Imperial College London

MSci EXAMINATION May 2013

This paper is also taken for the relevant Examination for the Associateship

GENERAL RELATIVITY

For 4th-Year Physics Students

Monday 20th May 2013: 14:00 to 16:00

The paper consists of two *sections: A and B*
Section A contains one *question [40 marks total].*
Section B contains four *questions [30 marks each].*

Candidates are required to:
Answer **ALL** *parts of Section A and* **TWO QUESTIONS** *from Section B.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-, +, +, +)$ signature.

You may find the following formulae useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right)$$

The covariant derivative of a vector field is,

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha$$

and for a covector field is,

$$\nabla_\mu w_\nu \equiv \partial_\mu w_\nu - \Gamma^{\alpha}{}_{\mu\nu} w_\alpha$$

For a Lagrangian of a curve $x^\mu(\lambda)$ of the form,

$$L = \int d\lambda \, \mathcal{L}(x^\mu, \frac{dx^\mu}{d\lambda})$$

the Euler-Lagrange equations are,

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial (\frac{dx^\mu}{d\lambda})} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

Section A

Answer all of section A.

SECTION A

1. This question concerns accelerated motion in curved spacetimes.

(i) Suppose we have a massive particle following a trajectory $x^\mu(\tau)$ in a general spacetime, where τ is the particle's proper time. The particle's 4-velocity v^μ is defined as $v^\mu = dx^\mu/d\tau$. Why is $v^\mu v_\mu = -1$?

ANSWER:

Testing material given in lectures.

In an infinitesimal time $d\tau$ the spacetime interval will be,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} d\tau^2 \tag{1}$$

Now proper time for a particle is defined to be $ds^2 = -d\tau^2$ and so,

$$-1 = g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu} v^\mu v^\nu \tag{2}$$

Probable mark assignment: 4 for method, 1 for accuracy.

[5 marks]

(ii) Use the chain rule property of derivatives to show that the 4-velocity transforms as a vector.

ANSWER:

Testing material seen in lectures.

Under a coordinate transform, so that $x'^{\mu'} = x'^{\mu'}(x^\nu)$, then using the chain rule,

$$v'^{\mu'} = \frac{dx'^{\mu'}}{d\tau} = \frac{\partial x'^{\mu'}}{\partial x^\mu}\frac{dx^\mu}{d\tau} = \frac{\partial x'^{\mu'}}{\partial x^\mu} v^\mu \tag{3}$$

and hence this does indeed transform as a vector.

Probable mark assignment: 2 for knowing vector transformation, 2 for chain rule, 2 for general method, 1 for accuracy.

[7 marks]

(iii) The 4-acceleration a^μ is defined as $a^\mu = v^\nu \nabla_\nu v^\mu$. Show that in Minkowski spacetime this can be written as $a^\mu = d^2 x^\mu / d\tau^2$.

ANSWER:

Testing material given in lectures.

[This question continues on the next page . . .]

In Minkowski spacetime $\Gamma^{\alpha}{}_{\mu\nu} = 0$ and so,

$$a^{\mu} = v^{\nu}\nabla_{\nu}v^{\mu} = v^{\nu}\partial_{\nu}v^{\mu} = \frac{dx^{\nu}}{d\tau}\frac{\partial v^{\mu}}{\partial x^{\nu}} = \frac{dv^{\mu}}{d\tau} = \frac{d^2x^{\mu}}{d\tau^2} \tag{4}$$

Probable mark assignment: 2 for general method, 2 for $\Gamma = 0$ in Minkowski, 1 for chain rule, 1 for accuracy.

[6 marks]

(iv) By considering $v^{\nu}\nabla_{\nu}\left(v^{\mu}v_{\mu}\right)$, show that a^{μ} and v^{μ} are orthogonal 4-vectors (ie. $a^{\mu}v_{\mu} = 0$).

ANSWER:

Testing material discussed in lectures and example sheets, although not in this exact format.

Start with $v^{\mu}v_{\mu} = -1$. Then act with $v^{\nu}\partial_{\nu}$ to obtain,

$$0 = v^{\nu}\partial_{\nu}\left(v^{\mu}v_{\mu}\right) = v^{\nu}\nabla_{\nu}\left(v^{\mu}v_{\mu}\right) = 2v^{\mu}v^{\nu}\nabla_{\nu}v_{\mu} = 2v^{\mu}a_{\mu} \tag{5}$$

Probable mark assignment: 1 for general method, 1 for understanding ∇ is ∂ on a scalar, 2 for understanding $v^{\nu}\partial_{\nu}(v^2) = 0$, 2 for product rule, 1 for accuracy.

[7 marks]

(v) Show that since $a^{\mu}v_{\mu} = 0$ then a^{μ} must be a spacelike vector.

ANSWER:

Material discussed in lectures and example sheets.

Go to the instantaneous local inertial frame of the particle, so that at some point p on its trajectory then $v^{\mu} = (1, 0, 0, 0)$ and $g_{\mu\nu} = \eta_{\mu\nu}$ at that point. Then since $a^{\mu}v_{\mu} = 0$ then, $a^t = 0$, so that,

$$a^{\mu} = (0, a^i) \tag{6}$$

for 3-vector a^i, and so,

$$g_{\mu\nu}a^{\mu}a^{\nu} = \delta_{ij}a^ia^j = (a^1)^2 + (a^2)^2 + (a^3)^2 \geq 0 \tag{7}$$

at the point p. Note that if $g_{\mu\nu}a^{\mu}a^{\nu} = 0$ then $a^{\mu} = 0$ and the acceleration vanishes. So for non-vanishing acceleration $g_{\mu\nu}a^{\mu}a^{\nu} > 0$ at p and so a^{μ} is spacelike there. But we could have chosen p to be any point on the trajectory, and hence a^{μ} must always be spacelike.

Probable mark assignment: 2 for method, 2 for correct use of local inertial frame, 1 for accuracy

[5 marks]

[This question continues on the next page . . .]

(vi) Now consider a particle moving in the Schwarzschild spacetime, with coordinates $x^\mu = (t, r, \theta, \phi)$ and metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{8}$$

Consider a particle accelerating to stay at constant spatial position, so that r, θ, ϕ remain constant. Use the fact that,

$$\Gamma^r{}_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right), \Gamma^t{}_{tt} = \Gamma^\theta{}_{tt} = \Gamma^\phi{}_{tt} = 0 \tag{9}$$

to calculate the norm $\sqrt{a^\mu a_\mu}$ of the 4-acceleration of the particle for $r > 2M$. What happens to this quantity at $r = 2M$ and why?

ANSWER:

Part of this has featured in an example sheet question, but not in the same format, so this will be unfamiliar to the great majority of students.

