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Symmetry and Unification

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Conventions

We use natural units

$$\hbar = 1, \;\; c = 1 \tag{0.0.1}$$

The charge on the electron will be denoted by $-e$ where

$$e = +1.6 \times 10^{-19}\mathrm{C}\,. \tag{0.0.2}$$

The Minkowskii metric used here is of the form

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g^{\mu}_{\nu} = \delta^{\mu}_{\nu}. \tag{0.0.3}$$

This choice means that we will have to add a minus sign to the definition of the Euclidean action relative to the definition of that used for the Minkowskii action.

We will define four vectors so that

$$\begin{aligned} K^\mu &= (k_0, +\vec{k}), \qquad K^\mu K_\mu = k_0^2 - \vec{k}^2 \\ P^\mu &= (p_0, +\vec{p}), \qquad P^\mu P_\mu = p_0^2 - \vec{p}^2 \\ X^\mu &= (\tau, +\vec{x}), \qquad X^\mu X_\mu = \tau^2 - \vec{x}^2 \end{aligned} \tag{0.0.4}$$

We define our zero of energy in the way normally encountered in relativistic systems, that is a particle at rest has energy equal to its rest mass. Thus for a particle in vacuo, the **dispersion relation** — the relationship between energy, ω, and three-momenta $\boldsymbol{k}$, for a physical, on-shell particle — is[1]

$$\omega = (\boldsymbol{k}^2 + m^2)^{1/2} \tag{0.0.5}$$

Indices, Subscripts and Superscripts

$a, b, \ldots$	Used for indices of generators (basis vectors) of Lie algebras, e.g. T^a
$A, B, \ldots$	Used for indices of unbroken generators, e.g. T'^A
$i, j, \ldots$	Used as indices in the group representation space, e.g. group elements U_{ij}, fields $\boldsymbol{\phi}_i$
$Z, Y \ldots$	Used for indices of broken generators, e.g. T''^Z
$\alpha, \beta, \ldots$	Spin indices running over $0, 1, 2, 3$ in four-dimensions, e.g. fermion fields ψ^α, Dirac gamma matrices $\gamma^{\alpha\beta}$
$\mu, \nu, \ldots$	Lorentz indices running over $0, 1, 2, 3$ in four-dimensions
T	Transpose of a matrix $(\mathsf{M}^\mathsf{T})_{ij} := (\mathsf{M})_{ji}$
$\dagger$	Hermitian conjugate of a matrix $(\mathsf{M}^\dagger)_{ij} := ((\mathsf{M})_{ji})^*$

[1] This form follows from Lorentz invariance but only when in vacuo. Physical particles do not have such form in many-body problems, such as hot relativistic plasmas and condensed matter systems.

Other Notation

$\boldsymbol{z}$	A general vector z
Z	A general matrix Z
$A^\mu(x)$	An abelian gauge field
A	A general element of a Lie algebra
b	Often the number of broken generators, usually $b = \dim(G) - \dim(H)$
$\mathcal{A}$	A lie Algebra (so $\{\mathsf{A}\} = \mathcal{A}$)
d	Often the dimension of a representation, so indices $i, j = 1, 2, \ldots, d$
D^μ, D^μ	Covariant derivatives, see (6.1.27), (6.2.24) and (**??**)
e	A general abelian gauge coupling *or*, specifically for EM, the *magnitude* of the charge on the electron $= +1.6 \times 10^{-19}$C
g	Sometimes a gauge field coupling constant e.g. in (6.1.27), (6.2.24) and (**??**), *or* sometimes a general coupling constant as in $g\phi^3$, *or* a generic element a group G, *or* the dimension of a group G, $g \equiv \dim(G)$
G	A group, usually in this context a Lie Group such as $U(1) \approx SU(2), SO(3) \approx SU(2)$ (see chapter 3 or appendix B)
h	Sometimes the dimension of a group or subgroup H, $h \equiv \dim(H)$
H	A subgroup, often the little or stability group (see section **??**) *or* the Hamiltonian, e.g. see (**??**)
$\mathcal{L} = \mathcal{L}[\phi, \ldots]$	A Lagrangian density, here a functional of fields, see (3.1.1) and (2.1.20)
M, M^2	Mass matrices see (**??**)
T^a	A generator of a Lie algebra (**??**)
T'^A	An unbroken generator (**??**)
T''^Z	A broken generator (**??**)
U	A unitary matrix (**??**), and if real, it is then orthogonal (**??**)
v	For real Higgs field, the size of the field in the physical vacuum For complex fields $v/\sqrt{2}$ is the modulus of the vacuum field.
$V \equiv V(\phi)$	Usually the potential energy for scalar fields
$W^{\mu a} \equiv W^{\mu a}(x)$	Non-abelian gauge boson field, a real function of space-time $x \equiv x^\mu$
W^μ	Lie algebra values non-abelian gauge boson field $\mathsf{W}^\mu = \mathsf{T}^a W^{\mu a}$
$\phi(x), \boldsymbol{\phi}(x)$	Real scalar field, a vector of real scalar fields
$\Phi(x), \boldsymbol{\Phi}(x)$	Complex scalar field, a vector of complex scalar fields
$\lvert\boldsymbol{\phi}_0\rvert$	Field value at minimum energy
$\psi(x) \equiv \psi^\alpha(x)$	Fermion field, the spin index α will often be supressed
$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma_0$	Conjugate to fermion field
$\boldsymbol{\psi}(x)$	Vector of fermion fields (spin index supressed)
$\boldsymbol{\tau}^i$	The three Pauli matrices, see (B.2.4).

Glossary

Abbreviation	**Meaning**
QM	**Q**uantum **M**echanics. (Not the same as QFT, see section **??**).
QFT	**Q**uantum **F**ield **T**heory. Not just a relativistic version of QMbut a distinct theory, see section **??**)
EM	**E**lectro**M**agnetism. Classical unification of electric and magnetic forces due to Maxwell, see section **??**)
QED	**Q**uantum **E**lectro **D**ynamics, the QFTgeneralisation of EM, see section **??**)
SQED	**S**calar **Q**uantum **E**lectro **D**ynamics, a toy model of QEDwhere the electron and positron are replaced by scalar particles, see section **??**)
QCD	**Q**uantum **C**hromo **D**ynamics, the QFTbelieved to describe the strong nuclear force, see section **??** and chapter 9)
EW	**ElectroWeak** theory, a.k.a. Glashow-Weinberg-Salam theory. The unified QFTof electromagnetic and weak forces, see section **??** and chapter 10.)
GR	**General Relativity**, the classical theory of gravity, see section **??**)

Preface

The aim of this book is to study the unification of our fundamental theories of nature. Inevitably, such a book will always be a product of its time, and even then can never cover all the relevant current topics. So the rather more modest objective is to cover the key ideas behind the **standard model** of particle physics. This describes the interactions of *all* the elementary particles known today, plus the one which is still missing in 2002, the Higgs particle. In this case, elementary means that our fundamental building blocks (the quarks, leptons, gauge bosons and Higgs particles) are elementary on scales of $\sim 10^{-18}$fm and larger, the limits of experiments in 2002. Some of these particles of the standard model could be composed of smaller particles, but we have no compelling evidence of this and would not expect any direct evidence until we could probe smaller distance scales.

In this case, elementary means on scales of $\sim 10^{-18}$fm and above, the limits of experiments in 2003, the particles of the standard model are fundamental and are not composed of any smaller particles.

From a theoretical point of view, the standard model, with a limited number of parameters, fits all the known data. Over the last twenty years since t'Hooft and Veltman solved the last big theoretical problem, gaps have been filled in our experimental (e.g. discovery of the top quark) and theoretical knowledge (e.g. improved predictions from QCD). In particular the parameters of the standard model have been measured more accurately, and some are now extremely precisely e.g. the mass of the Z boson is known to five significant figures. However the standard model remains consistent with all experimental results. Even the recent exciting discovery of neutrino masses can be included very naturally in the standard model. There are periodic scares but the theoretical calculations and the experiments can both be extremely complicated and it is a real triumph of particle physics that the standard model has survived these challenges for so many years. This is somewhat of a Pyrrhic victory for theoreticians and experimentalists alike, but it does suggest that the basic ideas behind the standard model are on firm ground and it is well worth studying them in detail.

The history of the standard model is one of the unification of ideas that through much of the twentieth century appeared to be disparate. Thus the real hope is that the theoretical methods underlying the standard model can be successful in a further unification of our ideas, leading to a more fundamental, hopefully, simpler, picture of the physical world. Indeed in many ways the standard model, while an extremely successful description of particle physics at energy scales of around 100GeV and below, is very unsatisfactory, leaving many big questions unanswered, e.g. why do all quarks and leptons come in threes, differing only in their mass? So another objective of this book is to discuss in general terms the successful principles behind the standard model in the hope these ideas may still be needed to gain deeper insights as new data comes in. In this sense, the standard model is only acting for us an important illustration of the techniques. Since we have not been successful in creating a sufficiently compelling yet more fundamental theory, for instance as embodied in a GUT(Grand Unified Theory), and since we already know the answer to standard model physics, I will try to convey the flavour of the true task ahead through toy models and examples. So in chapter 11 I will present several simple exercises, in which the reader is asked to reconstruct simple models from imaginary data, giving the reader a feeling of how one might proceed to find more fundamental theories.

Another reason for emphasising the ideas behind the standard model of particle physics is because the same ideas have great relevance to condensed matter physics. Indeed many ideas emerged first in this area. Certainly the advances in experimental techniques of condensed matter have meant that testing the theoretical ideas of particle physics and cosmology in the lab has become more and more feasible. Given the long time scales for experimental particle physics, the condensed matter world should appeal more and more to particle physicists. However, for the sake of clarity, condensed matter is an area topic we will refer to only in passing.

Finally, my aim is not to present these results in their full technical glory. Rather my objective is to present them at the level appropriate for final year undergraduate physics students, or students starting a research level degree, and in doing so keep the book to a manageable size. This has two consequences.

First, I have tried to limit the amount of background material needed. The part students seemed to be most worried about is group theory but the aim is to provide all that one needs in context in the chapter on symmetry, chapter 3, with a summary of the maths given in appendix B. The hope is that all one needs is a familiarity with vectors, vector spaces and matrices to follow this book. In particular, I avoid a detailed discussion of how to find and work with representations of Lie groups beyond the basic fundamental representations. While such representation theory is essential for practical GUTs, it is not needed for the standard model nor to understand the basic principles behind GUTs.

Likewise a basic understanding of Lagrangian mechanics and particle physics is required though brief reviews in context will be given in sections A.1 and **??** respectively. Finally special relativity pervades almost all that we do here and only a summary is given in appendix **??**.

