is compatible with this. We see,

$$V^2 - X^a X^a = \left(\frac{\ell}{z} t\right)^2 - \left(\frac{\ell}{z} x^a\right)^2 \tag{2}$$

$$= \left(\frac{\ell}{z}\right)^2 \left(t^2 - x^a x^a\right) \tag{3}$$

and,

$$U^2 = \frac{z^2}{4}\left(1 + \frac{1}{z^2}\left(\ell^2 + x^a x^a - t^2\right)\right)^2 \tag{4}$$

and

$$(X^d)^2 = \frac{z^2}{4}\left(1 - \frac{1}{z^2}\left(\ell^2 - x^a x^a + t^2\right)\right)^2 \tag{5}$$

so,

$$\begin{aligned} U^2 - (X^d)^2 &= \frac{z^2}{4}\left(1 + \frac{2}{z^2}\left(\ell^2 + x^a x^a - t^2\right) + \frac{1}{z^4}\left(\ell^2 + x^a x^a - t^2\right)^2\right) \\ &\quad - \frac{z^2}{4}\left(1 - \frac{2}{z^2}\left(\ell^2 - x^a x^a + t^2\right) + \frac{1}{z^4}\left(\ell^2 - x^a x^a + t^2\right)^2\right) \\ &= \frac{z^2}{4}\left(\frac{4}{z^2}\left(\ell^2\right) + \frac{2}{z^4}\left(2\ell^2\left(x^a x^a - t^2\right)\right)\right) \\ &= \ell^2 + \frac{\ell^2}{z^2}\left(x^a x^a - t^2\right) \end{aligned} \tag{6}$$

so indeed we confirm,

$$\begin{aligned} U^2 + V^2 - X^i X^i &= \left(V^2 - X^a X^a\right) + \left(U^2 - (X^d)^2\right) \\ &= \frac{\ell^2}{z^2}\left(t^2 - x^a x^a\right) + \left(\ell^2 + \frac{\ell^2}{z^2}\left(x^a x^a - t^2\right)\right) \\ &= \ell^2 \end{aligned} \tag{7}$$

Now to compute the induced metric. Firstly,

$$\begin{aligned} dV &= \frac{\ell}{z} dt - \frac{\ell}{z^2} t dz \\ dX^a &= \frac{\ell}{z} dx^a - \frac{\ell}{z^2} x^a dz \end{aligned} \tag{8}$$

so,

$$\begin{aligned} -dV^2 + dX^a dX^a &= -\left(\frac{\ell}{z} dt - \frac{\ell}{z^2} t dz\right)^2 + \left(\frac{\ell}{z} dx^a - \frac{\ell}{z^2} x^a dz\right)^2 \\ &= \frac{\ell^2}{z^2}\left(-dt^2 + dx^a dx^a\right) + \frac{2\ell^2}{z^3} t dt dz - \frac{2\ell^2}{z^3} x^a dx^a dz \\ &\quad + \frac{\ell^2}{z^4}\left(-t^2 + x^a x^a\right) dz^2 \end{aligned} \tag{9}$$

And,

$$\begin{aligned} dU &= \frac{dz}{2}\left(1 + \frac{1}{z^2}\left(\ell^2 + x^a x^a - t^2\right)\right) + \frac{z}{2}\left(-\frac{2dz}{z^3}\left(\ell^2 + x^a x^a - t^2\right)\right) \\ &\quad + \frac{z}{2}\left(\frac{1}{z^2}\left(2x^a dx^a - 2t dt\right)\right) \\ &= \frac{dz}{2}\left(1 - \frac{1}{z^2}\left(\ell^2 + x^a x^a - t^2\right)\right) \\ &\quad + \frac{1}{z}\left(x^a dx^a - t dt\right) \end{aligned} \tag{10}$$

and,

$$
\begin{aligned}
dX^d &= \frac{dz}{2}\left(1-\frac{1}{z^2}\left(\ell^2-x^a x^a+t^2\right)\right)+\frac{z}{2}\left(+\frac{2dz}{z^3}\left(\ell^2-x^a x^a+t^2\right)\right) \\
&\quad -\frac{z}{2}\left(\frac{1}{z^2}\left(-2x^a dx^a+2tdt\right)\right) \\
&= \frac{dz}{2}\left(1+\frac{1}{z^2}\left(\ell^2-x^a x^a+t^2\right)\right) \\
&\quad +\frac{1}{z}\left(x^a dx^a-tdt\right) \qquad (11)
\end{aligned}
$$

