AdS-CFT

Toby Wiseman (Imperial College)
QFFF Special topics 2017

Solutions for problem sheet 3: AdS-CFT

1. So we have,

$$ds^2 = \frac{\ell^2}{z^2}\left(-f dt^2 + dx^a dx^a + \frac{1}{f}dz^2\right) \ , \quad f = 1 - \left(\frac{z}{z_0}\right)^d \tag{1}$$

The spatial section of the horizon ie. the induced metric at t =const and $z = z_0$, is simply,

$$ds^2_{horiz} = g^{horiz}_{ab} dx^a dx^b = \frac{\ell^2}{(z_0)^2}\left(dx^a dx^a\right) \tag{2}$$

Hence the horizon area is,

$$A = \int d^{d-1}x \sqrt{\det g^{horiz}_{ab}} = \int d^{d-1}x \left(\frac{\ell}{z_0}\right)^{d-1} = \left(\frac{\ell}{z_0}\right)^{d-1} L^{d-1} \tag{3}$$

Now consider the Killing vector $K = \partial/\partial t$ so $K^A = (1, 0, \ldots, 0)$. Then it has norm,

$$\begin{aligned} |K| &= \sqrt{K^2} = \sqrt{(K^A K^B g_{AB})} = \sqrt{(K^t K^t g_{tt})} \\ &= \sqrt{g_{tt}} = \sqrt{-f(z)\frac{\ell^2}{z^2}} \end{aligned} \tag{4}$$

Then,

$$\begin{aligned}
\partial_z|K| &= \partial_z\sqrt{-f(z)\frac{\ell^2}{z^2}} = \frac{1}{2\sqrt{-f(z)\frac{\ell^2}{z^2}}}\partial_z\left(-f(z)\frac{\ell^2}{z^2}\right) \\
&= \frac{1}{2\sqrt{-f(z)\frac{\ell^2}{z^2}}}\left(2f(z)\frac{\ell^2}{z^3} - f'(z)\frac{\ell^2}{z^2}\right) \\
\partial_\mu|K| &= 0 \qquad (5)
\end{aligned}$$

as $|K|$ only depends on z.

$$\begin{aligned}
\kappa^2 &= -g^{AB}\left(\partial_A|K|\right)\left(\partial_B|K|\right)\Big|_{z=z_0} = -g^{zz}\left(\partial_z|K|\right)^2\Big|_{z=z_0} \\
&= -\frac{z^2 f(z)}{\ell^2}\left(\frac{1}{2\sqrt{-f(z)\frac{\ell^2}{z^2}}}\left(2f(z)\frac{\ell^2}{z^3} - f'(z)\frac{\ell^2}{z^2}\right)\right)^2\Bigg|_{z=z_0} \\
&= \frac{z^4}{4\ell^4}\left(2f(z)\frac{\ell^2}{z^3} - f'(z)\frac{\ell^2}{z^2}\right)^2\Bigg|_{z=z_0} \\
&= \frac{z^4}{4\ell^4}\left(f'(z)\frac{\ell^2}{z^2}\right)^2\Bigg|_{z=z_0} \\
&= \frac{f'(z)^2}{4}\Bigg|_{z=z_0} \qquad (6)
\end{aligned}$$

Now,

$$f'(z) = \partial_z\left(-\left(\frac{z}{z_0}\right)^d\right) = -d\frac{z^{d-1}}{z_0^d} \qquad (7)$$

so,

$$f'(z_0) = -\frac{d}{z_0} \qquad (8)$$

so finally we see,

$$\kappa^2 = \frac{1}{4}\left(-\frac{d}{z_0}\right)^2 = \frac{d^2}{4z_0^2} \qquad (9)$$

and hence,

$$\kappa = \frac{d}{2z_0} \tag{10}$$

Then using the relation to temperature,

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\frac{d}{2z_0} = \frac{d}{4\pi z_0} \tag{11}$$

Hence we have,

$$\begin{aligned} S &= \frac{A}{4G} = \frac{1}{4G}\left(\frac{\ell}{z_0}\right)^{d-1} L^{d-1} = \frac{1}{4G}\left(\frac{4\pi T\ell}{d}\right)^{d-1} L^{d-1} \\ &= 4\pi\frac{\ell^{d-1}}{16\pi G}\left(\frac{4\pi T}{d}\right)^{d-1} L^{d-1} \\ &= 4\pi C\left(\frac{4\pi T}{d}\right)^{d-1} L^{d-1} \end{aligned} \tag{12}$$

using the relation $C = \frac{\ell^{d-1}}{16\pi G}$.