The second consequence is that in striving to produce an introductory text, there are several important areas not touched upon in any detail. As a theoretical work, the input from experiment is discussed only in broad outline, and the true ingenuity of experimental physicists, the complexity and fascination of their work will not come through here. Several important theoretical topics are also side lined. The most important one is QFT— **Quantum Field Theory**. As it is the foundation of all theoretical work in this area, it may seem strange to leave this to a brief summary in appendix A. However QFTcan be a lifetime's study in itself, and there are certainly no end of books on QFTat all levels. However, by the very depth of QFT, the ideas of unification have to play second fiddle in basic QFTtexts. The objective in this book is to put the focus on the ideas behind attempts to unify our view of the forces and matter of nature. I hope to show that this can be done, in the first instance, without quantising field theory, and part of this introduction will try to make the link between classical and quantum field theory (section 2.1). However, as I tackle ideas in a different order and from a different perspective from basic texts on QFT, I also hope that this book will provide illumination on both subjects for those working on QFTat the same time as this text.

Chapter 1

Basic Particle Physics

Before we start the real job of studying the role of symmetry in particle physics, we need to look at the different types of fundamental particle we know of in the world, and their properties. In doing so we will highlight some of the issues we would like to address when constructing a unified description of the fundamental forces. This we do in this chapter. We will also need to look at field theory, the theoretical tool used to describe the dynamics of all these particles, and that will be the subject of the following chapter. We will then start on symmetry in chapter 3.

This chapter is designed to give a very brief introduction to the main features of particle physics as we understand today. There are many books giving an in depth guide to the subject, for instance [15, 16, 17, 18, 19], and talks or lectures at graduate schools can provide good overviews too, for example [22]. The Particle Data Group, [20, 21], not only presents the latest data but also provides a set of short articles on many different aspects of particle physics. Basic quantum field texts may also have good descriptions of particle physics but most will focus on the formalism of QFT(quantum field theory, see chapter 2), used to describe the dynamics of particles.

1.1 The Forces of Nature

Feature/Force	EM	Weak	Strong	Gravity
Particle	Photon γ	W^+, W^-, Z^0	8 gluons	Graviton
Range	Long	Short $\lesssim$ 1fm	Short $\lesssim$ 1fm	Long
Mass	$m_\gamma = 0$	$m_w = 80.4$GeV $m_z = 91.2$GeV	$m_{\text{gluon}} = 0$	$m_{\text{graviton}} = 0$
Associated Charge Types	Electric	Weak Isospin and Weak Hypercharge	Colours	Energy
Charge Strength	$\alpha(0) \sim 1/137$ $\alpha(M_Z) \sim ?$	$\alpha_W \sim \alpha$ $\alpha_W(M_Z) \sim ?$	$\alpha_s(0) \sim ?$ $\alpha_s(M_Z) \sim 0.12$	Very Weak Very Weak
Charge of Universe	0	broken	0	> 0
Classical Theory	Maxwell EM 18??			General Relativity 191?
Quantum 30's		Fermi Theory	Yukawa Theory	
Theories 40's, 50's	QED	V-A Theory		
60's,70's	Unified as EW		QCD	(none)
Future	Unified as GUT			
	Unified as Theory of Everything			

The concept of a *force carrying particle* is not very precise. At different distance or energy scales one may be able to find different pictures, or more technically **effective field theories** which can provide a good description of the physics at different energy scales, e.g. Yukawa's use of pions to model low energy strong interactions. In this section we refer to the particles and forces needed to describe physics at the highest energy scales available in 2003.

The **range** of a force refers to the range when investigating the fundamental interactions of particles, so involving just a few test particles. This fundamental range is the connected with the mass of the particles mediating the force, with range proportional to the inverse of the mass.

The range of the fundamental interactions should be distinguished from the effective range in real materials or in the universe as a whole, as this involves interactions with many particles. Thus the effective range of EMis usually lab scales, $\sim$ 1m, because overall the universe is full of electrically charged particles, though almost invariably they combine to give packets which are overall charge neutral. So while physical processes can create local imbalances in charge over short distances, over long distances we must account for the presence of these particles and effects like screening.[1]

The **charges** refer to the various conserved numbers associated with all matter, and not just the electric charge[2]. We will discuss these in more detail below. Each force will only act between particles carrying the relevant charges for that force. However note that the we have indicated that the total charge of the universe for the weak force charges is described as **broken**. The precise meaning of this is one of the key topics to be discussed in the Unification course, but in effect it means that today we can not associate a definite weak charge to physical particles, and we can not use this charge in the way as electrical charge.

Finally we have noted the best classical theory followed by a list, in approximate historical order, of the quantum theories.

1.2 Conserved Numbers

A precise link can be made between conserved quantities and symmetries. For certain types of symmetry (continuous symmetries) the precise link is given by **Noether's theorem**, again a major topic of this book, covered in chapter 3. However, let us split the charges into two types, depending on whether they are linked to space-time symmetries or particle symmetries.

Charges of Space-Time Symmetries

Associated with space-time symmetries and described by the Poincaré group, or its subgroup the Lorentz group. Thus we imagine comparing the results of two experiments which are identical except the initial space-time coordinates or velocities of the particles in each experiment are different. Perhaps all the particles in one experiment are moving faster than in the other, or the positions in one experiment are the mirror image of the positions in the second experiment. When the results are identical, or at least simply related, then we have a space-time symmetry. These symmetries are intimately linked with **unitarity** — the fact that probabilities for all possible outcomes in an experiment must add up to one. It is perhaps not immediately obvious that these lead to the following conservation laws or charges

- *Energy.* Physical energies of all particles must be zero or a positive value relative to the energy of having no particle. This includes anti-particles. Conservation of energy is related to time-translation sysmmetry.

  Confusingly, it is usually convenient to use a mathematical label in a theory, also called energy, but which can be negative. This is usually the zero-th component k_0 of an energy-momentum four-vector

[1] The discharges in electric storms are examples of EMcharge imbalances over some of the largest distances on earth, yet they also show that such imbalances do not persist over time.

[2] In some articles, *charge* may refer specifically to *electric charge* but this will have to be deduced from the context.

$k_\mu = (k_0, \boldsymbol{k})$. The physical values might be $k_0 = +\sqrt{\boldsymbol{k}^2 + m^2}$ for particles and $k_0 = -\sqrt{\boldsymbol{k}^2 + m^2}$ for anti-particle solutions even though their physical energy is also $+\sqrt{\boldsymbol{k}^2 + m^2}$ and so positive.

- *Three-momentum* conservation, related to space translation.

- *Angular momentum* conservation comes from rotation symmetry.

- *Rest mass.* Space-time symmetry tells us that every stable particle has a single rest mass value which must be zero or positive but is otherwise unconstrained. When ever you see particles with special values of mass, e.g. same mass as another particle, or a zero mass, then another symmetry principle is at work. Corollary, without any further principles at work, expect particles to have different masses. Patterns in masses will be a particular focus here.

- *Spin.* A quantum effect since it appears in units of $\hbar/2$. Also intimately connected with space-time symmetry as its existence can be seen in the representation theory of the Lorentz group. Major distinctions: **bosons** have integer spin, **fermions** have half-integer spin. Spin zero particles are also called **scalar particles**.

- *CPT.* Three discrete space-time symmetries.

  (i) C — Charge conjugation under which particles are switched with their anti-particles. Not a space-time symmetry in the sense given above but see below.

  (ii) P — Parity under which positions and momenta are reversed but spin is left unchanged.

  (iii) T — Time reversal, where the time argument is reverse.

  The key point is that the combined CPT is *always* a symmetry, a result ensured by unitarity. This then means that particles are

  - either their own anti-particle e.g. the photon, Z^0, π^0, fermions called **Majorana spinors**,
  - or they have a distinct partner, their anti-particle e.g. $e^\pm, p^\pm$. In this case the anti-particle has the same mass, lifetime, spin but carry the opposite electric charge.

  Thus C-symmetry is not a space-time symmetry in the sense given above but a more careful study shows we should extend our definition of space-time symmetries to include C.

Particle Quantum Numbers

Conserved quantities associated with **internal symmetries**, symmetries associated with the particles themselves rather than space-time. Again we imagine two experiments where the initial positions and velocities of the particles are identical but we change the type of particle at each point. If the results are simply related, again we have a symmetry. Note that all conserved quantities are usually called charges, with an adjective to indicate which one. If no adjective is given with a charge, the context will tell you what type of quantum number is referred to, or failing that it probably means electric charge. We will study these later but examples are

- *Electric Charge.* Exactly conserved. Universe has no net electric charge.

- *Strangeness.* Essentially the number of strange quarks minus the number of strange anti-quarks. Conserved in strong interactions but not in weak interactions.

- *Baryon number.* Number of baryons (see below) minus the number of anti-baryons. Exactly conserved in present proven theories. Net baryon number in the universe.

- *Lepton number*. Number of leptons (see below) minus the number of anti-leptons. Also exactly conserved in present proven theories. Net lepton number in the universe.

1.3 The Particles

We can split the known fundamental particles into four distinct types. There are also some bound states of these fundamental particles which we will often encounter.

- **Leptons**. From the greek meaning 'light ones', spin 1/2, no interactions with strong force (no colour charge). Two types:
  - e (electron), μ (muon) and τ (tau). Massive, $m_e = 0.5\text{MeV} < m_\mu = 0.1\text{GeV} \ll m_\tau = 1.8\text{GeV}$. They all have the same electric charge -e. Their anti-particles are distinct particles with the same mass but opposite electric charge. For the electron, this is called the **positron**, the only anti-particle to be honoured with its own unique name.
  - ν_e, ν_μ, ν_τ (neutrinos), one type each for each of e, μ, τ plus each has a distinct anti-particle. The τ neutrino was only found at Fermilab in 2000. There is now evidence for these having a non-zero mass difference[3], perhaps of the order 10^{-2}eV. This means that at least one neutrino has a non-zero mass.

Generation 1	e	ν_e	u	d
mass	0.5MeV	< 3eV	~ 3MeV	~ 7MeV
Generation 2	μ	ν_μ	c	s
mass	0.1GeV	< 0.2MeV	~ 1.2GeV	~ 0.1GeV
Generation 3	τ	ν_τ	t	b
mass	1.8GeV	< 18eV	174GeV	~ 4.2GeV
Electric Charges / e	-1	0	+2/3	-1/3

Table 1.1: Basic properties of the fundamental Fermions. Each has an anti-particle of the same mass but with all other charges of the same size but opposite sign.

- **Quarks**. Spin 1/2, interact via all forces especially the strong force, so carry electric charge and colour. One of their most important properties is that quarks are never seen individually because of a property of the strong force called **confinement**.

  There are six types or **flavour** of quark and they are

  - u (up), c (charm), t (top) with electric charge +2/3 e
  - d (down), s (strange), b (bottom) with electric charge -1/3 e.

  Again, each quark has a distinct anti-particle, same mass, opposite charges. The flavour charge or quantum number, that is the number of each type of quark minus the number of anti-quarks of that flavour, is conserved in strong interactions, but not in interactions mediated by the weak interactions. Since the latter are weaker, they happen on longer time scales and it can be a good approximation to think of flavour conservation. Thus while free quarks are never observed, we can deduce the flavours of quarks in any particle, by noting various approximate (short time) conservation laws in particle

[3] This comes from observations at SuperKamiokande obervatory in Japan, (200?) and SNO in Canada (200?+) of *neutrino mixing*, a property similar to that seen with the quarks. The data in January 2002 is consistent with squares of the neutrino masses differing by about 10^{-4}eV^2 [21].

interactions. These were some of the earliest conservation laws noted: **isospin** — linked to the number of up and down quarks[4], **strangeness** — conservation of minus the number of strange quarks, and **charm conservation** — conservation of the number of charm quarks. With the development of the quark model in the 60's (see chapter 9 can these experimental laws were translated into the language of quark flavours and their approximate conservation, to which we will return below.