Then,

$$
\begin{aligned}
-dU^2+(dX^d)^2 &= -\left(\frac{dz}{2}\left(1-\frac{1}{z^2}\left(\ell^2+x^a x^a-t^2\right)\right)+\frac{1}{z}\left(x^a dx^a-tdt\right)\right)^2 \\
&\quad +\left(\frac{dz}{2}\left(1+\frac{1}{z^2}\left(\ell^2-x^a x^a+t^2\right)\right)+\frac{1}{z}\left(x^a dx^a-tdt\right)\right)^2 \\
&= \frac{dz^2}{4}\left(-\left(1-\frac{1}{z^2}\left(\ell^2+x^a x^a-t^2\right)\right)^2+\left(1+\frac{1}{z^2}\left(\ell^2-x^a x^a+t^2\right)\right)^2\right) \\
&\quad -\frac{1}{z^2}\left(x^a dx^a-tdt\right)^2+\frac{1}{z^2}\left(x^a dx^a-tdt\right)^2 \\
&\quad -\frac{1}{z}\left(1-\frac{1}{z^2}\left(\ell^2+x^a x^a-t^2\right)\right)\left(x^a dx^a dz-tdtdz\right) \\
&\quad +\frac{1}{z}\left(1+\frac{1}{z^2}\left(\ell^2-x^a x^a+t^2\right)\right)\left(x^a dx^a dz-tdtdz\right) \qquad (12)
\end{aligned}
$$

and continuing,

$$
\begin{aligned}
-dU^2 + (dX^d)^2 &= \frac{dz^2}{4}\left(-\left(1-\frac{1}{z^2}\left(\ell^2+x^a x^a - t^2\right)\right)^2 + \left(1+\frac{1}{z^2}\left(\ell^2 - x^a x^a + t^2\right)\right)^2\right) \\
&\quad + 2\frac{\ell^2}{z^3}\left(x^a dx^a dz - t dt dz\right) \\
&= \frac{dz^2}{4}\left(-\frac{1}{z^4}\left(\ell^2+\left(x^a x^a - t^2\right)\right)^2 + \frac{1}{z^4}\left(\ell^2 - \left(x^a x^a - t^2\right)\right)^2\right) \\
&\quad + \frac{dz^2}{2z^2}\left(\left(\ell^2 + x^a x^a - t^2\right) + \left(\ell^2 - x^a x^a + t^2\right)\right) \\
&\quad + 2\frac{\ell^2}{z^3}\left(x^a dx^a dz - t dt dz\right) \\
&= -dz^2\frac{\ell^2}{z^4}\left(x^a x^a - t^2\right) + \frac{\ell^2 dz^2}{z^2} + 2\frac{\ell^2}{z^3}\left(x^a dx^a dz - t dt dz\right)
\end{aligned} \tag{13}
$$

So that,

$$
\begin{aligned}
-dU^2 - dV^2 + (dX^i)^2 &= -dV^2 + dX^a dX^a - dU^2 + (dX^d)^2 \\
&= \left(\frac{\ell^2}{z^2}\left(-dt^2 + dx^a dx^a\right) + \frac{2\ell^2}{z^3} t dt dz - \frac{2\ell^2}{z^3} x^a dx^a dz + \frac{\ell^2}{z^4}\left(-t^2 + x^a x^a\right) dz^2\right) \\
&\quad + \left(-dz^2\frac{\ell^2}{z^4}\left(x^a x^a - t^2\right) + \frac{\ell^2 dz^2}{z^2} + 2\frac{\ell^2}{z^3}\left(x^a dx^a dz - t dt dz\right)\right) \\
&= \frac{\ell^2}{z^2}\left(-dt^2 + dx^a dx^a\right) + \frac{\ell^2 dz^2}{z^2}
\end{aligned} \tag{14}
$$

which is what we wanted to show. Phew.

