Imperial College London

MSci EXAMINATION May 2013

This paper is also taken for the relevant Examination for the Associateship

GENERAL RELATIVITY

For 4th-Year Physics Students

Monday 20th May 2013: 14:00 to 16:00

The paper consists of two *sections: A and B*
Section A contains one *question [40 marks total].*
Section B contains four *questions [30 marks each].*

Candidates are required to:
Answer **ALL** *parts of Section A and* **TWO QUESTIONS** *from Section B.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature.

You may find the following formulae useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right)$$

The covariant derivative of a vector field is,

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha$$

and for a covector field is,

$$\nabla_\mu w_\nu \equiv \partial_\mu w_\nu - \Gamma^{\alpha}{}_{\mu\nu} w_\alpha$$

For a Lagrangian of a curve $x^\mu(\lambda)$ of the form,

$$L = \int d\lambda \, \mathcal{L}(x^\mu, \frac{dx^\mu}{d\lambda})$$

the Euler-Lagrange equations are,

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial (\frac{dx^\mu}{d\lambda})} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

Section A

Answer all of section A.

SECTION A

1. This question concerns accelerated motion in curved spacetimes.

(i) Suppose we have a massive particle following a trajectory $x^\mu(\tau)$ in a general spacetime, where τ is the particle's proper time. The particle's 4-velocity v^μ is defined as $v^\mu = dx^\mu/d\tau$. Why is $v^\mu v_\mu = -1$?

[5 marks]

(ii) Use the chain rule property of derivatives to show that the 4-velocity transforms as a vector.

[7 marks]

(iii) The 4-acceleration a^μ is defined as $a^\mu = v^\nu \nabla_\nu v^\mu$. Show that in Minkowski spacetime this can be written as $a^\mu = d^2 x^\mu / d\tau^2$.

[6 marks]

(iv) By considering $v^\nu \nabla_\nu \left(v^\mu v_\mu\right)$, show that a^μ and v^μ are orthogonal 4-vectors (ie. $a^\mu v_\mu = 0$).

[7 marks]

(v) Show that since $a^\mu v_\mu = 0$ then a^μ must be a spacelike vector.

[5 marks]

(vi) Now consider a particle moving in the Schwarzschild spacetime, with coordinates $x^\mu = (t, r, \theta, \phi)$ and metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

Consider a particle accelerating to stay at constant spatial position, so that r, θ, ϕ remain constant. Use the fact that,

$$\Gamma^r{}_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) , \Gamma^t{}_{tt} = \Gamma^\theta{}_{tt} = \Gamma^\phi{}_{tt} = 0 \tag{2}$$

to calculate the norm $\sqrt{a^\mu a_\mu}$ of the 4-acceleration of the particle for $r > 2M$. What happens to this quantity at $r = 2M$ and why?

[10 marks]

[Total 40 marks]

Section B

Answer 2 out of the 4 questions in the following section.

SECTION B

2. This question concerns the Einstein equations for a star made of perfect fluid.

(i) State the stress tensor $T_{\mu\nu}$ for a perfect fluid in terms of the fluid energy density ρ, pressure P and 4-velocity u^μ (recall $u^\mu u_\mu = -1$). Take n^μ to be orthogonal to u^μ and consider $n^\mu \nabla^\nu T_{\mu\nu}$ to derive one of the fluid equations,

$$n^\mu \left(\partial_\mu P + (\rho + P)\, u^\nu \nabla_\nu u_\mu \right) = 0 \qquad (1)$$

[8 marks]

(ii) Consider a time independent, spherically symmetric metric describing a star. We take coordinates $x^\mu = (t, r, \theta, \phi)$ and a metric,

$$ds^2 = -e^{2f(r)} dt^2 + \frac{1}{h(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \qquad (2)$$

where $f(r)$ and $h(r)$ are functions of r. The star is made of perfect fluid. Since it is static then $u^\mu = (T(r), 0, 0, 0)$. Firstly determine the function $T(r)$. Then using part i) above, choose $n^\mu = (0, 1, 0, 0)$ and compute the necessary $\Gamma^\alpha{}_{\mu\nu}$ components to show that,

$$\frac{dP}{dr} = -(\rho + P)\frac{df}{dr} \qquad (3)$$

[9 marks]