However, even when the flavour of quarks changes in a weak interaction, the total number of quarks minus anti-quarks is always conserved. This is the modern interpretation of **baryon number conservation**.

Each quark can also appear in one of three **colours**, a more complicated type of charge than familiar electric charges of EMforces. Colours are the charges acted on by the gluons and are a strong force phenomenon only. No other quark property varies with the colour, e.g. a quark with a given mass, 'flavour' and electric charge appears in three copies, which differ only in their colour. We also have no experimental probe of colour due to the property of confinement since this means we can not create a probe of a definite colour. We can easily create beams of positrons or electrons and use the different results to make deductions about electric charges e.g. find the charges on the constituents of a nucleus.

This means that colour charges of individual quarks, while conserved, can not be deduced from particle properties. The best we can do is to count the number of colours by seeing that some particles can appear in several forms, differing by the colour charges of their constituent quarks.

Confinement also means that it is hard to specify the mass of an individual quark, and the definitions above, let alone the numerical estimates are still a subject for debate (except for that of the very heavy top quark). Confinement also ensures that quarks always appear in nature as the fundamental constituents of bound states called **Hadrons** - quark bound states. The hadrons can be classified by their quark constituents, i.e. by the flavour of the bound quarks, but they also contain significant gluon components.

The nature of the strong interactions is such that there are only two types of hadrons seen today:[5]

- **Mesons**. The name comes from the fact that the first examples found (the lightest mesons) had a mass in between the lightest leptons and baryons (see below), all that were known at the time. Mesons are bound state of one quark and one anti-quark, so they are *always bosons*. The number of mesons is not a conserved number.
  Examples include the three **pions**, π^+, π^0, π^-, which are the lowest energy bound states of up and down quark/anti-quark pairs. They have approximately the same mass $m_{\pi^+} = m_{\pi^-} = 140\text{MeV}, m_{\pi^0} = 135\text{MeV}$. They are the lightest hadrons by a long way[6] and accords with the observation that the up and down quarks are very light. The π^+ and π^- must have the same mass as they are a particle/anti-particle pair. The fact that the π^0 has almost the same mass is an example of the approximate equality of up and down quark masses, and this leads one to **isospin** symmetry. The pions differ in their electric charges of $\pm e$ and 0. They all have spin zero and so their low energy behaviour ($E \lesssim m_\pi$) can be described by scalar fields — the simplest fields encountered in field theories.
- **Baryons**. The name comes from the fact that the first examples found (the lightest baryons) had a mass greater than that of the leptons and mesons. Bound states of three quarks (or three anti-quarks), and so they are *always fermions*. Baryon number is conserved in all tested theories, and

[4]The approximate equality in up and down quark masses leads to a more complicated type of symmetry so the conservation law for the numbers of up and down quark than just conserving each number in same way as EMcharge conservation. See section ???.

[5]Perhaps more types exist as evidence in 2003 suggested.

[6]The next lightest are the kaons, which have a mass around 490MeV. They are bound states of an up or down quark with an anti-strange quark.

is equivalent to the conservation of total quark number, i.e. each quark carries a baryon number of 1/3. This symmetry must be broken at higher energies as world is made of baryons only, there are no anti-baryons in nature, a property known as **baryon asymmetry**. So assuming that we didn't start with any imbalance, the baryon asymmetry must have been created at some point in the big bang in a process known as **baryogenesis**. This must have involved a process that breaks baryon number but it probably happened at very high energies.
The lightest baryons are the p^+ (proton) and n (neutron), collectively known as **nucleons**. The proton and neutron are made from bound states of three up/down quarks in their lowest energy configurations. They have approximately the same mass of $m_p = 938\text{MeV}, m_n = 940\text{GeV}$, again indicating that the up and down quarks must have almost identical masses.

- **Gauge Bosons**. Force carriers. Always spin 1, and intrinsically linked to **gauge symmetry**. This symmetry ensures that they are massless in vacuo unless a **symmetry breaking mechanism** is operating, see chapter 7. We have

  - The **photon** γ. Massless and chargeless mediator of electromagnetic force.
  - W^+, W^-, Z^0. Needed to describe weak forces. They can also interact directly with themselves unlike photons. Both are massive with $m_W = 80.4\text{GeV}, m_Z = 91.2\text{GeV}$ and this is explained using the Higgs-Kibble mechanism. The fact that the W-bosons mediate the weak force yet carry electric charges is an clue that these two forces might be interrelated
  - **Gluons**. Eight of them, mediating the strong force, carry no electric or weak charges, just colour. Also interact with themselves.

  This leaves gravity which is thought of as being mediated by the **graviton**. However gravity does not seem to fit into this pattern, part of the failure to quantise gravity. If it did it would be mediated by the graviton which would carry spin two.

- **Higgs**. The 'mass givers'. Always scalar particles, i.e. spin zero. Essential part of the same symmetry breaking mechanism needed to give mass to gauge bosons. Higgs fields are also vital for understanding masses of leptons and quarks. Required for the standard model but not yet detected. The simplest model of the Higgs particle(s) fits all known data at 95% confidence level if $114\text{GeV} < m_{\text{higgs}} < 250\text{GeV}$.[7]

  In case one is sceptical about a model built around a fundamental particle of a type never observed in particle physics, equivalents are well known in condensed matter physics. The same symmetry breaking mechanism is responsible for superconductivity and superfluidity, e.g. the Cooper Pairs in superconductivity are Higgs particles. Note that Cooper pairs are spin zero bound states of two electrons and there is no reason why the Higgs of the standard model shouldn't be a bound state rather than a fundamental particle. However, such compositeness would have to be obvious only at much higher energy scales, i.e. data today is consistent with a point-like Higgs particle on scales of $E \sim 100\text{GeV}$, i.e. distances greater than $10^{-18}m$.

Generations

The fundamental fermions, the leptons and quarks, appear in threes, where the properties of each member of the triplet are the same except for the mass, as shown in table 1.1. Thus we can group the fermions in three **generations**, each one heavier than the last. In order of increasing mass, the three generations consist of

[7] At the time of writing, October 2004, these values are still shifting. For instance a reanalysis of the Fermilab Tevatron data led to an improved value for the top quark mass. This in turn led to the the calculations of the upper bound on the Higgs mass rising to this value in 2004.

- The electron, its neutrino, the up and the down quarks, e, ν_e, u, d, and their anti-particles,
- The muon, its neutrino, the charm and strange quarks, μ, ν_μ, c, s, and their anti-particles,
- The tau, its neutrino, the top and bottom quarks, τ, ν_τ, t, b, and their anti-particles,

In fact there are weak force processes which lead to mixing between different generations. This means that the number of particles in each generation is not strictly conserved but we will neglect this until section 10.8.[8]

1.4 Problems which Unification may solve

Over the last hundred years particle physics has thrown up several questions, including

- Why are the forces of nature linked to the symmetry group $U(1) \times SU(2) \times SU(3)$. As we shall see this is not a natural group to choose.
- Can one theory with one coupling constant describe the many coupling constants arising in particle physics: $\alpha = e^2/4\pi$ of QED, Fermi's constant for the weak nuclear force $G_F \sim e^2/M_W$, α_S of QCD, and Newton's constant of gravity G?
- Can one theory with perhaps one mass scale describe the range of masses seen today? Why should the electron have a mass of only 0.5MeV yet the Higgs, tau lepton and W/Z gauge bosons all have masses which are about 10^5 times heavier? This is an example of the **mass hierarchy** problem.
- Can we describe patterns seen in particles; e.g. why do the π^+, π^0 and π^- particles have (nearly) equal masses but different electric charges?
- The problem of generations: why are there three of every fermion which differ only in their mass?
- Why is electric charge always in units of $e/3$?
- Why are we unable to quantise gravity in any simple manner?

Some of these questions have been answered over the years and many remain open. Successes though may point the way to answer more of these questions in future.

Which ideas have been successful so far?

The foundation of particle physics is **Quantum Field Theory**, (QFT). It remarkably good for describing the quantum effects of particles. QFTis *not* quantum mechanics, nor is it relativistic quantum mechanics but we will leave a study of QFTfor other texts and just review the key points below in section **??**. The primary aim of this book is to study the other important set of ideas used in unification, those of *symmetry*. Its mathematical description using group theory will be reviewed in chapter 3. So let us start by reviewing the history of the unification programme, to see what progress has been made.

The first example of unification was Maxwell's unification of electric and magnetic phenomena into one consistent classical theory. In the twentieth century it became obvious that quantum physics was needed for further progress. The classical EM theory of Maxwell became QED — **quantum electrodynamics** — developed through the forties. This theory can be tested to great accuracy, with the magnetic moment measured and calculated to twelve significant figures and in complete agreement.

[8] This also makes it rather harder to say what exactly we mean when we talk about some of these fundamental particles but we shall ignore this detail until section **??**.

If EMprovided the impetus for the development of QFTand QED, it was investigations of the strong force which stimulated the application of group theory and symmetry to particle physics. From the forties onwards, reaching a climax in the sixties, a veritable zoo of particles with odd properties were found. Much of this was understood though first flavour symmetries, the eight-fold way. the similarities between the pions mentioned above are one example. The quark model followed on from this and led to the development of QCD— **quantum chromodynamics** — our current QFTtheory of the strong interactions built around a QFTwith $SU(3)$ symmetry and the subject of chapter 9.

One of the greatest successes of the late sixties and seventies was to combine QEDin the larger theory of EW — **electroweak theory**, and Glashow, Weinberg and Salam were awarded for their work on this. EWdescribes both weak nuclear forces and EMin one theory. The EWexemplifies the types of unification we are striving for, though in many ways it is not as good as we would like. Many fundamental parameters, such as the masses of the fermions are still input parameters for EWand are not predicted. The idea of local symmetry breaking, discussed in chapter 7, is central to the success of the electroweak (EW) theory as we will show in chapter 10. Some predictions of this theory have been verified to several significant figures in particle physics experiments such as those at CERN, Fermilab, DESY etc. Explaining the origin of the masses of the force carrying particles, and finding the massive ones, the $W^{\pm}$ and Z^{0}, was the crowning success of this theory.

The current theory of the EM and weak forces, EW, and that of the strong forces, QCD, together form what is called the **standard model of particle physics**. This has been incredibly successful. It was essentially complete as a theory in the early seventies and it has survived rigourous experimental verification over thirty years. We will therefore use the standard model as our ultimate example in this text. However, it is also given as an exemplary model to show how our ideas of symmetry have led to a deeper understanding of particle physics in the hope that the same ideas may lead to a deeper understanding in the future and answer some of the questions listed above.