The 4-velocity is $v^\mu = (f, 0, 0, 0)$ for some function f since the particle is kept at fixed position in space. Then since $g_{\mu\nu} v^\mu v^\nu = -1$ then,

$$-\left(1 - \frac{2M}{r}\right) f^2 = -1 \tag{10}$$

so that,

$$f = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \tag{11}$$

The 4-acceleration is,

$$a^\mu = v^\nu \nabla_\nu v^\mu = v^\nu \partial_\nu v^\mu + v^\nu v^\alpha \Gamma^\mu{}_{\nu\alpha} = v^t \partial_t v^\mu + v^t v^t \Gamma^\mu{}_{tt} = f^2 \Gamma^\mu{}_{tt} = \frac{1}{1 - \frac{2M}{r}} \Gamma^\mu{}_{tt} \tag{12}$$

Using the Christoffel components given in the question we have, $a^t = a^\theta = a^\phi = 0$ and,

$$a^r = \frac{1}{1 - \frac{2M}{r}} \Gamma^r{}_{tt} = \frac{1}{1 - \frac{2M}{r}} \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) = \frac{M}{r^2} \tag{13}$$

Then the norm,

$$a^\mu a_\mu = g_{rr} (a^r)^2 = \frac{1}{1 - \frac{2M}{r}} \frac{M^2}{r^4} \tag{14}$$

[This question continues on the next page ...]

so that,

$$\sqrt{a^{\mu}a_{\mu}} = \frac{1}{\sqrt{1-\frac{2M}{r}}}\frac{M}{r^2} \tag{15}$$

This is indeed spacelike (ie. > 0) for $r > 2M$ and diverges, $\sqrt{a^{\mu}a_{\mu}} \to \infty$ at $r = 2M$. This is the horizon of the black hole, and an infinite acceleration is required to keep a timelike particle sitting at the horizon.

Probable mark assignment: 2 for method, 2 for correctly getting velocity, 4 for correctly computing acceleration and norm, 2 for interpretation

[10 marks]

[Total 40 marks]

Section B

Answer 2 out of the 4 questions in the following section.

SECTION B

2. This question concerns the Einstein equations for a star made of perfect fluid.

(i) State the stress tensor $T_{\mu\nu}$ for a perfect fluid in terms of the fluid energy density ρ, pressure P and 4-velocity u^μ (recall $u^\mu u_\mu = -1$). Take n^μ to be orthogonal to u^μ and consider $n^\mu \nabla^\nu T_{\mu\nu}$ to derive one of the fluid equations,

$$n^\mu \left(\partial_\mu P + (\rho + P)\, u^\nu \nabla_\nu u_\mu \right) = 0 \tag{1}$$

ANSWER:

Perfect fluid equations were discussed in lectures and example sheets, but not in this exact way.

The stress tensor is;

$$T_{\mu\nu} = \rho u_\mu u_\nu + P \left(u_\mu u_\nu + g_{\mu\nu} \right) \tag{2}$$

Conservation is;

$$\begin{aligned} 0 = \nabla^\mu T_{\mu\nu} &= (\nabla^\mu \rho) u_\mu u_\nu + \rho \left(u_\mu \nabla^\mu u_\nu + u_\nu \nabla^\mu u_\mu \right) \\ &\quad + (\nabla^\mu P) \left(u_\mu u_\nu + g_{\mu\nu} \right) + P \left(u_\mu \nabla^\mu u_\nu + u_\nu \nabla^\mu u_\mu \right) \end{aligned} \tag{3}$$

where we recall $\nabla^\mu g_{\mu\nu} = 0$. Then contracting with n^ν and using $n^\nu u_\nu = 0$ gives,

$$\begin{aligned} 0 = u^\nu \nabla^\mu T_{\mu\nu} &= \rho \left(n^\nu u_\mu \nabla^\mu u_\nu \right) + (\nabla^\mu P) \left(n^\nu g_{\mu\nu} \right) + P \left(n^\nu u_\mu \nabla^\mu u_\nu \right) \\ &= n^\nu (\nabla_\mu P) + n^\nu (\rho + P) \left(u_\mu \nabla^\mu u_\nu \right) \end{aligned} \tag{4}$$

and $\nabla_\mu P = \partial_\mu P$ as it is a scalar, and hence this gives the result.

Probable mark assignment: 3 for stress tensor, 3 for method, 2 for accuracy

[8 marks]

(ii) Consider a time independent, spherically symmetric metric describing a star. We take coordinates $x^\mu = (t, r, \theta, \phi)$ and a metric,

$$ds^2 = -e^{2f(r)} dt^2 + \frac{1}{h(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{5}$$

where $f(r)$ and $h(r)$ are functions of r. The star is made of perfect fluid. Since it is static then $u^\mu = (T(r), 0, 0, 0)$. Firstly determine the function $T(r)$. Then using part i) above, choose $n^\mu = (0, 1, 0, 0)$ and compute the necessary ${\Gamma^\alpha}_{\mu\nu}$ components to show that,

$$\frac{dP}{dr} = -(\rho + P) \frac{df}{dr} \tag{6}$$

[This question continues on the next page . . .]

ANSWER:

The stellar metric was not covered in lectures or example sheets, so this is unseen material.

Now since $g_{\mu\nu}u^\mu u^\nu = -1$ we have,

$$-1 = g_{\mu\nu}u^\mu u^\nu = g_{tt}u^t u^t = -e^{2f(r)}T^2 \tag{7}$$

and so,

$$T = e^{-f(r)} \tag{8}$$

Consider the equation from part i) with $n^\mu = (0, 1, 0, 0)$, then,

$$\begin{aligned}
0 &= n^\mu \left(\partial_\mu P + (\rho + P)\, u^\nu \nabla_\nu u_\mu\right) \\
&= \partial_r P + (\rho + P)\, u^\nu \nabla_\nu u_r \\
&= \partial_r P + (\rho + P)\left(u^\nu \partial_\nu u_r + \Gamma^\alpha{}_{\nu r} u^\nu u_\alpha\right) \\
&= \partial_r P + (\rho + P)\left(u^t \partial_t u_r + \Gamma^t{}_{tr} u^t u_t\right) \\
&= \partial_r P + (\rho + P)\,\Gamma^t{}_{tr} g_{tt} u^t u^t \\
&= \partial_r P + (\rho + P)\,\Gamma^t{}_{tr} e^{2f(r)} T^2 \\
&= \partial_r P + (\rho + P)\,\Gamma^t{}_{tr}
\end{aligned} \tag{9}$$

Now we require $\Gamma^t{}_{tr}$;

$$\begin{aligned}
\Gamma^t{}_{tr} &= \frac{1}{2} g^{t\nu}\left(\partial_t g_{r\nu} + \partial_r g_{t\nu} - \partial_\nu g_{tr}\right) \\
&= \frac{1}{2} g^{tt}\left(\partial_t g_{rt} + \partial_r g_{tt} - \partial_t g_{tr}\right) \\
&= \frac{1}{2} g^{tt} \partial_r g_{tt} \\
&= \frac{1}{2} e^{-2f(r)} \partial_r e^{2f(r)} \\
&= \partial_r f
\end{aligned} \tag{10}$$

and then,

$$\begin{aligned}
0 &= \partial_r P + (\rho + P)\,\Gamma^t{}_{tr} \\
&= \partial_r P + (\rho + P)\,\partial_r f
\end{aligned} \tag{11}$$

as required.

Probable mark assignment: 2 marks for getting T, 4 marks for overall method, 3 for accuracy

[9 marks]

[This question continues on the next page . . .]

(iii) The non-zero components of Ricci with one index up and one down are,

$$
\begin{aligned}
R^t{}_t &= -\frac{2h}{r}\frac{df}{dr} - \frac{1}{2}\frac{dh}{dr}\frac{df}{dr} + L(r) \,, \quad R^r{}_r = -\frac{1}{r}\frac{dh}{dr} - \frac{1}{2}\frac{dh}{dr}\frac{df}{dr} + L(r) \\
R^\theta{}_\theta &= R^\phi{}_\phi = \frac{1}{r^2}(1-h) - \frac{1}{2r}\frac{dh}{dr} - \frac{h}{r}\frac{df}{dr}
\end{aligned} \tag{12}
$$

where $L(r)$ is a function of f and h you will not need to know explicitly.