(iii) The non-zero components of Ricci with one index up and one down are,

$$\begin{aligned} R^t{}_t &= -\frac{2h}{r}\frac{df}{dr} - \frac{1}{2}\frac{dh}{dr}\frac{df}{dr} + L(r)\,, \quad R^r{}_r = -\frac{1}{r}\frac{dh}{dr} - \frac{1}{2}\frac{dh}{dr}\frac{df}{dr} + L(r) \\ R^\theta{}_\theta &= R^\phi{}_\phi = \frac{1}{r^2}(1-h) - \frac{1}{2r}\frac{dh}{dr} - \frac{h}{r}\frac{df}{dr} \end{aligned} \qquad (4)$$

where $L(r)$ is a function of f and h you will not need to know explicitly.

Calculate the Einstein tensor components, G_{tt} and G_{rr}, and then the corresponding tt and rr components of the Einstein equations. Define,

$$h(r) = 1 - \frac{2m(r)}{r} \qquad (5)$$

and then show these Einstein equations yield,

$$\frac{dm}{dr} = 4\pi G_N r^2 \rho\,, \qquad \frac{df}{dr} = \frac{m + 4\pi G_N r^3 P}{r^2 - 2mr} \qquad (6)$$

[These are the *Tolman-Oppenheimer-Volkoff* equations for a relativistic star.]

[8 marks]

(iv) If the star has a surface at $r = R$, then outside this surface for $r > R$ there is no fluid matter ie. $\rho = P = 0$. Solve the equations to find $m(r)$ and show $e^{2f(r)} = h(r)$ is a solution. Hence determine the metric in the star's exterior. What is this exterior spacetime? What is its mass in terms of $m(r)$?

[5 marks]

[Total 30 marks]

3. This question concerns scalar fields and FLRW spacetime.

(i) Consider a scalar field $\phi(t, x^i)$ with potential $V(\phi)$ on a general spacetime. Its stress tensor is given as,

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\alpha\phi\nabla_\alpha\phi\right) - g_{\mu\nu}V(\phi) \qquad (1)$$

Using the equation of motion of this scalar field,

$$\nabla^\alpha\nabla_\alpha\phi = \frac{dV(\phi)}{d\phi} \qquad (2)$$

show that the stress energy is conserved.

[8 marks]

(ii) Take spacetime to be FLRW, with coordinates $x^\mu = (t, x^i)$ with $i = 1, 2, 3$, and,

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j \qquad (3)$$

Compute all the Christoffel symbol components $\Gamma^\alpha{}_{\mu\nu}$ for this metric.

[8 marks]

(iii) Take the scalar to have the symmetries of FLRW, so that ϕ is only a function of time t. Also take its potential to vanish, $V(\phi) = 0$ - this is a *massless* scalar field. Solve the massless scalar equation of motion to show that,

$$\phi(t) - \phi(t_0) = k\int_{t_0}^{t} dt' \frac{1}{a(t')^3} \qquad (4)$$

where k is a constant of integration.

[8 marks]

(iv) A comoving perfect fluid with equation of state $P = w\rho$, for constant w, obeys,

$$\rho(t) = \frac{c}{a(t)^{3(1+w)}} \qquad (5)$$

in FLRW where c is a constant. Show the stress tensor for the massless scalar in FLRW is the same as that for a perfect fluid with $w = +1$ (a 'stiff fluid'). Find the relation between the constants c and k.

[6 marks]

[Total 30 marks]

4. Before Einstein completed his equations of General Relativity, an alternative theory was proposed by Nordström. As with Einstein's theory, in Nordström's theory gravity is due to curvature of spacetime. However, the theory is much simpler as the spacetime metric cannot be general, but is given in terms of one function $\phi(t, x^i)$, as,

$$ds^2 = \phi^2 \left(-dt^2 + dx^i dx^i\right) \tag{1}$$

where we have taken coordinates $x^\mu = (t, x^i)$ with $i = 1, 2, 3$. Particle motion is then just as for GR but in this particular curved spacetime.