Chapter 2

Field Theory

The derivations in this book are based almost entirely on classical field theory. In many problems the classical description gives a good qualitative account of what the full quantum theory predicts and indeed the classical analysis often forms the starting point for the quantum analysis. Thus it is useful to sideline the complications of QFTso we can concentrate in this book on the symmetry aspects. So in the first section we give a brief description of classical field theory.

However the full theories are quantum so while we leave the details of QFTto other texts, we still need an outline of — a recipe for — the link between our classical results and the full quantum field theory. We will look at the link between this classical analysis and the quantum theory in section 2.2 below and make some additional comments as we go.

2.1 Classical Field Theory

What is a field theory? How do we calculate the behaviour of classical fields? How does a field relate to what we call particles. Let us try to answer these questions in this section starting with some examples.

Examples of Classical Fields

When we solve for the motion of a classical particle in classical mechanics, the answer is given as the position, $\boldsymbol{x}$ as a function of time t, or simply written as $\boldsymbol{x}(t)$. The classical particle is at one point in space at any one time. Classical fields on the other hand are things which spread through large amounts of space, and which can also vary in time. Thus to specify the field we must give its value (amplitude), say f, at every point in space and at every time. They are functions of space and time, $f(\boldsymbol{x}, t)$. Familiar examples of fields in the classical world are the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields.

The best known example of a classical field theory is therefore that of classical electromagnetism. There the physically observed evolution of electric and magnetic fields solutions satisfy Maxwell's equations which are

$$\nabla.\boldsymbol{E} = \rho, \quad \nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla.\boldsymbol{B} = 0, \quad \nabla \wedge \boldsymbol{B} = \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{j} \tag{2.1.1}$$

where ρ is the electric charge density and $\boldsymbol{j}$ its current. It is possible to rewrite these equations in a manner more appropriate for a relativistic problem

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial_\mu \mathcal{F}^{\mu\nu} = 0 \tag{2.1.2}$$

where

$$
\begin{aligned}
j^\mu &= (\rho, \boldsymbol{j}), && (2.1.3)\\
A^\mu &= (\phi, \boldsymbol{A}), \quad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi, \quad \boldsymbol{B} = \nabla \wedge \boldsymbol{A} && (2.1.4)\\
F^{\mu\nu} &:= \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} && (2.1.5)\\
\mathcal{F}^{\mu\nu} &:= \frac{1}{2}\epsilon^{\mu\nu\eta\lambda} F_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} && (2.1.6)
\end{aligned}
$$

The components of the four-vector A^μ are the electrostatic potential and the vector potential. The $F^{\mu\nu}$ is the **field-strength tensor** and $\mathcal{F}^{\mu\nu}$ is its **dual**. In this form the field for EMis now the function $A^\mu(\boldsymbol{x}, t)$ and not the familiar electric and magnetic fields. This form is more useful for particle physicists as the space-time transformation properties, in particular under general Lorentz transformations, are much simpler in this form. The field A^μ is a four-vector, that is it transforms like a vector under Lorentz transformations. For this reason it is often called a **vector field**. Such fields always correspond to particles with spin one though this connection requires a study of the group theory behind the Lorentz transformations. These fields also appear in what is called **local symmetry** or **gauge symmetry**, as we will see in chapter 6. Thus they are also called **gauge bosons**. While EMprovides the best known example of classical fields, we will see later that it is not the simplest example.

Having noted the link with spin, from the list of fundamental particles given in section 1.3, its clear most of them are spin one-half fermions. The Dirac equation (see (8.1.1)) describes how such particles evolve, at least when there are no interactions. The spin one-half property can again be linked with non-trivial behaviour under Lorentz transformations.

However, many issues can be illustrated with the simplest examples of fields which are those representing spin zero particles. These are called **scalar fields** as they remain unchanged under Lorentz transformations. We will encounter two types of scalar field, real $\phi(x) \in \mathbb{R}$ and complex $\Phi(x) \in \mathbb{C}$ though the latter can always be reexpressed as two real fields $\Phi = (\phi_1 + i\phi_2)/(\sqrt{2})$. No fundamental scalar particles are yet known, but the undiscovered Higgs particle of the EWmodel is such a particle. The pions, π^+, π^0, π^-, are composite scalar particles (made from two up/down quarks) and there are many other spin-zero mesons. In a non-relativistic context, Cooper pairs, two electron bound states in superconductors, are also scalar particles. Thus scalar fields can be useful in the real world. However their mathematical simplicity means that they are the ubiquitous example in QFTtexts, and indeed many examples in this book will use them. They do have one unique role that no other field can play and that is in **symmetry breaking**. The existence of superconducting and superfluid states are examples of this phenomena and we are sure that it is responsible for the mass of the $W^\pm$ and Z^0 gauge bosons too. Symmetry breaking is only possible with a scalar field (be it fundamental or otherwise) and one of the main goals of this book is to study this process. This is another reason why scalar fields will be central to our discussion. A typical equation of motion would be

$$(\Box + m^2)\phi(x) = -\lambda\,(\phi(x))^3 \qquad (2.1.7)$$

where the left-hand side is the **Klein-Gordon** equation for a non-interacting scalar particle. Parameters called **coupling constants** control the strength of interactions. Here λ on the right hand side is a coupling constant and this term describes interactions between the scalar particles.

Principle of least action

A classical theory, say for a mass on a spring moving in d-dimensions with position $\boldsymbol{x}(t) = (x_1(t), x_2(t), \dots, x_d(t))$, will be able to give the allowed motion of the mass as functions of time $\boldsymbol{x}(t)$. To do this one needs equations of motion, such as derived from Newton's laws of motion, and boundary conditions, two in simple cases (say $\dot{\boldsymbol{x}}(0)$ and $\boldsymbol{x}(0)$). The equations of motion can also be derived from the **principle of least action**, where the action S is given in terms of either a Hamiltonian H or a Lagrangian L

$$S := \int dt\, H\left[\boldsymbol{p}(t), \boldsymbol{x}(t)\right] = \int dt\, L\left[\dot{\boldsymbol{x}}(t), \boldsymbol{x}(t)\right] \tag{2.1.8}$$

Here p is the momentum, the conjugate variable to position x

$$p_i(t) = \frac{\mathrm{d}L}{\mathrm{d}\dot{x}_i(t)} \tag{2.1.9}$$

For instance for a mass on a perfect spring we might have

$$L\left[\dot{\boldsymbol{x}}(t), x(t)\right] = \frac{1}{2m}(\dot{\boldsymbol{x}}(t))^2 - \frac{k}{2}(\boldsymbol{x}(t))^2, \quad H\left[\boldsymbol{p}(t), \boldsymbol{x}(t)\right] = \frac{1}{2m}(\boldsymbol{p}(t))^2 + \frac{k}{2}(\boldsymbol{x}(t))^2 \tag{2.1.10}$$

Note that the Hamiltonian is the total energy, kinetic plus potential energy and it is easiest to define this. The Lagrangian can then be defined through a Legendre transform as

$$L[\boldsymbol{x}, \dot{\boldsymbol{x}}] = \boldsymbol{p}.\dot{\boldsymbol{x}} - H[\boldsymbol{x}, \boldsymbol{p}] \tag{2.1.11}$$

The principle of least action states that the possible classical behaviours, $\bar{\boldsymbol{x}}(t)$, are functions which extremise the action. That is if we compare the action for $\bar{\boldsymbol{x}}(t)$ against a solution which is slightly different, $\bar{\boldsymbol{x}}(t) + \delta\bar{\boldsymbol{x}}(t)$, then the difference is not first order but second order in these small perturbations

$$\delta S = S[\bar{\boldsymbol{x}} + \delta\bar{\boldsymbol{x}}, \dot{\bar{\boldsymbol{x}}} + \delta\dot{\bar{\boldsymbol{x}}}] - S[\bar{\boldsymbol{x}}, \dot{\bar{\boldsymbol{x}}}] = O(\delta\bar{\boldsymbol{x}}^2), \tag{2.1.12}$$

$$\rightarrow \quad \frac{\partial S}{\partial x_i}\delta x_i + \frac{\partial S}{\partial \dot{x}_i}\delta \dot{x}_i = 0 \tag{2.1.13}$$

In the second form we are using **functional differentiation** not ordinary differentiation, as $x_i = x_i(t)$ is a function not a simple variable. Equation (2.1.13) is more normally seen written in terms of the Lagrangian

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\boldsymbol{x}}(t)} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{x}(t)} = 0. \tag{2.1.14}$$

These are the **Euler-Lagrange** equations. They are the **equations of motion** or **eom**. Solutions of these equations give all the possible classical behaviours for this system. So, for a classical theory, the equations of motion contain all the physics. The equations are exactly those which are obtained using Newton's laws and principle of least action is an alternative or equivalent starting point to Newton's laws. So if our system described a mass on a spring, the equations of motion will be exactly those equations obtained from Newton's laws of motion,

$$\ddot{\boldsymbol{x}} = -k\boldsymbol{x} \tag{2.1.15}$$

The behaviour of a classical system is *always* one solution to the equation of motion. There are many different solutions, typically of different energies, momentum etc. The equation of motion are satisfied by solutions $x(t)$ which describe the possible behaviour of the system. Different initial conditions will select which of these the system actually follows. Since it is classical, then these are deterministic equations, i.e. once initial conditions are given, the behaviour is completely fixed[1].

This principle of least action is also at the heart of the path integral approach to QFTpioneered by Feynman. This emphasises the close link between classical and quantum analysis often (but not always) present.

[1]We are leaving aside any practicabilities and any worries about classical chaos.

Equations of Motion for Fields

Classical fields are functions of space and time. There may be d of them, f_i ($i = 1, 2, \ldots d$), and assume they take real values $f_i(x) \in \mathbb{R}$. Their classical behaviour is usually given in terms of a **Lagrangian density** $\mathcal{L}$, which is simply related to the Lagrangian L and action S through

$$S = \int dt\, L, \quad L = \int d^3x\, \mathcal{L} \tag{2.1.16}$$

The equations of motion are again defined uniquely by the condition that the action S defined as

$$S \;=\; \int d^4x\, \mathcal{L}(f_i, \partial_\mu f_i) \tag{2.1.17}$$

should be extremised. Consider a small (infinitesimal) variation $\delta f_i(x)$ from a classical solution $\bar{f}_i(x)$ so

$$\begin{aligned} 0 = \delta S &= \int d^4x\, \delta\mathcal{L} = \int d^4x\, \left[\frac{\partial \mathcal{L}}{\partial f_i}\delta f_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu f_i)}\delta(\partial_\mu f_i)\right] & (2.1.18)\\ &= \int d^4x\, \left[\left(\frac{\partial \mathcal{L}}{\partial f_i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu f_i)}\right)\right)\delta f_i + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu f_i)}\delta f_i\right)\right] & (2.1.19)\end{aligned}$$

where this is to be evaluated at $f_i(x) = \bar{f}_i(x)$. The last term gives a contribution only on the boundary but we will assume such contributions are zero. Thus for action to be extremised for any and all variations δf_i we see that

$$\frac{\partial \mathcal{L}}{\partial f_i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu f_i)}\right) = 0 \quad \forall\, i \in 1, 2, \ldots, d. \tag{2.1.20}$$

These are again called the **Euler-Lagrange** field equations.