Calculate the Einstein tensor components, G_{tt} and G_{rr}, and then the corresponding tt and rr components of the Einstein equations. Define,

$$h(r) = 1 - \frac{2m(r)}{r} \tag{13}$$

and then show these Einstein equations yield,

$$\frac{dm}{dr} = 4\pi G_N r^2 \rho \,, \qquad \frac{df}{dr} = \frac{m + 4\pi G_N r^3 P}{r^2 - 2mr} \tag{14}$$

[These are the *Tolman-Oppenheimer-Volkoff* equations for a relativistic star.]

ANSWER:

Again unseen material.

The Ricci scalar is,

$$
\begin{aligned}
R &= R^t{}_t + R^r{}_r + R^\theta{}_\theta + R^\phi{}_\phi \\
&= R^t{}_t + R^r{}_r + 2R^\theta{}_\theta \\
&= \frac{2}{r^2} - \frac{2h(r)}{r^2} - \frac{2h'(r)}{r} - f'(r)\left(\frac{4h(r)}{r} + h'(r)\right) + 2L
\end{aligned}
$$

so that,

$$\frac{1}{2}R = \frac{1}{r^2} - \frac{h(r)}{r^2} - \frac{h'(r)}{r} - f'(r)\left(\frac{2h(r)}{r} + \frac{h'(r)}{2}\right) + L$$

Then,

$$G_{tt} = R_{tt} - \frac{1}{2}g_{tt}R = g_{tt}R^t{}_t - \frac{1}{2}g_{tt}R = g_{tt}\left(R^t{}_t - \frac{1}{2}R\right) \tag{15}$$

where,

$$R^t{}_t - \frac{1}{2}R = -\left(\frac{1}{r^2} - \frac{h(r)}{r^2}\right) \tag{16}$$

so,

$$G_{tt} = e^{2f(r)}\left(\frac{1}{r^2} - \frac{h(r)}{r^2} - \frac{h'(r)}{r}\right) \tag{17}$$

[This question continues on the next page . . .]

And,

$$G_{rr} = R_{rr} - \frac{1}{2}g_{rr}R = g_{rr}R^r{}_r - \frac{1}{2}g_{rr}R = g_{rr}\left(R^r{}_r - \frac{1}{2}R\right) \tag{18}$$

where,

$$R^r{}_r - \frac{1}{2}R = -\left(\frac{1}{r^2} - \frac{h(r)}{r^2}\right) + 2\frac{f'(r)h(r)}{r} \tag{19}$$

so,

$$\begin{aligned} G_{rr} &= \frac{1}{h(r)}\left(-\left(\frac{1}{r^2} - \frac{h(r)}{r^2} - \frac{h'(r)}{r}\right) + 2\frac{f'(r)h(r)}{r}\right) \\ &= \frac{1}{r^2} - \frac{1}{h(r)r^2} + \frac{2}{r}f'(r) \end{aligned} \tag{20}$$

The Einstein equation is,

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{21}$$

Now,

$$T_{tt} = \rho u_t^2 + P\left(u_t u_t + g_{tt}\right) \tag{22}$$

and $u_t = g_{tt}u^t = -e^{2f(r)}T = -e^{f(r)}$, so,

$$T_{tt} = \rho e^{2f(r)} \tag{23}$$

and,

$$T_{rr} = \rho u_r^2 + P\left(u_r u_r + g_{rr}\right) = \frac{P}{h(r)} \tag{24}$$

Then the *tt* component of the Einstein equation is;

$$e^{2f(r)}\left(\frac{1}{r^2} - \frac{h(r)}{r^2} - \frac{h'(r)}{r}\right) = 8\pi G_N\left(\rho e^{2f(r)}\right) \tag{25}$$

so we find,

$$\frac{1}{r^2} - \frac{h(r)}{r^2} - \frac{h'(r)}{r} = 8\pi G_N\rho \tag{26}$$

And for the *rr* component,

$$\frac{1}{r^2} - \frac{1}{h(r)r^2} + \frac{2}{r}f'(r) = 8\pi G_N\left(\frac{P}{h(r)}\right) \tag{27}$$

[This question continues on the next page . . .]

so,

$$-\frac{1}{r^2} + \frac{h(r)}{r^2} + \frac{2h(r)}{r} f'(r) = 8\pi G_N P \tag{28}$$

Now if,

$$h(r) = 1 - \frac{2m(r)}{r} \tag{29}$$

then,

$$h'(r) = -\frac{2m'(r)}{r} + \frac{2m(r)}{r^2} \tag{30}$$

Substituting into the tt component,

$$\begin{aligned} 8\pi G_N \rho &= \frac{1}{r^2} - \frac{h(r)}{r^2} - \frac{h'(r)}{r} \\ &= \frac{1}{r^2} - \frac{1}{r^2}\left(1 - \frac{2m(r)}{r}\right) - \frac{1}{r}\left(-\frac{2m'(r)}{r} + \frac{2m(r)}{r^2}\right) \\ &= +\frac{2m'(r)}{r^2} \end{aligned} \tag{31}$$

Hence we obtain the required result,

$$m'(r) = 4\pi G_N \rho(r) \tag{32}$$

Substituting into the rr component,

$$\begin{aligned} 8\pi G_N P &= -\frac{1}{r^2} + \frac{h(r)}{r^2} + \frac{2h(r)}{r} f'(r) \\ &= -\frac{1}{r^2} + \frac{1}{r^2}\left(1 - \frac{2m(r)}{r}\right) + \frac{2}{r} f'(r)\left(1 - \frac{2m(r)}{r}\right) \\ &= -\frac{2m(r)}{r^3} + \frac{2}{r} f'(r)\left(1 - \frac{2m(r)}{r}\right) \end{aligned} \tag{33}$$

Hence,

$$2m(r) + 8\pi G_N r^3 P = 2r^2 f'(r)\left(1 - \frac{2m(r)}{r}\right) \tag{34}$$

and so we obtain the required result,

$$f'(r) = \frac{m(r) + 4\pi G_N r^3 P}{r^2 - 2m(r) r} \tag{35}$$

Probable mark assignment: 3 marks for method, 1 for Einstein equations, 4 for accuracy

[8 marks]

[This question continues on the next page . . .]

(iv) If the star has a surface at $r = R$, then outside this surface for $r > R$ there is no fluid matter ie. $\rho = P = 0$. Solve the equations to find $m(r)$ and show $e^{2f(r)} = h(r)$ is a solution. Hence determine the metric in the star's exterior. What is this exterior spacetime? What is its mass in terms of $m(r)$?

ANSWER:

Again unseen material.