2. So taking,

$$ds^2 = \frac{\ell^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \tag{15}$$

we must compute,

$$\Gamma^C{}_{AB} = \frac{1}{2}g^{CD}\left(\partial_A g_{BD} + \partial_B g_{AD} - \partial_D g_{AB}\right) \tag{16}$$

Thus,

$$\begin{aligned}\Gamma^z{}_{zz} &= \frac{1}{2}g^{zz}\left(\partial_z g_{zz}\right)\\ &= \frac{1}{2}\frac{z^2}{\ell^2}\partial_z\left(\frac{\ell^2}{z^2}\right)\\ &= -\frac{1}{z}\end{aligned} \tag{17}$$

and,

$$\begin{aligned}\Gamma^z{}_{\mu\nu} &= \frac{1}{2}g^{zz}\left(-\partial_z g_{\mu\nu}\right)\\ &= -\frac{1}{2}\frac{z^2}{\ell^2}\partial_z\left(\frac{\ell^2}{z^2}\eta_{\mu\nu}\right)\\ &= +\frac{1}{z}\eta_{\mu\nu}\end{aligned} \tag{18}$$

and,

$$\begin{aligned}\Gamma^\mu{}_{z\nu} &= \frac{1}{2}g^{\mu\alpha}\left(\partial_z g_{\nu\alpha}\right)\\ &= \frac{1}{2}\frac{z^2}{\ell^2}\eta^{\mu\alpha}\partial_z\left(\frac{\ell^2}{z^2}\eta_{\nu\alpha}\right)\\ &= -\frac{1}{z}\delta^\mu_\nu\end{aligned} \tag{19}$$

Due to the symmetries all other components vanish.
3. Consider a particle at fixed spatial location. Then the tangent vector to its worldline $v \propto \partial/\partial t$, and using proper time parameterization, so $v^A = dx^A/d\tau$ with τ the particle proper time then $v^2 = -1$. Hence,

$$v^t = \frac{z}{\ell}\,, \quad v^i = v^z = 0 \tag{20}$$

so that $v^2 = g_{tt}(v^t)^2 = -1$.

Now the acceleration is,

$$\begin{aligned} a^A &= v^B \nabla_B v^A = v^B \left(\partial_B v^A + \Gamma^A{}_{BC} v^C \right) \\ &= \Gamma^A{}_t v^t v^t \\ &= \frac{z^2}{\ell^2} \Gamma^A{}_{tt} \end{aligned} \tag{21}$$

since $\partial_\mu v^A = 0$ and $v^z = v^i = 0$. Hence,

$$\begin{aligned} a^\mu &= \frac{z^2}{\ell^2} \Gamma^\mu{}_{tt} = 0 \\ a^z &= \frac{z^2}{\ell^2} \Gamma^z{}_{tt} = \frac{1}{z} \eta_{tt} \frac{z^2}{\ell^2} = -\frac{z}{\ell^2} \end{aligned} \tag{22}$$

Thus we see the acceleration of the trajectory is in the negative z sense, ie. towards the conformal boundary. If the force providing the acceleration were switched off, and the particle subsequently followed a geodesic, this would therefore be away from the boundary to larger z. Then the norm is,

$$a^2 = g_{zz}(a^z)^2 = \frac{\ell^2}{z^2} \left(\frac{z}{\ell^2} \right)^2 = \frac{1}{\ell^2} \tag{23}$$

which is indeed constant and so independent of the position of the particle.