To make contact with the Hamiltonian, we define the canonical momentum $\pi_i(x)$ as before via

$$\pi_i(x) = \frac{\partial \mathcal{L}}{\partial(\dot{f}_i(x))} \tag{2.1.21}$$

Then the Lagrangian and Hamiltonian are related through the same Legendre transformation

$$\mathcal{H} = \pi_i \dot{f}_i - \mathcal{L}, \qquad H(f_i, \pi_i) := \int d^3x\, \mathcal{H} \tag{2.1.22}$$

Again it is the Hamiltonian and not the Lagrangian which is related to the total energy of the system of fields.

Particles and Degrees of Freedom

So far we have worked with a single generic field with many components $f_i(x)$. The discussion is so general that in principle different components could be describing different types of particle. However, it is much more natural to use a different symbol for every type of particle and so to work with a collection of different fields, say $A^\mu(x)(\mu = 0, 1, 2, 3)$ for photons, $\psi^\alpha(x)(\alpha = 0, 1, 2, 3)$ for a electrons/positrons, $\pi_i(x)(i = 1, 2, 3)$ for the three pions. In chapter 3 this split into different types of field will be done in a more precise definition manner based on symmetry considerations.

However, as these examples show, we will still encounter fields representing one type of particle yet which have several components. This is essential because sometimes what we call a single type of particle, a photon, an electron and which we represent with a single field, actually comes in several variants, known as **degrees of freedom**. There is one degree of freedom for every distinct way that a particle can transport energy and momentum along any one given direction in space.

In terms of the field theory each degree of freedom requires its own real field to describe it. Thus if a scalar particle or field has d degrees of freedom, then we normally use a d-component real vector of space-time variable functions, $\phi_i(x)(x = 1, 2, \ldots, d)$[2]. For simple fields, the number of degrees of freedom is indeed going to be the same as the number of independent real functions of space-time needed to describe the field.

The pions illustrate several of these points. Perhaps we tend to think of the three pions as distinct particles, with different electric charges. We might be tempted to describe the presence of each type of pion using its own real field. However all three have closely related properties, in particular the same mass (this is relevant only if we ignore the small mass differences). It turns out to be advantageous to always use a single field with several components to describe closely related particles. Thus in the case of pions we will need a three-component real field to describe these three pions, one real field for each distinct way energy and momentum can be transferred through the system. It is really a matter of choice whether we want to think of the π^+, π^0, π^- as separate particles, or as three degrees of freedom of a single pion particle. The former is probably how pions are usually discussed while the latter reflects the way they are described in QFT.

In relativistic contexts, it is always best to use a field that also describes both the modes of one particle and the modes of the corresponding anti-particles. The pion example shows this as the ϕ^+/π_- particle/anti-particle pair are described together in the three component field. The electron is usually described by the same field as the positron, though we normally think of them as distinct particles. In this case each are spin one-half particle with two distinct modes, spin up or down. Thus the electron field has four degrees of freedom in relativistic QFT, and describes all the electron/positron modes.

In both the electron and pion examples, the number of degrees of freedom matched the number of components in the field vector used to describe them. The photon illustrates one last problem. For a given direction of travel (the direction of the Poynting vector and the direction of energy transfer) a photon in a vacuum has two modes corresponding to the two possible independent polarisations of light. Thus a photon in vacuo therefore has two degrees of freedom.[3] However, the relativistic field theory for the photon is given in terms of a four-component real-field, A^μ, which has simple Lorentz transformation properties. As the massless photon has only two physical modes (the two polarisations) this field has two redundant, that is unphysical, components. One could, in principle, work with a simple two-component field for the photon but the Lorentz transformation properties are much more complicated. In practice it is preferred to have simple Lorentz properties but to complicate the relationship between the fields and the physical modes of physical particles. In the case of electromagnetism, the inclusion of unphysical modes in the fields is the issue of **gauge invariance**.

The grouping of the different physical modes in the system will be made precisely when we start to use group theory to study symmetry in chapter 3. However, as we start to look at more advanced topics, we will see that symmetry can be hidden and there may be ways of grouping particles together, describing them with single fields, that don't reflect their obvious properties but reflects deeper underlying truths. The simple relationships outlined here become more complicated. However, the most important point is that whatever way we choose to group the physical modes, there must always be one real function per degree of freedom. Thus whenever we decide to change our mathematical representation of the physics, when we choose to work in a second set of fields, we must always make sure that the new fields contain the original real functions used for the degrees of freedom, no more, no less. Changing the way we describe the problem mathematically must not alter the physical content. Likewise, while a change of field definitions may highlight different aspects of the physics, it must never be more than a shuffling of the physical modes.

One final remark is to note that one can use a single complex function to represent two independent real functions $f_1(x)+if_2(x)$. For instance the Klein-Gordon equation with mass parameter m can be an equation

[2]Note that it is common to be lazy and use x rather than x^μ, for example in the arguments of functions.

[3]Interestingly it can have one additional longitudinal polarisation in some materials and we will see that when symmetry is 'broken' the photon like fields - gauge fields, have mass and have three degrees of freedom.

of motion for one real field representing a single particle of mass m, or it can be an equation of motion for a complex field. In the latter case we are describing two scalar modes, both of mass m, and further that these two modes indicate that there is a particl/anti-particle pair. We will look at this relationship more precisely in chapter 4.

Vacuum energy of the unbroken theory

For unbroken symmetry, $\boldsymbol{\phi}$ is function to use to describe the physics, rather than $\eta(x) = \phi(x) + a$ or $\eta(x) = sh(\phi(x))$ or some other redefinition which has the same mathematical information. The fact that $\boldsymbol{\phi}$ as given is a good choice is clear because its Lagrangian, (2.3.2), satisfies all the criteria set out above, e.g. its a simple polynomial of the fields and derivatives, etc. etc. The classical Lagrangian in terms of ϕ (2.3.7) can reflect the real quantum physics, meaning that a study of the classical theory can be a good guide to essential aspects of the full QFT.

However, there is one last criterion, namely that $\boldsymbol{\phi} = 0$ be a lowest energy solution for the theory, so that the field $\boldsymbol{\phi}$ represents fluctuations around the lowest energy or **vacuum** energy solution. This is the case of **unbroken symmetry** and in the case of (2.3.2) occurs if $m^2 > 0, \lambda > 0$. It corresponds to having a vacuum expectation value in the full QFTfor all components of the $\boldsymbol{\phi}$ equal to zero $\langle 0|\boldsymbol{\phi}_i(x)|0\rangle = 0$ (and therefore constant in space and time). We call this the **vacuum** solution for $\boldsymbol{\phi}$. If we rewrote the Lagrangian density of a Hamiltonian density (the latter is roughly kinetic plus potential energy, the former is just the difference) it is then clear that zero field is also the lowest energy value classically too *provided* $m^2 > 0, \lambda > 0$. It therefore makes sense to think of $\boldsymbol{\phi}$ being small and representing small quantum fluctuations about this vacuum solution. Then, provided λ is small, terms $O(\phi^3)$ or $O(\phi^4)$ etc are going to be very small and won't alter the qualitative picture given by the larger $O(\boldsymbol{\phi}^2)$ terms, *provided* the **coupling constant** λ is small too.

Spin models and field theories

Another useful analogy starts with quantum spin models of the type often discussed in quantum mechanics courses. A typical example, used as a simple model of ferromagnetism in iron and other materials, considers the spin of a single electron at each lattice site of a material. Classically the spin at lattice point a might be represented as a vector $\boldsymbol{S}(a)$ of fixed length $|\boldsymbol{S}| = 1/2$ but free to point in any direction in space. Each electron a interacts with its nearest neighbours, $b \in \mathrm{nn}_a$, with the potential energy proportional to vector product of the spins, so that the Hamiltonian takes the form

$$H = \sum_a \sum_{b \in \mathrm{nn}_a} g S_j(a) S_j b, \quad j = 1, 2, \ldots d \tag{2.1.23}$$

One can consider variations of this model. Perhaps the spin is confined in one direction $d = 1$ so only one component S_1 is involved. This model is the Ising model. Perhaps we add some simple quantum mechanics and allow the size of the spin vector to vary, say $S_1 = |S|, |S| - 1, \ldots, -|S|$. Finally we could let the spins take continuous values $S_1 \in \mathbb{R}$ — a continuous spin model. In this case the theory becomes equivalent to a theory of a scalar field theory $\phi_i(a) \equiv S_i(a)$ on a Euclidean space-time lattice, a common approximation used in practical QFTcalculations.

However, one should be *every* careful with this spin/field analogy. In this case it is not obvious what the particle side of the picture should be — the scalar field represents a spin zero particle in QFTyet we started with objects in our spin models of higher spins than that. The length of the scalar field vector $\boldsymbol{\phi}$ is not to be thought of as a physical spin as in the original model with $\boldsymbol{S}$. Following on from this we see that the spins $\boldsymbol{S}$ are limited to live in at most the three-dimensional space of our physical experience. However, in the 'continuous spin' version, the field $\boldsymbol{\phi}$ need not be pointing in any direction of real space. For instance we could have many scalar fields, many scalar particles, each field component with a value at each space-time point for each scalar particle, just as the density of particles varies from point to point. In this case the index of the fields, d, runs over the number of scalar particles *not* the space-time dimension. These fields live in a different space, called the **internal space**. At each space-time point we have a new label or coordinate but this has nothing to do with the space (or space-time) of the model. The axes of the internal space are labelled by different types of particle, e.g. we would need a seven dimensional space to describe a system of pions, protons, neutrons and their anti-particles. As we will see, we tend to split these internal spaces up into subspaces, each for the particles of closely related properties. For instance we might focus on the three-dimensional subspace needed for the three pions. These three scalar fields would be better represented as three components of a scalar field vector $\boldsymbol{\phi}$ living in a three dimensional internal space.

2.2 From Classical to Quantum Theory

In a quantum system, the classical solution for a field is only one of the most probable values for a field but as such it can still play a central role in QFT. Further the quantum expectation value of the classical eom is also satisfied in QFTeven though many field configurations other than the classical field configurations contribute. It is therefore worthwhile studying the classical eom and their solutions as many results will remain true in the full QFTand as such they often form a starting point for a study in the full QFT.

The Lagrangian density $\mathcal{L}$ by itself gives us a complete description of a classical theory of the dynamics of fields $\boldsymbol{\phi}$, e.g. we can derive classical equations of motion from it. From this view point it is not obvious why we should associate the d fields $\phi_j(x)$, which are just some real functions of space-time variables x, with particles — point like objects characterised by certain definite numbers such as energy, momentum, charge etc. In fact this link is *only* possible if we look at the full QFT. Only QFTcan show us why m is related to the mass of some particle, λ a measure of the strength of interactions between particles.