So for $r > R$ we have $\rho = P = 0$, and so,

$$m'(r) = 0 \ , \quad f'(r) = \frac{m(r)}{r^2 - 2m(r)r} \tag{36}$$

Firstly then $m(r)$ is constant, say $m(r) = M$. Then,

$$f'(r) = \frac{M}{r^2 - 2Mr} = \left(+\frac{M}{r^2}\right)\frac{1}{1 - \frac{2M}{r}} = \frac{1}{2}\frac{h'(r)}{h(r)} \tag{37}$$

so,

$$k\, e^{2f(r)} = h(r) \tag{38}$$

for some constant k. Hence $e^{2f(r)} = h(r)$ is a solution. Then the metric is,

$$\begin{aligned} ds^2 &= -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ &= -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned} \tag{39}$$

which is Schwarzschild - see Qu A1 - with mass M. Note that since $m'(r) = 0$ for $r > R$, and then $M = m(r)$, then at $r = R$ we must have $M = m(R)$.

Probable mark assignment: 2 for method, 1 for accuracy, 2 for interpretation

[5 marks]

[Total 30 marks]

3. This question concerns scalar fields and FLRW spacetime.

(i) Consider a scalar field $\phi(t, x^i)$ with potential $V(\phi)$ on a general spacetime. Its stress tensor is given as,

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\alpha\phi\nabla_\alpha\phi\right) - g_{\mu\nu}V(\phi) \tag{1}$$

Using the equation of motion of this scalar field,

$$\nabla^\alpha\nabla_\alpha\phi = \frac{dV(\phi)}{d\phi} \tag{2}$$

show that the stress energy is conserved.

ANSWER:

The stress tensor for a scalar field was not discussed in lectures, so this is new material.

Consider,

$$\begin{aligned}\nabla^\mu T_{\mu\nu} &= \nabla^\mu\left(\nabla_\mu\phi\nabla_\nu\phi\right) - \frac{1}{2}g_{\mu\nu}\nabla^\mu\left(\nabla^\alpha\phi\nabla_\alpha\phi + 2V(\phi)\right) \\ &= (\nabla^2\phi)(\nabla_\nu\phi) + (\nabla^\mu\nabla_\nu\phi)(\nabla_\mu\phi) - g_{\mu\nu}(\nabla^\mu\nabla^\alpha\phi)(\nabla_\alpha\phi) - g_{\mu\nu}\nabla^\mu V(\phi)\end{aligned} \tag{3}$$

Now recall that $\nabla_\mu\nabla_\nu\phi = \nabla_\nu\nabla_\mu\phi$, so,

$$\begin{aligned}\nabla^\mu T_{\mu\nu} &= (\nabla^2\phi)(\nabla_\nu\phi) + (\nabla^\mu\nabla_\nu\phi)(\nabla_\mu\phi) - (\nabla^\alpha\nabla_\nu\phi)(\nabla_\alpha\phi) - \partial_\nu V(\phi) \\ &= (\nabla^2\phi)(\nabla_\nu\phi) - \partial_\nu V(\phi) \\ &= (\nabla^2\phi)(\nabla_\nu\phi) - \frac{dV(\phi)}{d\phi}\frac{\partial\phi}{\partial x^\nu} \\ &= (\nabla_\nu\phi)\left(\nabla^2\phi - \frac{dV(\phi)}{d\phi}\right) \\ &= 0\end{aligned} \tag{4}$$

due to scalar equation of motion.

Probable mark assignment: 2 for method, 1 for product rule (in expanding $\nabla^\mu T_{\mu\nu}$), 2 for chain rule (in writing $\partial_\nu V = dV/d\phi\partial\phi/\partial x^\nu$), 3 for accuracy.

[8 marks]

(ii) Take spacetime to be FLRW, with coordinates $x^\mu = (t, x^i)$ with $i = 1, 2, 3$, and,

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j \tag{5}$$

Compute all the Christoffel symbol components $\Gamma^\alpha{}_{\mu\nu}$ for this metric.

[This question continues on the next page . . .]

ANSWER:

An example sheet question covered computing Christoffel components for FLRW, so conscientious students will have done this exercise before.

We use,

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right) = 0 \tag{6}$$

to compute the components;

$$\Gamma^t{}_{tt}\ , \quad \Gamma^t{}_{it} = \Gamma^t{}_{ti}\ , \quad \Gamma^t{}_{ij} = \Gamma^t{}_{ji}\ , \quad \Gamma^j{}_{it} = \Gamma^j{}_{ti}\ , \quad \Gamma^i{}_{jk} \tag{7}$$

The inverse metric is,

$$g^{tt} = -1\ , \quad g^{ij} = \frac{1}{a^2}\delta^{ij} \tag{8}$$

with other components zero.

Firstly,

$$\Gamma^t{}_{tt} = \frac{1}{2} g^{tt} \left(\partial_t g_{tt} + \partial_t g_{tt} - \partial_t g_{tt} \right) = 0 \tag{9}$$

Then,

$$\Gamma^t{}_{it} = \frac{1}{2} g^{tt} \left(\partial_i g_{tt} + \partial_t g_{it} - \partial_t g_{it} \right) = 0 \tag{10}$$

Then,

$$\begin{aligned} \Gamma^t{}_{ij} &= \frac{1}{2} g^{tt} \left(\partial_i g_{jt} + \partial_j g_{it} - \partial_t g_{ij} \right) = -\frac{1}{2} g^{tt} \partial_t g_{ij} \\ &= -\frac{1}{2}(-1)\partial_t \left(a(t)^2 \delta_{ij} \right) \\ &= \delta_{ij} a \partial_t a \end{aligned} \tag{11}$$

Then,

$$\begin{aligned} \Gamma^i{}_{jt} &= \frac{1}{2} g^{ik} \left(\partial_j g_{tk} + \partial_t g_{jk} - \partial_k g_{jt} \right) = \frac{1}{2}(\frac{1}{a^2}\delta^{ik}) \left(\partial_t g_{jk} \right) \\ &= \frac{1}{2}\frac{1}{a^2}\delta^{ik}\partial_t \left(a^2 \delta_{jk} \right) \\ &= \frac{1}{a^2}\delta^{ik}\delta_j k a \partial_t a \\ &= \frac{1}{a}\delta^i_j \partial_t a \end{aligned} \tag{12}$$

Finally,

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im} \left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij} \right) = 0 \tag{13}$$

[This question continues on the next page . . .]

Probable mark assignment: 1 for inverse metric, 4 marks for method, 3 for accuracy

[8 marks]

(iii) Take the scalar to have the symmetries of FLRW, so that ϕ is only a function of time t. Also take its potential to vanish, $V(\phi) = 0$ - this is a *massless* scalar field. Solve the massless scalar equation of motion to show that,

$$\phi(t) - \phi(t_0) = k \int_{t_0}^{t} dt' \frac{1}{a(t')^3} \tag{14}$$

where k is a constant of integration.

ANSWER:

This calculation is not covered in lectures or example sheets.