For $\mathcal{N} = 4$ Yang-Mills we have $d = 4$ and $C = N^2/(8\pi^2)$. Hence,

$$\begin{aligned} S &= 4\pi\frac{N^2}{8\pi^2}\left(\pi T\right)^3 L^3 \\ &= \frac{N^2\pi^2}{2}T^3 V \end{aligned} \tag{13}$$

2. We have,

$$g = -f dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2_{(d-1)}\ , \quad f = \frac{r^2}{\ell^2} + 1 - \frac{k}{r^{d-2}} \tag{14}$$

for constant k which has a conformal boundary as $r \to \infty$. We may see this by considering,

$$g' = -j dt^2 + \frac{1}{jr^4}dr^2 + d\Omega^2_{(d-1)}\ , \quad j = \frac{f}{r^2} = \frac{1}{\ell^2} + \frac{1}{r^2} - \frac{k}{r^d} \tag{15}$$

so that,

$$g = r^2 g' \tag{16}$$

Now take $r = 1/z$ so,

$$g = \frac{1}{z^2} g' \tag{17}$$

and noting that $dr = -dz/z^2$ so $dr^2 = dz^2/z^4$ and so $dr^2/r^4 = dz^2$, then,

$$g' = -j dt^2 + \frac{1}{j} dz^2 + d\Omega^2_{(d-1)} \ , \quad j = \frac{1}{\ell^2} + z^2 - k z^d \tag{18}$$

We see that since j is smooth near $z = 0$, then g' is a regular metric there. Then we may take z to be the defining function for g with respect to the regular g'. Now the geometry of the conformal boundary is just the geometry of g' at $z = 0$, so,

$$\begin{aligned} g_{boundary} = g'|_{z=0} &= -j dt^2 + d\Omega^2_{(d-1)} \ , \quad j = \frac{1}{\ell^2} + z^2 - k z^d \\ &= -\frac{1}{\ell^2} dt^2 + d\Omega^2_{(d-1)} \\ &= -d\tau^2 + d\Omega^2_{(d-1)} \end{aligned} \tag{19}$$

taking $\tau = t/\ell$.

Now let us compute the horizon area and surface gravity. We write k in terms of another constant r_0,

$$k = r_0^{d-2} \left(\frac{r_0^2}{\ell^2} + 1 \right) \tag{20}$$

so that the horizon is at $r = r_0$, ie. $f(r_0) = 0$. We are interested in the temperature $T = \kappa/2\pi$ as measured by the boundary field theory with time translation generator $\partial/\partial\tau$. Hence we need to compute the surface gravity κ with respect to the Killing field $K = \partial/\partial\tau$ (and not $\partial/\partial t$). Then as in the previous question,

$$\kappa^2 = -g^{AB} \left(\partial_A |K| \right) \left(\partial_B |K| \right) \big|_{r=r_0} \tag{21}$$

Let us firstly compute $|K|$,

$$|K| = \sqrt{K^2} = \sqrt{g_{tt} K^t K^t} \tag{22}$$

Now since $K = \partial/\partial\tau = \ell \partial/\partial t$ then we have,

$$K^t = \ell \tag{23}$$

and so,

$$|K| = \ell\sqrt{g_{tt}} = \ell\sqrt{-f} \tag{24}$$

Since $f = f(r)$ we have,

$$\begin{aligned}
\kappa^2 &= -g^{rr}\left(\partial_r|K|\right)\left(\partial_r|K|\right)\big|_{r=r_0} \\
&= -f\ell^2\left(\partial_r\sqrt{-f}\right)\left(\partial_r\sqrt{-f}\right)\Big|_{r=r_0} \\
&= -f\ell^2\left(-\frac{1}{2\sqrt{-f}}\partial_r f\right)^2\Bigg|_{r=r_0} \\
&= \frac{\ell^2}{4}\left(\partial_r f\right)^2\Bigg|_{r=r_0}
\end{aligned} \tag{25}$$

Now,

$$\begin{aligned}
\partial_r f\big|_{r=r_0} &= 2\frac{r}{\ell^2} + \frac{d-2}{r^{d-1}}r_0^{d-2}\left(\frac{r_0^2}{\ell^2}+1\right)\Bigg|_{r=r_0} \\
&= d\frac{r_0}{\ell^2} + (d-2)\frac{1}{r_0}
\end{aligned} \tag{26}$$

and so,

$$\begin{aligned}
\kappa^2 &= \frac{\ell^2}{4}\left((d+1)\frac{r_0}{\ell^2} + (d-1)\frac{1}{r_0}\right)^2 \\
&= \left(\frac{d}{2}\frac{r_0}{\ell} + \frac{d-2}{2}\frac{\ell}{r_0}\right)^2
\end{aligned} \tag{27}$$

so we find,

$$\kappa = \frac{d}{2}\frac{r_0}{\ell} + \frac{d-2}{2}\frac{\ell}{r_0} \tag{28}$$

and temperature $T = \kappa/2\pi$ is,

$$T = \frac{d}{4\pi}\frac{r_0}{\ell} + \frac{d-2}{4\pi}\frac{\ell}{r_0} \tag{29}$$