However there is a good rationale for studying in this book the classical field theories rather than the full QFT. What we are implicitly assuming in this course is that the full quantum solution, $\bar{f}_{\mathrm{q}}(x)$, is of the form

$$\bar{f}_{\mathrm{q}}(x) = \bar{f}_{\mathsf{Cl}}(x) + \delta \bar{f}(x) \tag{2.2.1}$$

where the classical field, $\bar{f}_{\mathsf{Cl}}(x)$, is the solution to the classical equations of motion, and $\delta \bar{f}(x)$ is a small quantum correction, i.e $\sim O(\hbar)$. If this is true then when we study classical field theory and its symmetries we will get a good qualitative picture of the physics of the full quantum theory as the quantum effects generate only 'small' changes. Of course this is merely hope at this stage. The full quantum theory must be studied to see if its valid, or at least classical predictions compared against experimental data. In both cases it is known that there are many problems where this classical approximation does at least give use a good overview and its a genuinely useful tool.

At the same time there are several situations where the classical picture is misleading and the quantum solution may not always be obtained by a small-order ($O(\hbar)$) correction to the classical solution. These include

- Bound states — such as the mesons and baryons in QCD(quantum chromodynamics), the theory of strong nuclear forces. Their existence can not be checked directly from the classical analysis, e.g. we can not calculate their masses. However, the symmetry in classical theory is preserved in the quantum theory and from this we can predict some of the properties of the bound states, such as their conserved charges.

- Phase transitions, where thermal fluctuations balance quantum fluctuations.

- Anomalies, where quantum effects destroy classical symmetry.

The last point is particularly important. Symmetry is at the core of what we are doing in this book, and there is no way in the classical theory to see if the quantum theory has all the symmetry of the classical theory. Luckily, it is relatively simple in QFTto see if anomalies are present or not.

We will need to draw on quantum theory for a few aspects. The most important is to make a connection between particles and fields; e.g.

$$\text{pions } \pi \longleftrightarrow \pi(x), \qquad \text{photons } \gamma \longleftrightarrow A^{\mu}(x) \tag{2.2.2}$$

One way of thinking of particle/field duality is to imagine that peaks in a field can be thought of as corresponding to the positions of a particles. This follows, assuming that the zero field value is the lowest energy state, because a peak in a field represents at least a localised lump of energy, one property at least of what we call a particle.

The propagation of quanta (packets of conserved quantities) is described in QFT by *propagators*. A **free propagator**, i.e. no interactions, is a solution to the equations of motion of the quadratic part of the Lagrangian. The quadratic requirement implies no terms of three or more fields. The full solution to QFT is usually built on top of the free non-interacting theory and its solutions, so we shall focus on the quadratic part of the Lagrangian.

Degrees of Freedom

Another better way of saying that a field has a N independent ways of moving energy in any given direction is to say it represents N particles. To make the link with particles, one really needs to turn to QFTand in particular talk about how many different types of annihilation and creation operators are needed to build the quantum version of the field. A field made up of N independent real fields require N distinct types of annihilation/creation operator pairs $\hat{a}_i(\boldsymbol{k}), \hat{a}_i^{\dagger}(\boldsymbol{k})$, $i = 1, 2, \ldots, N$ where $\boldsymbol{k}$ is the momentum label. Distinct here means that for a given energy and momentum, there are different operators, and we distinguish them with the label i. We do not count here the fact that there are different operators for each momentum k, afterall the same particle can move at different momenta. More precisely, distinct annihilation and creation operators in QFTmeans they commute with each other, even for the same momentum k,

$$[\hat{a}_i(\boldsymbol{k}), \hat{a}_j^{\dagger}(\boldsymbol{k}')] \propto \delta_{ij}\delta^3(\boldsymbol{k} - \boldsymbol{k}'). \tag{2.2.3}$$

Thus this field represents d distinct particles which can run around at any momentum k.

There are many different ways mathematically of rewriting the field $\boldsymbol{\phi}(x)$ but we will always need d-independent real functions to describe it fully. It can be quite hard to check that an given redefinition does indeed allow you to describe *all* possible values of $\boldsymbol{\phi}(x)$. For instance trying to use the d fields $\eta_i(x) := \exp(\phi_i(x)\boldsymbol{\phi}^2)$ will be confusing and complicated. Any appropriate redefinition can not change the physics, but it can help reveal (usually it will just obscure) the true physics. However, we will almost always work with appropriate and simple field definitions and such field redefinitions will not be a concern. An exception is the use of the unitary gauge in chapter 7.

Vacuum energy of the unbroken theory

For unbroken symmetry, $\boldsymbol{\phi}$ is function to use to describe the physics, rather than $\eta(x) = \phi(x) + a$ or $\eta(x) = sh(\phi(x))$ or some other redefinition which has the same mathematical information. The fact that $\boldsymbol{\phi}$ as given is a good choice is clear because its Lagrangian, (2.3.2), satisfies all the criteria set out above, e.g. its a simple polynomial of the fields and derivatives, etc. etc. The classical Lagrangian in terms of ϕ (2.3.7) can reflect the real quantum physics, meaning that a study of the classical theory can be a good guide to essential aspects of the full QFT.

However, there is one last criterion, namely that $\boldsymbol{\phi} = 0$ be a lowest energy solution for the theory, so that the field $\boldsymbol{\phi}$ represents fluctuations around the lowest energy or **vacuum** energy solution. This is the case of **unbroken symmetry** and in the case of (2.3.2) occurs if $m^2 > 0, \lambda > 0$. It corresponds to having a vacuum expectation value in the full QFTfor all components of the $\boldsymbol{\phi}$ equal to zero $\langle 0|\boldsymbol{\phi}_i(x)|0\rangle = 0$ (and therefore constant in space and time). We call this the **vacuum** solution for $\boldsymbol{\phi}$. If we rewrote the Lagrangian density of a Hamiltonian density (the latter is roughly kinetic plus potential energy, the former is just the difference) it is then clear that zero field is also the lowest energy value classically too *provided* $m^2 > 0, \lambda > 0$. It therefore makes sense to think of $\boldsymbol{\phi}$ being small and representing small quantum fluctuations about this vacuum solution. Then, provided λ is small, terms $O(\phi^3)$ or $O(\phi^4)$ etc are going to be very small and won't alter the qualitative picture given by the larger $O(\boldsymbol{\phi}^2)$ terms, *provided* the **coupling constant** λ is small too.

2.3 Typical Lagrangians in Particle Physics

The form of the Lagrangian needed to describe relativistic particles are very specific, the form specified by symmetries such as space-time symmetries, and by quantum ideas such as renormalisability. Different parts are related to different types of physical properties of the particles being described. As a result the same terms appear again and again and are referred to using standard terminology. This we will outline below as a full understanding requires a proper QFTtreatment. We will also focus only on scalar fields in 3+1 dimensions. Similar ideas apply to other fields in four-dimensions, and we will note these as these are introduced in later sections, gauge fields in section 6 and fermions in section 8. Generalisations to other dimensions or to non-relativistic theories are also straight forward.

Fields of Particles of similar properties

When particles have the same spin and mass and they interact at relatives strengths which are simple multiples of each other, then there is some deeper relationship between the particles. For instance the three pions are all scalar particles of essentially the same mass, their EMcharges are $+1, 0$ and -1 so their EMinteractions are closely related, and one finds that their strong and weak interactions are closely related. When particles have such similar properties, there is a **symmetry** relating these particles and there is a mathematical **group** associated with this theory. A major part of this book is to see how to express these similarities between particles in terms of Lagrangians and fields. At this stage, all we need to note is that when particles share common features in this way, it will turn out to be extremely convenient to put the fields associated with all the particles into a single *vector* of fields. For instance, for the pions we might define a three component field

$$\boldsymbol{\Phi}(x) = \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \Phi_3(x) \end{pmatrix} \tag{2.3.1}$$

where $\Phi^1(x)$ could be the field describing the π^+ particle, $\Phi^2(x)$ might be for the π^0 and the π^- is linked to the third component $\Phi^3(x)$.

Free and Interacting parts

The first division and most important division of any Lagrangian is into free and interacting terms. The **free field** part of any Lagrangian is that part made up of terms which are at most quadratic in the fields, $O(F^2)$. The remaining cubic and higher terms are called **interaction terms**. We will see one reason for these names in a moment. In this case then we have

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi).(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2}_{\text{Free field terms}} - \underbrace{\lambda\phi^4}_{\text{Interaction term}} \tag{2.3.2}$$

The **e.o.m.** (equation of motion) is[4]

$$\partial^\mu\partial_\mu\phi + m^2\phi + 4\lambda\phi^3 = 0 \tag{2.3.3}$$

The terms of two or less fields in the Lagrangian appear as linear terms in the eom. These represent propagating wave solutions. Since one can add two solutions to a linear equation to get another solution, these represent waves that pass through each other without noticing other solutions. They are non-interacting terms and represent what we would identify as an isolated partilce or wave in a detector. Terms which have more than two fields in the Lagrangian are interactions as these lead to non-linear terms in the eom. Such non-linear terms mean that the sum of two solutions to the linear parts, i.e. non-interacting proagating waves, are no longer guaranteed to be a solution. That is the non-linear terms distrupt the solutions to the linear part, i.e. the non-linearity is an interaction.

The naming is clearer in Fourier space where the equation of motion becomes

$$(k^2 - m^2)\phi(k) = 4\lambda \sum_{k_1,k_2} \phi(k_1)\phi(k_2)\phi(k - k_1 - k_2) \tag{2.3.4}$$

$$\phi(k) := \int d^4x \, e^{ikx}\phi(x) \tag{2.3.5}$$

We see that it is only the term proportional to lambda which mixes the different energies and three-momenta. In a particle language such mixing is caused by the scalar particles interacting. The term with coefficient λ controls the mixing or *interactions* of the different components of the fields, hence the name.

There are other divisions of the Lagrangian which will be encountered, namely

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi).(\partial^\mu\phi)}_{\text{Kinetic Terms}} - \underbrace{V(\phi)}_{\text{Potential}} , \tag{2.3.6}$$

$$V(\phi) := \underbrace{\frac{1}{2}m^2\phi^2}_{\text{Mass term}} + \underbrace{\lambda\phi^4}_{\text{Quartic interaction term}} \tag{2.3.7}$$

These are discussed below.

2.3.1 Free Fields

The terms which are no more than quadratic in the fields are called the **free field terms**. These describe the propagation of particles (i.e. lumps of conserved measurable quantities such as energy, momentum, spin, charges) without any interactions. This can be seen in the classical theory from the eom in fourier space (2.3.4). For each k, the free field eom ($\lambda = 0$) is satisfied only is $k^2 = m^2$. Thus each $\phi(k)$ describes some sort of amplitude of a wave of momentum $\boldsymbol{k}$ and energy $\sqrt{(\boldsymbol{k}^2 + m^2)}$.

More importantly for QFT, with zero interactions one can build on these classical solutions **QFT** (Quantum Field Theory) exactly in any dimensions if these are the only terms present. With interactions present,

[4] **EFS 2.3.1**: Derive the eom of (2.3.3) for the single real self-interacting scalar field ϕ of (2.3.2).

there are virtually no exactly solvable QFT theories[5]. As a result, free theories play a central role in all QFTapproximate solutions to the physically interacting theories.