If $V = 0$ then the scalar equation of motion is,

$$0 = \nabla^2\phi \quad = \quad g^{\mu\nu}\nabla_\mu\partial_\nu\phi = g^{\mu\nu}\left(\partial_\mu\partial_\nu\phi + \Gamma^\alpha{}_{\mu\nu}\partial_\alpha\phi\right) \tag{15}$$

Now if $\phi = \phi(t)$, then,

$$\begin{aligned} 0 &= g^{\mu\nu}\left(\partial_\mu\partial_\nu\phi - \Gamma^\alpha{}_{\mu\nu}\partial_\alpha\phi\right) \\ &= g^{tt}\partial_t\partial_t\phi - g^{\mu\nu}\Gamma^t{}_{\mu\nu}\partial_t\phi \end{aligned} \tag{16}$$

and using the fact that $\Gamma^t{}_{tt} = 0$ and $g^{ti} = 0$ then,

$$\begin{aligned} 0 &= -\partial_t\partial_t\phi - g^{ij}\Gamma^t{}_{ij}\partial_t\phi \\ &= -\partial_t^2\phi - \frac{1}{a^2}\delta^{ij}\left(\delta_{ij}a\partial_t a\right)\partial_t\phi \\ &= -\partial_t^2\phi - \frac{3}{a}\partial_t a\partial_t\phi \end{aligned} \tag{17}$$

where we use $\delta^{ij}\delta_{ij} = 3$. Then,

$$\frac{\partial_t(\partial_t\phi)}{\partial_t\phi} = -3\frac{\partial_t a}{a} \tag{18}$$

and so,

$$\partial_t\phi = k\frac{1}{a^3} \tag{19}$$

for some constant k, so integrating from a time t_0,

$$\phi(t) = \phi(t_0) + k\int_{t_0}^{t} dt' \frac{1}{a(t')^3} \tag{20}$$

as required.

Probable mark assignment: 4 marks for method, 1 for $\delta^{ij}\delta_{ij} = 3$, 3 for accuracy

[8 marks]

[This question continues on the next page . . .]

(iv) A comoving perfect fluid with equation of state $P = w\rho$, for constant w, obeys,

$$\rho(t) = \frac{c}{a(t)^{3(1+w)}} \tag{21}$$

in FLRW where c is a constant. Show the stress tensor for the massless scalar in FLRW is the same as that for a perfect fluid with $w = +1$ (a 'stiff fluid'). Find the relation between the constants c and k.

ANSWER:

This calculation is not covered in lectures or example sheets.

The velocity of a perfect fluid at rest in FLRW is $v^\mu = (1, 0, 0, 0)$ for our metric (so $v^\mu v_\mu = -1$). Then,

$$T_{\mu\nu} = (\rho + P)\, v_\mu v_\nu + P g_{\mu\nu} \tag{22}$$

so if $P = w\rho$ then,

$$T_{tt} = \rho\,(1 + w)\, v_t v_t + w\rho g_{tt} = \rho\,(1 + w) - w\rho = \rho \tag{23}$$

and,

$$T_{ti} = \rho\,(1 + w)\, v_t v_i + w\rho g_{ti} = 0 \tag{24}$$

and,

$$T_{ij} = \rho\,(1 + w)\, v_i v_j + w\rho g_{ij} = a^2 w\rho\delta_{ij} \tag{25}$$

For the massless scalar field we have,

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\alpha\phi\nabla_\alpha\phi\right) \\ &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{tt}(\partial_t\phi)^2 \\ &= \nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}g_{\mu\nu}(\partial_t\phi)^2 \end{aligned} \tag{26}$$

so,

$$T_{tt} = (\partial_t\phi)^2 + \frac{1}{2}g_{tt}(\partial_t\phi)^2 = (\partial_t\phi)^2 - \frac{1}{2}(\partial_t\phi)^2 = \frac{1}{2}(\partial_t\phi)^2 \tag{27}$$

and,

$$T_{ti} = \nabla_t\phi\nabla_i\phi + \frac{1}{2}g_{ti}(\partial_t\phi)^2 = 0 \tag{28}$$

and,

$$T_{ij} = \nabla_i\phi\nabla_j\phi + \frac{1}{2}g_{ij}(\partial_t\phi)^2 = \frac{1}{2}a^2\delta_{ij}(\partial_t\phi)^2 \tag{29}$$

[This question continues on the next page . . .]

so we see that the massless scalar has the same stress tensor as the fluid if,

$$\frac{1}{2}(\partial_t\phi)^2 = \rho \ , \quad \frac{1}{2}a^2\delta_{ij}(\partial_t\phi)^2 = a^2 w\rho\delta_{ij} \tag{30}$$

which implies that $w = 1$ and,

$$\frac{1}{2}(\partial_t\phi)^2 = \rho \tag{31}$$

Let us check this. Recall,

$$\phi(t) - \phi(t_0) = k\int_{t_0}^{t} dt' \frac{1}{a(t')^3} \ , \quad \rho(t) = \frac{c}{a(t)^{3(1+w)}} \tag{32}$$

so that,

$$\partial_t\phi(t) = k\frac{1}{a(t)^3} \tag{33}$$

Then,

$$\rho = \frac{1}{2}(\partial_t\phi)^2 = \frac{k^2}{2}\frac{1}{a(t)^6} \tag{34}$$

and this equals,

$$\rho = \frac{c}{a(t)^{3(1+w)}} \tag{35}$$

provided we have $w = 1$, and then,

$$c = \frac{k^2}{2} \tag{36}$$

Probable mark assignment: 3 marks for method, 1 for stress tensor of fluid, 2 for accuracy

[6 marks]

[Total 30 marks]

4. Before Einstein completed his equations of General Relativity, an alternative theory was proposed by Nordström. As with Einstein's theory, in Nordström's theory gravity is due to curvature of spacetime. However, the theory is much simpler as the spacetime metric cannot be general, but is given in terms of one function $\phi(t, x^i)$, as,

$$ds^2 = \phi^2 \left(-dt^2 + dx^i dx^i \right) \tag{1}$$

where we have taken coordinates $x^\mu = (t, x^i)$ with $i = 1, 2, 3$. Particle motion is then just as for GR but in this particular curved spacetime.

(i) A massive particle in the spacetime follows the timelike geodesic $x^\mu = (T(\tau), X^i(\tau))$ where τ is its proper time. Assume the Nordstöm scalar ϕ is time independent, so $\phi = \phi(x^i)$. Use the Euler-Lagrange equations to vary the Lagrangian,

$$L = \int d\tau \, \phi^2(X) \left(-\left(\frac{dT}{d\tau}\right)^2 + \frac{dX^i}{d\tau}\frac{dX^i}{d\tau} \right) \tag{2}$$

with respect to X^i and hence determine that the geodesic is governed by,

$$\frac{d^2 X^i}{d\tau^2} = -\frac{1}{\phi^3}\frac{\partial \phi(X)}{\partial X^i} \tag{3}$$

ANSWER:

Nordstöms theory was not covered in lectures or example sheets so this is new material.

So,

$$L = \int d\tau \mathcal{L} = \int d\tau \, \phi^2(X) \left(-\left(\frac{dT}{d\tau}\right)^2 + \frac{dX^i}{d\tau}\frac{dX^i}{d\tau} \right) \tag{4}$$

Note that since τ is proper time we have,

$$\mathcal{L} = \phi^2(X) \left(-\left(\frac{dT}{d\tau}\right)^2 + \frac{dX^i}{d\tau}\frac{dX^i}{d\tau} \right) = -1 \tag{5}$$

Vary with respect to X; then,

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \frac{dX^i}{d\tau}}\right) = \frac{\partial \mathcal{L}}{\partial X^i} \tag{6}$$

so,

$$\begin{aligned}
\frac{d}{d\tau}\left(+2\phi^2 \frac{dX^i}{d\tau}\right) &= \frac{\partial(\phi(X)^2)}{\partial X^i}\left(-\left(\frac{dT}{d\tau}\right)^2 + \frac{dX^i}{d\tau}\frac{dX^i}{d\tau} \right) \\
2\phi^2 \frac{d}{d\tau}\left(\frac{dX^i}{d\tau}\right) &= -\phi^{-2}\frac{\partial(\phi(X)^2)}{\partial X^i} \\
2\phi^2 \frac{d^2 X^i}{d\tau^2} &= -2\phi^{-1}\frac{\partial \phi(X)}{\partial X^i}
\end{aligned} \tag{7}$$

[This question continues on the next page . . .]

so then,

$$\frac{d^2X^i}{d\tau^2} = -\frac{1}{\phi^3}\frac{\partial\phi(X)}{\partial X^i} \tag{8}$$

Probable mark assignment: 2 marks for method, 3 for correctly using Euler, 1 for correctly using $\mathcal{L} = -1$, 2 for accuracy.