4. Consider a spacelike curve in Poincare AdS in a constant t slice. Let us parameterize the curve using the z coordinate. For simplicity assume the curve is monotonic in z - if this gives an infinite result it will obviously be true for any curve even if it is not monotonic. Then, it has tangent $v^A = dx^A/dz$. Since it is in a constant t slice, $v^t = dt/dz = 0$. The length of a curve is,

$$s = \int_0^\epsilon dz \sqrt{v^A v_A} \tag{24}$$

where we assume the curve starts at $z = 0$ and ends at $z = \epsilon$. Now we have,

$$v^A v_A = g_{\mu\nu} v^\mu v^\nu + g_{zz}(v^z)^2 = g_{ij} v^i v^j + g_{zz}(v^z)^2 \tag{25}$$

But $v^z = dz/dz = 1$ so,

$$v^A v_A = \frac{\ell^2}{z^2} \left(1 + \delta_{ij} \frac{dx^i}{dz} \frac{dx^j}{dz} \right) \geq \frac{\ell^2}{z^2} \tag{26}$$

since $\delta_{ij}\frac{dx^i}{dz}\frac{dx^j}{dz} = \left|\frac{dx^i}{dz}\right|^2 > 0$.

Hence the distance is,

$$s = \int_0^\epsilon dz\sqrt{v^A v_A} > \int_0^\epsilon dz\sqrt{\frac{\ell^2}{z^2}} = \ell^2\int_0^\epsilon \frac{dz}{z} \tag{27}$$

This diverges at the low limit of the integral (ie. the boundary) giving an infinite result.

5. The geodesic curves in,

$$g = \frac{\ell^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \tag{28}$$

and just the same as for $(d+1)$-Minkowski,

$$g' = \eta_{\mu\nu}dx^\mu dx^\nu + dz^2 = \eta_{AB}dx^A dx^B \tag{29}$$

since, $g' = \frac{z^2}{\ell^2}g$. The latter are simply curves,

$$x^A = a^A\lambda + b^A \tag{30}$$

for constants a^A and b^A with $\eta_{AB}a^A a^B = 0$ so that the tangent $v^A = dx^A/d\lambda = a^A$ is null. Then λ is an affine parameter for the geodesic. Note that if $a^z \neq 0$ we can always choose the affine parameter to be $\lambda = z$, in which case $a^z = 1$ and $b^z = 0$.

Thus these are also the null geodesics in Poincare-AdS. We see immediately that such curves reach the boundary $z = 0$ at finite coordinate t given explicitly by $t = a^t\left(-\frac{b^z}{a^z}\right) + b^t$.

Showing the affine parameter is finite is less obvious. Certainly the affine parameter in g' is just $\lambda = -b^z/a^z$ when the curve hits $z = 0$. However the affine parameter in g' is not the same as for g. Let us call the affine parameter in AdS α, and the tangent $u^A = dx^A/d\alpha$ then for a null geodesic we will have,

$$\frac{du^A}{d\alpha} + \Gamma^A{}_{BC}u^B u^C = 0 \tag{31}$$

But,

$$u^A = \frac{dx^A}{d\alpha} = \frac{dx^A}{d\lambda}\frac{d\lambda}{d\alpha} = a^A\frac{d\lambda}{d\alpha} \tag{32}$$

and so,

$$\frac{du^A}{d\alpha} = \frac{d}{d\alpha}\left(a^A \frac{d\lambda}{d\alpha}\right) = a^A \frac{d^2\lambda}{d\alpha^2} \tag{33}$$

Then,

$$\begin{aligned} 0 &= \frac{du^A}{d\alpha} + \Gamma^A{}_{BC} u^B u^C \\ &= a^A \frac{d^2\lambda}{d\alpha^2} + \Gamma^A{}_{BC} a^B a^C \left(\frac{d\lambda}{d\alpha}\right)^2 \end{aligned} \tag{34}$$