First let us split the free part of the Lagrangian into two more pieces

$$\mathcal{L}_0 \;=\; \underbrace{\frac{1}{2}(\partial_\mu \phi).(\partial^\mu \phi)}_{\text{Kinetic Terms}} - \underbrace{\frac{1}{2}m^2\phi^2}_{\text{Mass terms}} \tag{2.3.8}$$

Note that a subscript 0 is often (but not always) used to denote the free part of the Lagrangian[6].

Kinetic vs. Potential Terms

The $\frac{1}{2}\partial_\mu \boldsymbol{\phi}.\partial^\mu \boldsymbol{\phi}$ derivative terms are the **kinetic terms** as no propagation of energy, momentum, charge etc. occurs without these terms. They have two derivatives and are quadratic in the fields. The remaining terms, with no derivatives only in simple cases, are called the potential terms.

The naming is best understood by looking at the Hamiltonian density $\mathcal{H}$ defined to be

$$\mathcal{H} \;:=\; \Pi_i \frac{\partial \phi_i}{\partial t} - \mathcal{L} \tag{2.3.9}$$

$$\Pi_i \;:=\; \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \tag{2.3.10}$$

in terms of the canonical momenta Π_i. For our O(N) example we have

$$\mathcal{H} \;:=\; \frac{1}{2}\Pi_i\Pi_i + \frac{1}{2}\nabla_x \boldsymbol{\phi}_i \nabla_x \boldsymbol{\phi}_i + \underbrace{V(\boldsymbol{\phi}.\boldsymbol{\phi})}_{\text{Potential}}\,, \tag{2.3.11}$$

The Hamiltonian is the total energy, kinetic plus potential energy. The derivative terms are the kinetic energy terms, e.g. the $(\nabla\boldsymbol{\phi})^2$ is the momentum squared term which in the non-relativistic limit would be the usual kinetic term $p^2/2m = -(i\hbar\nabla)^2/2m$. The rest, here V, must be the potential energy terms.

Mass Terms

The quadratic term with no derivatives $\frac{1}{2}m^2\phi^2$ is the **mass term**. If there are only kinetic and mass terms for the fields ϕ in the Lagrangian density, then *and only then* may one interpret m as the mass of the particle described by the real field $\phi(x)$. This m is a classical mass and in the full quantum theory, the value of the physical mass will be different as the mass gets **renormalised** in the quantum theory.

Free Field Solutions

The free field case, when all interaction terms are zero ($\lambda = 0$ in (2.3.2)), is exactly solvable in any number of space-time dimensions. In the case of our scalar field example here, the equation of motion we obtain for the free field is called the **Klein-Gordon** equation

$$\partial^\mu\partial_\mu\phi(x) + m^2\phi(x) = 0 \qquad \textbf{Klein-Gordon equation} \tag{2.3.12}$$

In fact we can always solve the free field case exactly, both in quantum and classical theories and for any fields (in flat space-time). For this mathematical reason, *almost all* analytic solutions for field theories,

[5] Some special examples exist in 1+1 dimensions where conformal symmetry plays a vital role in allowing one to solve these theories.

[6] When discussing renormalisation in QFT, it is used to denote 'bare' (usually infinite) quantities.

classical or quantum, are built around free field solutions. The solutions are best seen by working with the Fourier transform of the fields $\phi(k)$, which gives for the free part

$$(k^\mu k_\mu - m^2)\phi_j(k) = 0 \Rightarrow \phi_j(k) \sim \exp\{\pm i\omega t\}\exp\{\pm i\boldsymbol{k}.\boldsymbol{x}\} \tag{2.3.13}$$

where

$$k_0 = \pm\omega, \quad \omega = |\sqrt{\boldsymbol{k}^2 + m^2}|. \tag{2.3.14}$$

Boundary conditions decide precisely what combinations of these exponentials form the solution. This tells us several things. First the free field solution $\phi(x)$ is a combination of packets of definite energy ω and three-momentum $\boldsymbol{k}$. Therefore it makes sense to represent (approximate) particles seen in detectors by free fields carrying the same energy and momentum as the particles. From this analogy it is also clear from (2.3.14) that m is then to be linked with the mass of the particle represented by these classical fields, since (2.3.14) is the standard energy momentum relationship for a relativistic particle..

2.3.2 Interacting theories

When we have $\lambda \neq 0$ we have an interacting theory, not just a free theory. The interactions spoil the simple free field solution. One must resort to other techniques to find a solution. For small λ, **perturbation theory** can be used to produce *approximate* solutions giving ϕ as a power series in λ. For large λ **non-perturbative** methods[7] must be used, such as straight numerical approximations.

To make sense, we want the **coupling constant** λ to be positive so that the potential energy density is **bounded below**, it has an absolute minimum. Thinking of $\phi_j(x)$ as small (quantum fluctuations) then, provided λ is small, cubic, quartic and higher[8] terms - $O(\phi^3)$ or $O(\phi^4)$, are going to be very small and won't alter the qualitative picture given by the larger quadratic $O(\phi^2)$ terms, *provided* the **coupling constant** λ is small too. The terms $O(\phi^3)$ or higher are usually called the **interaction terms**[9].

2.3.3 Other terms no higher than Quadratic

Initially, the Lagrangians will be of the form given in (2.3.2). However, other terms will be encountered, both in symmetry breaking and when sources are introduced. It is important to know how to deal with terms which are linear or quadratic as they spoil the interpretation of the m coefficient as a mass. Terms which are cubic or higher will invariably be treated as interactions.

Linear Terms

There must be *no linear term* in the fields if one is to interpret the quadratic terms in the way suggested above, namely propagation and mass terms for particles. If you have them, get rid of them.

The QFTremains completely solvable with such linear terms. For instance an appropriate redefinition of the fields using a constant shift, $\tilde{\phi}(x) = \phi(x) + v$ for some appropriate constant v, can always be found which removes linear terms, but it will change the coefficients of the other terms both quadratic (mass) and any the interactions too. This will be used in **symmetry breaking** where v will be linked to the **vacuum** state (lowest energy state) changing from that which is empty of real particles (though full of virtual particles in the quantum theory) to one full of real particles.

[7] In field theory, "perturbation theory" invariably means expansions in these coupling constants coefficients λ. Most analytic "non-perturbative solutions" are also perturbation series but in some other parameter.

[8] In four space-time dimensions, a renormalisable (finite) *quantum* theory will not have anything higher than quadratic field terms. Even though we are studying classical theories here, we will tend to stick to this limitation.

[9] More advanced considerations - renormalisation, or going beyond weak coupling perturbation theory ($\lambda \gtrsim 1$) - leads to a weakening of this simple division of what is an 'interaction' and what is a 'free' term. It is usually only involves adding quadratic or linear field terms to the interaction part, the cubic and quartic terms stay in the interaction part almost always.

Another area where such linear terms are encountered will be in the full quantum theory. There one will soon encounter a term like $\boldsymbol{j}(x).\phi(x)$ term, where $\boldsymbol{j}(x)$ are N external **sources**. I will mention it in this section as it is strictly less than quadratic and it does not alter the solubility of the quadratic QFTas mentioned above. However this linear $j\phi$ term is best thought of as representing an interaction between particles described by the ϕ field and external 'classical' sources $\boldsymbol{j}$. Something *like* the three different components of a magnetic field applied by experimentalists, say to bend charged particles, but this specific example requires a more sophisticated term to be added.

Note though that a $\boldsymbol{j}.\phi$ term will usually be encountered where it is to be thought of as *unphysical*, added only for mathematical convenience and has no physical meaning. Clearly in this case one only gets sensible physics if $\boldsymbol{j} = 0$. The mathematical trick is to take derivatives with respect to the components of the current $\boldsymbol{j}_i(x)$ (functional derivatives technically) and then to set $\boldsymbol{j} = 0$. It allows one to generate useful quantities in a mathematically concise way, i.e. its a trick to summarise all QFTas a generating functional $Z[j]$[10]. This is just the same idea as generating functions in mathematics, simple functions of a 'dummy' variable whose multiple derivatives with respect to the dummy variable, followed by setting the dummy variable to zero, generate series of well known functions e.g. Bessel's functions.

Quadratic mixing terms

There must also be *no mixing* of different field components in one quadratic term if one is to interpret the mass and kinetic terms as those describing the propagation of real physical particles of mass m. So no $\phi_i(x)\phi_j(x)$ $(i \neq j)$ terms if the propagation terms are to be interpreted as I have said above.

Should you encounter such mixing terms, then a redefinition of fields through a linear unitary transformation between the fields will always remove such terms but again this will change any cubic and quartic interactions, usually making them more complicated.

2.3.4 Other restrictions on Lagrangians

When constructing a classical Lagrangian to be used for QFT, it is common to restrict the choice of terms further, reflecting deeper ideas from the quantum physics.

Real Lagrangian

Actions and Lagrangians ought to be real. If they are not the simple probabilistic interpretation of the action S though $\exp\{-(i/\hbar)S\}$, e.g. in the path integral, fails in QFT. It indicates an instability in the QFT.

Renormalisability

In QFTthe famous ultra-violet infinities, which plague all these theories, can be removed systematically in perturbation theory to leave finite answers for physical quantities provided that all coupling constants (constant coefficients of terms in the Lagrangian) have positive or no dimensions, $[M]^n, n \geq 0$,[11] in natural units. The scalar field has natural units of $M^{d/2-1}$ in d space-time dimensions[12]

Thus in $d = 4$ dimensions, one is allowed a $g\phi^3$ type term in scalar field theories by this argument, but a $g\phi^6$ term is not allowed.[13]

[10]Usually its called Z but not always!

[11]M indicates some appropriate mass scale.

[12] **EFS 2.3.2**: By assuming that the kinetic term is always of the same form, prove that the scalar field has natural units of $M^{d/2-1}$ in d space-time dimensions. Hence deduce that form of the mass term is always $m^2\phi^2$ for a scalar field in any dimension. Why is $1+1$ dimensional space-time special?

[13] **EFS 2.3.3**: Which is the largest power of scalar fields allowed in any term in (a) $d = 3$, (b) $d = 6$ dimensions if the theory is to be renormalisable?

By default, you should always try to construct a Lagrangian which is renormalisable, or at least highlight the fact that your Lagrangian is not renormalisable. Non-renormalisable theories may can give good descriptions of low energy processes and, while not fundamental, they have many practical applications.

Locality

A key requirement in most QFT, is that the physics is local. That is physics at one space-time point is only effected by information in the neighbourhood that point. This is equivalent to demanding that each term is a polynomial of finite order in derivatives. Thus the kinetic term is a second order polynomial in derivatives ∂^μ, but a term like $\ln[M^4 + (\partial^\mu\phi)(\partial_\mu\phi)]$ is not (think of the expansion of the logarithm, it has terms of an infinite number of derivatives requiring information from an infinite distance from one point if we are to calculate them. A term like $[(\partial^\mu\phi)(\partial_\mu\phi)]^2$ is allowed by this rule[14].