[8 marks]

(ii) Nordström proposed a field equation governing ϕ to be,

$$\frac{1}{\phi^3}\left(-\partial_t^2 + \partial_i^2\right)\phi = \kappa\rho \tag{9}$$

where ρ is the matter energy density and κ is a constant. Consider a Newtonian limit similar to that in GR by taking $\phi = 1 + \epsilon\Phi$ and time independent with $\epsilon \ll 1$. Use your answer to part i) to identify the Newtonian gravitational potential and hence determine the constant κ in terms of Newton's constant G_N.

ANSWER:

Nordstöms theory was not covered in lectures or example sheets so this is new material.

We may expand the geodesic equation we derived in ϵ so that,

$$\begin{aligned}\frac{d^2X^i}{d\tau^2} &= -\frac{1}{\phi^3}\frac{\partial\phi(X)}{\partial X^i} = -\frac{1}{(1+\epsilon\Phi+...)^3}\frac{\partial}{\partial X^i}(1+\epsilon\Phi+...)\\ &= -\epsilon\frac{\partial\Phi}{\partial X^i} + ...\end{aligned} \tag{10}$$

to lowest order in ϵ. Hence to identify with Newtonian motion then $\epsilon\Phi$ must be the usual Newtonian potential, where,

$$\delta^{ij}\partial_i\partial_j(\epsilon\Phi) = 4\pi G_N\rho \tag{11}$$

Probable mark assignment so far: 2 mark for method, 1 for accuracy. 2 marks for identifying Newtonian potential.

We may expand Nordström's field equation,

$$\begin{aligned}\kappa\rho &= \frac{1}{\phi^3}\left(-\partial_t^2 + \partial_i^2\right)\phi = \frac{1}{(1+\epsilon\Phi+...)^3}\left(-\partial_t^2+\partial_i^2\right)(1+\epsilon\Phi+...)\\ &= \frac{1}{(1+\epsilon\Phi+...)^3}\partial_i^2(\epsilon\Phi+...)\\ &= \partial_i^2(\epsilon\Phi) + ...\end{aligned} \tag{12}$$

[This question continues on the next page . . .]

to lowest order, and hence we see that for this to be compatible with Newton's equation above we require,

$$4\pi G_N = \kappa \tag{13}$$

Probable mark assignment: 1 mark for method, 1 for accuracy. 1 mark for getting constant κ.

[8 marks]

(iii) Like GR, Nordström's theory predicts a gravitational redshift. Suppose a particle is at fixed position x_1^i and emits radiation with frequency ω in its rest-frame. At what frequency does a particle at fixed position x_2^i receive it, assuming that ϕ is time independent? Consider this redshift in the Newtonian limit - can it be used to distinguish Einstein's GR from Nordström's theory?

ANSWER:

Nordstöm's theory was not covered in lectures or example sheets; however clearly the calculation is close to the GR Newtonian redshift calculation which was covered in detail in lectures and example sheets.

The emitter particle has 4-velocity,

$$v^\mu_{(1)} = \left(\frac{1}{\phi(x_1)}, 0, 0, 0\right) \tag{14}$$

and the receiver has 4-velocity,

$$v^\mu_{(2)} = \left(\frac{1}{\phi(x_2)}, 0, 0, 0\right) \tag{15}$$

so that $v^2_{(1)} = v^2_{(2)} = -1$. Let the emitter have particle proper time τ_1 and the receiver τ_2. Then,

$$\frac{dt}{d\tau_1} = v^t_{(1)} = \frac{1}{\phi(x_1)} , \quad \frac{dt}{d\tau_2} = v^t_{(2)} = \frac{1}{\phi(x_2)} \tag{16}$$

so that,

$$\frac{d\tau_2}{d\tau_1} = \frac{\phi(x_2)}{\phi(x_1)} \tag{17}$$

So the observed frequency ω_{obs} is related to the emitted frequency ω as,

$$\frac{\omega_{obs}}{\omega} = \frac{d\tau_1}{d\tau_2} = \frac{\phi(x_1)}{\phi(x_2)} \tag{18}$$

In the Newtonian limit this gives,

$$\frac{\omega_{obs}}{\omega} = \frac{\phi(x_1)}{\phi(x_2)} \simeq \frac{1 + \epsilon\Phi(x_1)}{1 + \epsilon\Phi(x_2)} \simeq 1 + \epsilon\left(\Phi(x_1) - \Phi(x_2)\right) + ... \tag{19}$$

[This question continues on the next page . . .]

and since $\epsilon\Phi$ is the Newtonian potential Φ_N, then,

$$\frac{\omega_{obs}}{\omega} = 1 + \Phi_N(x_1) - \Phi_N(x_2) + ... \tag{20}$$

which is exactly the usual GR result. So the gravitational redshift predicted by Nordström's theory is the same as that for GR.

Probable mark assignment: 1 mark for the correct 4-velocities, 1 mark for identifying the relation between coordinate time and the two proper times, 1 mark for method and 1 for accuracy for computing the frequency redshift. 1 mark for method and 1 for accuracy in taking the $\epsilon \to 0$ limit and 2 for the comparison with GR.

[8 marks]

(iv) Assuming ϕ is time independent, perform the T variation of the Lagrangian in part i) to give a conserved quantity for the motion. Show how this conserved quantity can be written in terms of the particle's 4-velocity and an appropriate Killing vector K^μ which you should determine.

ANSWER:

This is similar to some discussion in lectures and example sheet questions in different contexts (such as Schwarzschild and Newtonian spacetime).

Vary the Lagrangian with respect to T; then,

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \frac{dT}{d\tau}}\right) = \frac{\partial \mathcal{L}}{\partial T} \tag{21}$$

but since Φ is independent of time $\frac{\partial \mathcal{L}}{\partial T} = 0$, then $\frac{\partial \mathcal{L}}{\partial \frac{dT}{d\tau}}$ is constant, and so,

$$\frac{\partial \mathcal{L}}{\partial \frac{dT}{d\tau}} = -2\phi^2 \frac{dT}{d\tau} = c \tag{22}$$

for a constant c.

The appropriate Killing vector is $K^\mu = (1, 0, 0, 0)$ as this vector field generates the time translation symmetry of the metric (and hence is Killing). Then,

$$K_\mu v^\mu = g_{\mu\nu} v^\mu K^\nu = g_{tt} v^t = -\phi^2 \frac{dT}{d\tau} \tag{23}$$

which proportional to the conserved quantity above, so,

$$K_\mu v^\mu = \frac{1}{2} c \tag{24}$$

[This question continues on the next page . . .]