(i) A massive particle in the spacetime follows the timelike geodesic $x^\mu = (T(\tau), X^i(\tau))$ where τ is its proper time. Assume the Nordstöm scalar ϕ is time independent, so $\phi = \phi(x^i)$. Use the Euler-Lagrange equations to vary the Lagrangian,

$$L = \int d\tau\, \phi^2(X) \left(-\left(\frac{dT}{d\tau}\right)^2 + \frac{dX^i}{d\tau}\frac{dX^i}{d\tau}\right) \tag{2}$$

with respect to X^i and hence determine that the geodesic is governed by,

$$\frac{d^2X^i}{d\tau^2} = -\frac{1}{\phi^3}\frac{\partial \phi(X)}{\partial X^i} \tag{3}$$

[8 marks]

(ii) Nordström proposed a field equation governing ϕ to be,

$$\frac{1}{\phi^3}\left(-\partial_t^2 + \partial_i^2\right)\phi = \kappa \rho \tag{4}$$

where ρ is the matter energy density and κ is a constant. Consider a Newtonian limit similar to that in GR by taking $\phi = 1 + \epsilon\Phi$ and time independent with $\epsilon \ll 1$. Use your answer to part i) to identify the Newtonian gravitational potential and hence determine the constant κ in terms of Newton's constant G_N.

[8 marks]

(iii) Like GR, Nordström's theory predicts a gravitational redshift. Suppose a particle is at fixed position x_1^i and emits radiation with frequency ω in its rest-frame. At what frequency does a particle at fixed position x_2^i receive it, assuming that ϕ is time independent? Consider this redshift in the Newtonian limit - can it be used to distinguish Einstein's GR from Nordström's theory?

[8 marks]

(iv) Assuming ϕ is time independent, perform the T variation of the Lagrangian in part i) to give a conserved quantity for the motion. Show how this conserved quantity can be written in terms of the particle's 4-velocity and an appropriate Killing vector K^μ which you should determine.

[6 marks]

[Total 30 marks]

5. This question concerns light bending in the Newtonian spacetime. Recall the Newtonian metric is,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \quad \text{with} \quad g_{\mu\nu} = \eta_{\mu\nu} - 2\epsilon\Phi(x^i)\,\delta_{\mu\nu} + O(\epsilon^{3/2})$$

where $x^\mu = (t, x^i)$ with $i = 1, 2, 3$ and we assume $\partial_t \Phi = 0$ so the spacetime is static. When $\epsilon \ll 1$ this is the Newtonian limit of GR with $\epsilon\Phi$ being the Newtonian gravitational potential.

(i) Parameterize a null geodesic in the Newtonian spacetime as $x^\mu(\lambda) = (T(\lambda), X^i(\lambda))$ with affine parameter λ. By varying

$$L = \int d\lambda g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{1}$$

with respect to X^i show that for a null geodesic,

$$\frac{d^2 X^i}{d\lambda^2} = 2\epsilon \left(\frac{\partial \Phi}{\partial X^k} \frac{dX^i}{d\lambda} \frac{dX^k}{d\lambda} - \delta^{ij} \delta_{kl} \frac{\partial \Phi}{\partial X^j} \frac{dX^k}{d\lambda} \frac{dX^l}{d\lambda} \right) \tag{2}$$

to leading order in ϵ.

[10 marks]

(ii) Take the Newtonian potential for a static point source with mass (ϵM) at position $x^i = (0, R, 0)$. Consider a light ray initially propagating along the x^1 axis, so that $x^\mu = (\lambda, \lambda, 0, 0)$ for $\lambda \to -\infty$. The trajectory of the ray is then

$$X^i(\lambda) = (X(\lambda), Y(\lambda), Z(\lambda)) = \left(\lambda + \epsilon G(\lambda) + O(\epsilon^{3/2}), \epsilon H(\lambda) + O(\epsilon^{3/2}), 0 \right) \tag{3}$$

Use the answer to part i) to show that,

$$\frac{d^2 H}{d\lambda^2} = -2 \frac{\partial \Phi(X^i)}{\partial Y} \tag{4}$$

[10 marks]

(iii) By using the explicit form of the Newtonian potential for the point mass, integrate twice to determine $H(\lambda)$. Hence show that light is deflected by an angle θ which to leading order in ϵ is,

$$\theta = \frac{4 G_N (\epsilon M)}{R} \tag{5}$$

Hint: You may find the following integral useful;

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 + x^2}} \tag{6}$$

[10 marks]

[Total 30 marks]