Firstly we can see there is a minimum temperature at $dT/dr_0 = 0$;

$$0 = \frac{dT}{dr_0} = \frac{d}{4\pi}\frac{1}{\ell} - \frac{d-2}{4\pi}\frac{\ell}{r_0^2} \tag{30}$$

so the minimum is for,

$$r_0 = \ell\sqrt{\frac{d-2}{d}} \tag{31}$$

so,

$$\begin{aligned} T_{min} &= \frac{d}{4\pi}\sqrt{\frac{d-2}{d}} + \frac{d-2}{4\pi}\sqrt{\frac{d}{d-2}} \\ &= \frac{1}{2\pi}\sqrt{d(d-2)} \end{aligned} \tag{32}$$

The area of the horizon is,

$$A = (r_0)^{d-1}\Omega_{(d-1)} \tag{33}$$

where $\Omega_{(d-1)}$ is the volume of a $(d-1)$-sphere, and hence entropy S is,

$$S = \frac{A}{4G} = \frac{1}{4G}(r_0)^{d-1}\Omega_{(d-1)} \tag{34}$$

Now using the relation $dE = TdS$ we see,

$$dS = \frac{d-1}{4G}(r_0)^{d-2}\Omega_{(d-1)}dr_0 \tag{35}$$

so,

$$\begin{aligned} dE = TdS &= \frac{d-1}{4G}(r_0)^{d-2}\left(\frac{d}{4\pi}\frac{r_0}{\ell} + \frac{d-2}{4\pi}\frac{\ell}{r_0}\right)\Omega_{(d-1)}dr_0 \\ &= (d-1)\frac{\ell^{d-1}}{16\pi G}\left(d\frac{r_0^{d-1}}{\ell^d} + (d-2)\frac{r_0^{d-3}}{\ell^{d-2}}\right)\Omega_{(d-1)}dr_0 \end{aligned} \tag{36}$$

Integrating,

$$E = E_0 + (d-1)\frac{\ell^{d-1}}{16\pi G}\left(\frac{r_0^d}{\ell^d} + \frac{r_0^{d-2}}{\ell^{d-2}}\right)\Omega_{(d-1)} \tag{37}$$

for constant E_0.

When $r_0 \to 0$ then $E = E_0 = E_{AdS}$, the energy of global AdS. Thus,

$$E = E_{AdS} + (d-1)\frac{\ell^{d-1}}{16\pi G}\left(\left(\frac{r_0}{\ell}\right)^d + \left(\frac{\ell}{r_0}\right)^{d-2}\right)\Omega_{(d-1)} \tag{38}$$

Then,

$$
\begin{aligned}
TS &= \left(\frac{d}{4\pi}\frac{r_0}{\ell} + \frac{d-2}{4\pi}\frac{\ell}{r_0}\right)\frac{1}{4G}(r_0)^{d-1}\Omega_{(d-1)} \\
&= \frac{\ell^{d-1}}{16\pi G}\left(d\left(\frac{r_0}{\ell}\right)^d + (d-2)\left(\frac{\ell}{r_0}\right)^{d-2}\right)\Omega_{(d-1)}
\end{aligned}
\tag{39}
$$

Then the free energy is,

$$
\begin{aligned}
F = E - TS &= E_{AdS} + \frac{\ell^{d-1}}{16\pi G}\left(-\left(\frac{r_0}{\ell}\right)^d + \left(\frac{\ell}{r_0}\right)^{d-2}\right)\Omega_{(d-1)} \\
&= E_{AdS} + C\left(-\left(\frac{r_0}{\ell}\right)^d + \left(\frac{\ell}{r_0}\right)^{d-2}\right)\Omega_{(d-1)}
\end{aligned}
\tag{40}
$$

Note the free energy of AdS is simply $F = E_{AdS}$ as $S = 0$ for global AdS with $r_0 = 0$. (Note that E_{AdS} is not necessarily zero due to the conformal anomaly for even d).

Hence we see $F - F_{AdS}$ becomes negative for $r_0 > \ell$. This corresponds to a temperature,

$$
\begin{aligned}
T_c &= \left.\frac{d}{4\pi}\frac{r_0}{\ell} + \frac{d-2}{4\pi}\frac{\ell}{r_0}\right|_{r_0=\ell} \\
&= \frac{d}{4\pi} + \frac{d-2}{4\pi} \\
&= \frac{d-1}{2\pi}
\end{aligned}
\tag{41}
$$