Probable mark assignment: 2 marks for method and 1 for accuracy of variation. 1 for method, 1 for Killing vector and 1 for accuracy in showing the conserved quantity is proportional to $v^\mu K_\mu$.

[6 marks]

[Total 30 marks]

5. This question concerns light bending in the Newtonian spacetime. Recall the Newtonian metric is,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \quad \text{with} \quad g_{\mu\nu} = \eta_{\mu\nu} - 2\epsilon \Phi(x^i)\, \delta_{\mu\nu} + O(\epsilon^{3/2})$$

where $x^\mu = (t, x^i)$ with $i = 1, 2, 3$ and we assume $\partial_t \Phi = 0$ so the spacetime is static. When $\epsilon \ll 1$ this is the Newtonian limit of GR with $\epsilon\Phi$ being the Newtonian gravitational potential.

(i) Parameterize a null geodesic in the Newtonian spacetime as $x^\mu(\lambda) = (T(\lambda), X^i(\lambda))$ with affine parameter λ. By varying

$$L = \int d\lambda g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{1}$$

with respect to X^i show that for a null geodesic,

$$\frac{d^2 X^i}{d\lambda^2} = 2\epsilon \left(\frac{\partial \Phi}{\partial X^k} \frac{dX^i}{d\lambda} \frac{dX^k}{d\lambda} - \delta^{ij} \delta_{kl} \frac{\partial \Phi}{\partial X^j} \frac{dX^k}{d\lambda} \frac{dX^l}{d\lambda} \right) \tag{2}$$

to leading order in ϵ.

ANSWER:

Light bending in Newtonian spacetime was covered in an example sheet question but specifically for a point mass, so the result here for general Φ is unseen material. In addition the null geodesic was obtained in the example sheet directly from the geodesic equation, not by action variation.

Begin with the Lagrangian,

$$\begin{aligned} L &= \int d\lambda \mathcal{L} = \int d\lambda g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \\ &= \int d\lambda \left(\eta_{\mu\nu} - 2\epsilon\Phi(X^i)\, \delta_{\mu\nu} + O(\epsilon^{3/2}) \right) \frac{dx^\nu}{d\lambda} \frac{dx^\nu}{d\lambda} \\ &= \int d\lambda \left(-1 - 2\epsilon\Phi(X^i) + O(\epsilon^{3/2}) \right) \frac{dT}{d\lambda} \frac{dT}{d\lambda} + \left(1 - 2\epsilon\Phi(X^i) + O(\epsilon^{3/2}) \right) \delta_{ij} \frac{dX^i}{d\lambda} \frac{dX^j}{d\lambda} \end{aligned} \tag{3}$$

Now for a null ray $\mathcal{L} = 0$, and hence,

$$\left(1 + 2\epsilon\Phi(X^i) + O(\epsilon^{3/2}) \right) \frac{dT}{d\lambda} \frac{dT}{d\lambda} - \left(1 - 2\epsilon\Phi(X^i) + O(\epsilon^{3/2}) \right) \delta_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0 \tag{4}$$

so,

$$\begin{aligned} \left(\frac{dT}{d\lambda} \right)^2 &= \left(\frac{1 - 2\epsilon\Phi(X^i) + O(\epsilon^{3/2})}{1 + 2\epsilon\Phi(X^i) + O(\epsilon^{3/2})} \right) \delta_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \\ &= \delta_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} + O(\epsilon) \end{aligned} \tag{5}$$

[This question continues on the next page . . .]

Consider the X^i Euler-Lagrange equation. Then,

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial(dX^i/d\lambda)}\right) &= \frac{d}{d\lambda}\left(2\left(1-2\epsilon\Phi(X)+O(\epsilon^{3/2})\right)\delta_{ij}\frac{dX^j}{d\lambda}\right)\\
&= 2\delta_{ij}\frac{d^2X^j}{d\lambda^2}-4\epsilon\frac{d\Phi(X)}{d\lambda}\delta_{ij}\frac{dX^j}{d\lambda}+O(\epsilon^{3/2})\\
&= 2\delta_{ij}\frac{d^2X^j}{d\lambda^2}-4\epsilon\frac{\partial\Phi(X)}{\partial X^k}\delta_{ij}\frac{dX^j}{d\lambda}\frac{dX^k}{d\lambda}+O(\epsilon^{3/2}) \qquad (6)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial X^i} &= (-2\epsilon\frac{\partial\Phi}{\partial X^i}+O(\epsilon^{3/2}))\left(\frac{dT}{d\lambda}\frac{dT}{d\lambda}+\delta_{mn}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}\right)\\
&= (-2\epsilon\frac{\partial\Phi}{\partial X^i}+O(\epsilon^{3/2}))\left(\frac{dT}{d\lambda}\frac{dT}{d\lambda}+\delta_{mn}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}\right)\\
&= (-2\epsilon\frac{\partial\Phi}{\partial X^i}+O(\epsilon^{3/2}))\left(2\delta_{mn}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}+O(\epsilon)\right)\\
&= -4\epsilon\frac{\partial\Phi}{\partial X^i}\delta_{mn}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}+O(\epsilon^{3/2}) \qquad (7)
\end{aligned}$$

So the Euler-Lagrange equations yield,

$$2\delta_{ij}\frac{d^2X^j}{d\lambda^2}-4\epsilon\frac{\partial\Phi(X)}{\partial X^k}\delta_{ij}\frac{dX^j}{d\lambda}\frac{dX^k}{d\lambda} = -4\epsilon\frac{\partial\Phi}{\partial X^i}\delta_{mn}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}+O(\epsilon^{3/2}) \qquad (8)$$

so that,

$$\frac{d^2X^j}{d\lambda^2} = 2\epsilon\left(\frac{\partial\Phi}{\partial X^k}\frac{dX^j}{d\lambda}\frac{dX^k}{d\lambda}-\delta^{ji}\delta_{mn}\frac{\partial\Phi}{\partial X^i}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}\right)+O(\epsilon^{3/2}) \qquad (9)$$

Probable mark assignment: 2 for general method, 1 mark for correct Lagrangian, 3 marks for Euler-Lagrange equations, 2 marks for using $\mathcal{L}=0$, 2 for accuracy.

[10 marks]

(ii) Take the Newtonian potential for a static point source with mass (ϵM) at position $x^i=(0,R,0)$. Consider a light ray initially propagating along the x^1 axis, so that $x^\mu=(\lambda,\lambda,0,0)$ for $\lambda\to-\infty$. The trajectory of the ray is then

$$X^i(\lambda)=(X(\lambda),Y(\lambda),Z(\lambda))=\left(\lambda+\epsilon G(\lambda)+O(\epsilon^{3/2}),\epsilon H(\lambda)+O(\epsilon^{3/2}),0\right) \qquad (10)$$

Use the answer to part i) to show that,

$$\frac{d^2H}{d\lambda^2} = -2\frac{\partial\Phi(X^i)}{\partial Y} \qquad (11)$$

[This question continues on the next page . . .]

ANSWER:

Light bending, and in particular this result, was covered in an example sheet question. However the derivation was different as discussed for part i).