Hence we can fix $\alpha = \alpha(\lambda)$ from any component - choose the t component. Then,

$$\begin{aligned} 0 &= a^t \frac{d^2\lambda}{d\alpha^2} + 2\Gamma^t{}_{z\mu} a^z a^\mu \left(\frac{d\lambda}{d\alpha}\right)^2 \\ &= a^t \frac{d^2\lambda}{d\alpha^2} - \frac{2}{z} a^z a^t \left(\frac{d\lambda}{d\alpha}\right)^2 \end{aligned} \tag{35}$$

so that,

$$\frac{d^2\lambda}{d\alpha^2} = \frac{2}{z} a^z \left(\frac{d\lambda}{d\alpha}\right)^2 \tag{36}$$

Let us take the choice that $\lambda = z$ discussed above, so $a^z = 1$. Then,

$$\frac{d^2 z}{d\alpha^2} = \frac{2}{z}\left(\frac{dz}{d\alpha}\right)^2 \tag{37}$$

so,

$$\frac{1}{z^2}\frac{d^2 z}{d\alpha^2} - \frac{2}{z^3}\left(\frac{dz}{d\alpha}\right)^2 = 0 \tag{38}$$

and,

$$\frac{d}{d\alpha}\left(\frac{1}{z^2}\frac{dz}{d\alpha}\right) = 0 \tag{39}$$

so that,

$$\frac{dz}{d\alpha} = cz^2 \tag{40}$$

for some constant c, and then,

$$z = \frac{1}{b - c\alpha} \tag{41}$$

for a constant b. Hence as $z \to 0$ we must have $\alpha \to \infty$. So indeed the affine parameter of the null geodesic diverges at the boundary.

6. We take $\phi = e^{ik_\mu x^\mu} f(z)$. Then,

$$\begin{aligned}
\nabla^2\phi &= g^{AB}\left(\partial_A\partial_B\phi - \Gamma^C{}_{AB}\partial_C\phi\right) \\
&= g^{\mu\nu}\partial_\mu\partial_\nu\phi + g^{zz}\partial_z\partial_z\phi - g^{AB}\Gamma^\mu{}_{AB}\partial_\mu\phi - g^{AB}\Gamma^z{}_{AB}\partial_z\phi \\
&= -\left(\frac{z^2}{\ell^2}\right)\eta^{\mu\nu}k_\mu k_\nu\phi + \left(\frac{z^2}{\ell^2}\right)\partial_z\partial_z\phi - g^{AB}\Gamma^z{}_{AB}\partial_z\phi \\
&= \left(\frac{z^2}{\ell^2}\right)\left(-k^2\phi + \partial_z\partial_z\phi\right) - g^{zz}\Gamma^z{}_{zz}\partial_z\phi - g^{\mu\nu}\Gamma^z{}_{\mu\nu}\partial_z\phi \\
&= \left(\frac{z^2}{\ell^2}\right)\left(-k^2\phi + \partial_z\partial_z\phi\right) - \left(\frac{z^2}{\ell^2}\right)\left(-\frac{1}{z}\right)\partial_z\phi - \left(\frac{z^2}{\ell^2}\right)\eta^{\mu\nu}\left(+\frac{1}{z}\eta_{\mu\nu}\right)\partial_z\phi \\
&= \left(\frac{z^2}{\ell^2}\right)\left(-k^2\phi + \partial_z\partial_z\phi + \frac{1}{z}\partial_z\phi - \eta^{\mu\nu}\eta_{\mu\nu}\frac{1}{z}\partial_z\phi\right) \\
&= \left(\frac{z^2}{\ell^2}\right)\left(-k^2\phi + \partial_z^2\phi - \frac{d-1}{z}\partial_z\phi\right)
\end{aligned} \tag{42}$$

Then,

$$0 = \nabla^2\phi - m^2\phi = \left(\frac{z^2}{\ell^2}\right)\left(-k^2\phi + \partial_z^2\phi - \frac{d-1}{z}\partial_z\phi\right) - m^2\phi \tag{43}$$

so,

$$0 = -k^2\phi + \partial_z^2\phi - \frac{d-1}{z}\partial_z\phi - \frac{\ell^2 m^2}{z^2}\phi \tag{44}$$

as required, given $\phi = e^{ik_\mu x^\mu} f(z)$.