The equation governing the trajectory is,

$$\frac{d^2X^i}{d\lambda^2} = 2\epsilon\left(\frac{\partial\Phi}{\partial X^k}\frac{dX^i}{d\lambda}\frac{dX^k}{d\lambda} - \delta^{ij}\delta_{mn}\frac{\partial\Phi}{\partial X^j}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}\right) + O(\epsilon^{3/2}) \qquad (12)$$

and we have,

$$x^i = (\lambda + \epsilon G(\lambda) + O(\epsilon^{3/2})), \epsilon H(\lambda) + O(\epsilon^{3/2})), 0) \qquad (13)$$

Let us call $x^i = (x, y, z)$, and $X^i = (X, Y, Z)$.

Now,

$$\delta_{mn}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda} = \frac{dX}{d\lambda}\frac{dX}{d\lambda} + \frac{dY}{d\lambda}\frac{dY}{d\lambda} + \frac{dZ}{d\lambda}\frac{dZ}{d\lambda} = \frac{dX}{d\lambda}\frac{dX}{d\lambda} + O(\epsilon) = 1 + O(\epsilon) \qquad (14)$$

Then we have,

$$\begin{aligned}\frac{d^2Y}{d\lambda^2} &= 2\epsilon\left(\frac{\partial\Phi}{\partial X^k}\frac{dY}{d\lambda}\frac{dX^k}{d\lambda} - \frac{\partial\Phi}{\partial Y}\delta_{mn}\frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}\right) + O(\epsilon^{3/2})\\ &= 2\epsilon\left(\frac{\partial\Phi}{\partial X^k}\frac{dY}{d\lambda}\frac{dX^k}{d\lambda} - \frac{\partial\Phi}{\partial Y}\right) + O(\epsilon^{3/2})\end{aligned} \qquad (15)$$

but as $dY/d\lambda = O(\epsilon)$, then,

$$\frac{d^2Y}{d\lambda^2} = -2\epsilon\frac{\partial\Phi}{\partial Y} + O(\epsilon^{3/2}) \qquad (16)$$

and so as $Y = \epsilon H(\lambda) + O(\epsilon^{3/2})$ then,

$$\frac{d^2H}{d\lambda^2} = -2\frac{\partial\Phi(X)}{\partial Y} + O(\epsilon^{1/2}) \qquad (17)$$

as required.

Probable mark assignment: 3 for understanding how to use relevant answer to part i), 4 marks for method in subsequent calucatlion, 3 for accuracy.

[10 marks]

(iii) By using the explicit form of the Newtonian potential for the point mass, integrate twice to determine $H(\lambda)$. Hence show that light is deflected by an angle θ which to leading order in ϵ is,

$$\theta = \frac{4G_N(\epsilon M)}{R} \qquad (18)$$

[This question continues on the next page . . .]

Hint: You may find the following integral useful;

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \tag{19}$$

ANSWER:

Again light bending, and in particular this result, was covered in an example sheet question. However the derivation was different as discussed for part i).

The Newtonian potential for the source is,

$$\epsilon\Phi(x^i) = -\frac{G_N \epsilon M}{\sqrt{x^2+(y-R)^2+z^2}} \tag{20}$$

so,

$$\Phi(x^i) = -\frac{G_N M}{\sqrt{x^2+(y-R)^2+z^2}} \tag{21}$$

so then,

$$\begin{aligned} \frac{\partial}{\partial Y}\Phi(X^i) &= \frac{\partial}{\partial Y}\left(-\frac{G_N M}{\sqrt{X^2+(Y-R)^2+Z^2}}\right) \\ &= \frac{G_N M(Y-R)}{(X^2+(Y-R)^2+Z^2)^{3/2}} \end{aligned} \tag{22}$$

Recall that,

$$X = \lambda + O(\epsilon)\ , \quad Y = O(\epsilon)\ , \quad Z = 0 \tag{23}$$

so that,

$$\frac{\partial}{\partial Y}\Phi(X^i) = \frac{-G_N MR}{(\lambda^2+R^2)^{3/2}} + O(\epsilon) \tag{24}$$

Then,

$$\frac{d^2H}{d\lambda^2} = \frac{2G_N MR}{(\lambda^2+R^2)^{3/2}} + O(\epsilon) \tag{25}$$

Using

$$\int d\lambda \frac{dx}{(R^2+x^2)^{3/2}} = \frac{x}{R^2\sqrt{R^2+x^2}} \tag{26}$$

then,

$$\begin{aligned} \frac{dH}{d\lambda} &= 2G_N MR\int \frac{d\lambda}{(\lambda^2+R^2)^{3/2}} + O(\epsilon) \\ &= c + 2G_N MR\frac{\lambda}{R^2\sqrt{R^2+\lambda^2}} + O(\epsilon) \\ &= c + \frac{2G_N M}{R}\frac{\lambda}{\sqrt{R^2+\lambda^2}} + O(\epsilon) \end{aligned} \tag{27}$$

[This question continues on the next page . . .]

where c is a constant of integration. We require $H \to 0$ as $\lambda \to -\infty$ so $dH/d\lambda \to 0$ too, and since,

$$\lim_{\lambda\to-\infty}\left(c + \frac{2G_N M}{R}\frac{\lambda}{\sqrt{R^2+\lambda^2}}\right) = c - \frac{2G_N M}{R} \tag{28}$$

then we have,

$$c = \frac{2G_N M}{R} \tag{29}$$

so,

$$\frac{dH}{d\lambda} = \frac{2G_N M}{R}\left(1 + \frac{\lambda}{\sqrt{R^2+\lambda^2}}\right) + O(\epsilon) \tag{30}$$

Integrate once more,

$$\begin{aligned} H &= \frac{2G_N M}{R}\int d\lambda\left(1 + \frac{\lambda}{\sqrt{R^2+\lambda^2}}\right) + O(\epsilon) \\ &= \frac{2G_N M}{R}\left(\lambda + \frac{1}{2}\int \frac{2\lambda d\lambda}{\sqrt{R^2+\lambda^2}}\right) + O(\epsilon) \\ &= \frac{2G_N M}{R}\left(d + \lambda + \sqrt{R^2+\lambda^2}\right) + O(\epsilon) \end{aligned} \tag{31}$$

for a constant of integration d. Now since,

$$\lim_{\lambda\to-\infty}\left(\frac{2G_N M}{R}\left(d + \lambda + \sqrt{R^2+\lambda^2}\right)\right) = \frac{2G_N M d}{R} \tag{32}$$

and we wish $H \to 0$ we require $d = 0$. So finally,

$$H = \frac{2G_N M}{R}\left(\lambda + \sqrt{R^2+\lambda^2}\right) + O(\epsilon) \tag{33}$$

For $\lambda \to +\infty$ we have,

$$H \sim \frac{4G_N M}{R}\lambda + O(\epsilon) \tag{34}$$

Then,

$$\lim_{\lambda\to\infty}\frac{Y(\lambda)}{X(\lambda)} \sim \frac{\epsilon H}{\lambda + \epsilon G} \sim \frac{4G_N(\epsilon M)}{R} + O(\epsilon^2) \tag{35}$$

and the deflection angle θ is,

$$\theta = \tan^{-1}\left(\lim_{\lambda\to\infty}\frac{Y(\lambda)}{X(\lambda)}\right) = \frac{4G_N(\epsilon M)}{R} + O(\epsilon^2) \tag{36}$$

[This question continues on the next page . . .]

for a point source with mass (ϵM).

Probable mark assignment: 3 marks for Newtonian potential, 4 for method and 3 for accuracy in subsequent calculation.

[10 marks]

[Total 30 marks]