Example 1 Single real scalar field
$\phi(x) \in \mathbb{R}$ describes a single scalar (spin zero) particle which is its own anti-particle such as the π^0. In 3+1 dimensions the most general renormalisable Lagrangian density for $\phi \in \mathbb{R}$ is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \equiv \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4(x) \tag{2.3.15}$$

We can make the following points:

(i) $\mathcal{L}$ is *real* if $\phi(x), m^2, \lambda \in \mathbb{R}$. If we do not limit our variables to be real then we would have a complex action and it would represent an unstable theory;

(ii) The first two terms are the terms quadratic in the fields. They form free part of the Lagrangian, $\mathcal{L}_0$. By themselves they would give the Klein-Gordon equation of motion.

(iii) The first component is called the kinetic term, even though it contains both $\dot{\phi}$ and $\nabla\phi$ expressions. Compare to the classical case where kinetic energy is only linked to time derivatives, e.g. $\frac{1}{2}m\dot{x}^2$. Without this term, the quanta do not propagate.

(iv) The m coefficient is the mass of the scalar particle, *provided* the quadratic terms have the standard form given in (2.3.15)

(v) Three or more fields per term are covered in the interaction terms. Here λ is a coupling constant, a measure of interaction strength, (cross-section $\propto \lambda^2$).

(vi) For the interaction term λ you will see in the texts, variously: λ, $\frac{\lambda}{4}$ or even $\frac{\lambda}{4!}$, as in our real case, (2.3.15). The precise term used is not important as long as one is consistent within the problem.

(vii) **renormalisability** can remove all the infinities of quantum field theory, provided all coefficients have units $[M]^n$, $n \geq 0$. This implies there are no $g\phi^6$ terms in equation (2.3.15).

(viii) $g\phi^3$ *is* allowed by these rules; however, we choose to leave it out for reasons of

- symmetry: $\phi \longleftrightarrow -\phi$;
- simplicity

(ix) $\mathcal{L}$ is a **Lorentz scalar**;

---

[14]**EFS 2.3.4**: Is this term ever renormalisable?

(x) this describes *one degree of freedom*: one distinct particle mode. What this means in quantum field theory is that $\hat{\phi}$ is described with *one* pair of annihilation/creation operators

$$\hat{a}_{\boldsymbol{k}} \qquad \hat{a}^{\dagger}_{\boldsymbol{k}} \tag{2.3.16}$$

That is, this particle is its own anti-particle;

(xi) if we wish to interpret m in equation (2.3.15) as mass, the first two terms must be quadratic, i.e. *free*. Thus *no linear terms*, such as $v(\phi(x))^1$, are allowed. Removal of linear terms is quite easy using a change of variables such as

$$\phi(x) = \eta^{(x)} + c \tag{2.3.17}$$

where c is chosen so that there are no $\left(\eta^{(x)}\right)^1$ terms;

(xii) this Lagrangian is *local* (in space-time), i.e. the behaviour of the field at x^μ depends upon its value throughout a small neighbourhood of x^μ. Thus $\phi(x)$ is only affected by field values an infinitesimal distance away from x^μ, and therefore only finite powers of ∂_μ are allowed.
We note that without this assumption, one can run into severe problems with, for example, special relativity and quantum gravity.

□

In practice, we will always want to reduce our problem to the quadratic form in (2.3.15). In fact, every formula can be written in this form. However, it is quite common to use complex fields to describe a problem.

Example 2 Complex Scalar Field
We could write

$$\Phi(x) = \frac{1}{\sqrt{2}}\big(\phi_1(x) + i\phi_2(x)\big) \in \mathbb{C} \tag{2.3.18}$$

and analyse $\phi_1, \phi_2 \in \mathbb{R}$ as above. However, it is common to work with Φ (for example, Φ can be an eigenstate of charge where the $\phi_i \in \mathbb{R}$ can not). We therefore need to obtain the canonical form

$$\mathcal{L} = (\partial_\mu \Phi(x))^\dagger \, (\partial^\mu \Phi(x)) - m^2 \Phi^\dagger(x)\Phi(x) - \frac{\lambda}{4}\left(\Phi^\dagger(x)\Phi(x)\right)^2 \tag{2.3.19}$$

where the third term is chosen to give $U(1)$ symmetry (that is, it gives a phase invariance). Once again, we note that the first two terms must be quadratic for m to be interpretable as a mass. This prohibits mathematically viable things like $\left(\Phi^\dagger\right)^2 + (\Phi)^2$, which *are* real scalars, but not particularly useful fields.

(1) The particle content of one $\Phi(x)$ field is *two particles* of equal mass m, thus giving rise to two degrees of freedom. These are a particle/anti-particle pair, e.g. π^+ and π^-.[15]
In quantum field theory, two pairs of annihilation/creation operators, $\hat{a}^\dagger$, $\hat{a}$ and $\hat{b}^\dagger$, $\hat{b}$ are needed to describe the quantum field $\hat{\Phi}$: a pair for each *distinct* particle, e.g.:

$$\left[\hat{a}, \hat{a}^\dagger\right] = i\hbar \qquad \left[\hat{b}, \hat{b}^\dagger\right] = i\hbar \qquad \left[\hat{a}, \hat{b}^\dagger\right] = 0 \tag{2.3.20}$$

(2) The factors of $\frac{1}{2}$ or $\frac{1}{\sqrt{2}}$, *et cetera*, come from standard required normalisation of commutation relations in quantum field theory.

[15] Contrast this to the case before, where the particle is its own anti-particle.

The relationship between real and complex scalar fields will be investigated in more detail and looking at the symmetry aspects in section **??**. □

Example 3 Several components
Consider $\boldsymbol{\phi}(x)$, equivalently $\phi_i(x)$, $i = 1, \ldots, d$; c.f.

$$\begin{array}{ccc} \psi^\alpha(x) & \text{Dirac spinor} & \alpha = 0,1,2,3 \\ A^\mu(x) & \text{EM gauge field}^{16} & \mu = 0,1,2,3 \end{array}$$

So to find the mass of such fields we must reduce $\mathcal{L}$ to the form[17]

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\phi}(x)\right)\left(\partial^\mu \boldsymbol{\phi}(x)\right) - \frac{1}{2}\sum_{j=1}^{d} m_j^2 \phi_j(x)\phi_j(x) + O\left(\phi^3\right) \tag{2.3.22}$$

where the $O\left(\phi^3\right)$ terms represent interactions. We usually work with

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\phi}\right)\left(\partial^\mu \boldsymbol{\phi}\right) - \frac{1}{2}\boldsymbol{\phi}(x)(\mathsf{M}^2)\boldsymbol{\phi}(x) + \text{interactions} \tag{2.3.23}$$

Here the constant matrix M^2 is real and can be chosen to be symmetric, so that $(\mathsf{M}^2)^\mathsf{T} = (\mathsf{M}^2)$.
From some basic results of **quadratic forms** we know there exists an orthogonal matrix O which diagonalises M, i.e.

$$\mathsf{O}\cdot\mathsf{O}^\mathsf{T} = \mathbb{1} \qquad\qquad \mathsf{O}\left(\mathsf{M}^2\right)\mathsf{O}^\mathsf{T} = \text{diag}\left(\lambda_1, \ldots\right) \tag{2.3.24}$$

where the λ_i are eigenvalues of M^2.
Thus, if we define $\boldsymbol{\eta}(x)$ such that

$$\boldsymbol{\eta}(x) = \mathsf{O}\boldsymbol{\phi}(x) \tag{2.3.25}$$

i.e. a linear redefinition of $\boldsymbol{\phi}$, then the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\eta}\right)\left(\partial^\mu \boldsymbol{\eta}\right) - \frac{1}{2}\mu_j \lambda_j \eta_j \eta_j + \text{interactions} \tag{2.3.26}$$

where the eigenvalues of M^2 are (mass)^2 parameters for the η_j fields. Now, all quadratic mixing terms, e.g. $\phi_1\phi_2$, are gone in the $\eta(x)$ variables.
Special case $\mathsf{M}^2 = m^2\mathbb{1}$, so that $\frac{1}{2}m^2\boldsymbol{\phi}\cdot\boldsymbol{\phi}$ gives rise to $O(d)$ symmetry.

□

[16] This has four components but two redundancies due to the polarisations of the two underlying fields. This is not important here, but may arise in quantum field theory.

[17] Note that since the vector space of all $\boldsymbol{\phi}(x)$ is, in essence, an artifice, the use of up and down indices on $\phi_i(x)$ is not necessary; although, it *does* become necessary when working with the group representation. The metric on the d-dimensional vector space of $\boldsymbol{\phi}$ is simply

$$g^{ij} = \text{diag}\left(+1, \ldots, +1\right) \tag{2.3.21}$$

Chapter 3

Symmetry

3.1 Symmetry and Groups in Classical Field Theory

Suppose that we have a set of d real fields, $\boldsymbol{f} = (f_1(x), f_2(x), f_3(x), \ldots, f_d(x))$ whose dynamics is described by some Lagrangian density $\mathcal{L}[\{f_i\}]$. If we denote the possible solutions to the equations of motion by capitals, $\boldsymbol{F} = (F_1(x), F_2(x), F_3(x), \ldots, F_d(x))$ then these satisfy the equations of motion

$$\partial_\mu \frac{\partial \mathcal{L}[\{f_i\}]}{\partial\left(\partial_\mu f_i\right)} - \frac{\partial \mathcal{L}}{\partial f_i} = 0 \tag{3.1.1}$$

There are many such solutions possible, the set $\mathcal{S} = \{\boldsymbol{F}\}$, which leads us to a definition of a symmetry for a classical field theory. A **symmetry** is a transformation, which always links any solution, say $\boldsymbol{F}$, to another equally good solution, say $\boldsymbol{F}'$, which has *similar* properties. For example a frictionless ball rolling along a flat line has an equation of motion of $d^2f/dt^2 = 0$ where f is the position of the ball[1]. The solutions have the ball moving at constant velocity v and its position is given by $F(t) = vt$. In this case, one symmetry is multiplying by -1 as $F'(t) = -F(t) = -vt$ is also a perfectly good solution to the same equation of motion. Both solutions have the same energy and speed but opposite velocities. This reflects the important property of our system which says reflecting the line about an origin gives us back the same problem, left and right are equivalent.

On the other hand we wish the symmetry transformation $F \leftrightarrow F'$ to be reversible. Consider multiplying F by zero. This again gives another solution, that of a stationary ball, but it has no special relation to the rolling ball solution. So this transformation tells us nothing about the special properties of our system and we do not want to call this a symmetry transformation.

Since a classically, only solutions to the equations of motion are relevant to physical problems, we could define a symmetry to be a reversible transformation between solutions to the equations of motion. However, in quantum theory, all field configurations have some contribution, not just those which are solutions to the classical equations of motion, are important[2].

So the appropriate definition of the symmetry for our purposes is given in terms of the action and the Lagrangian rather than the equation of motion and its solutions:

Definition 1 A symmetry transformation U is reversible and leaves the action invariant,

$$S[\{f_i\}] = S[\{f'_i\}] \quad \text{where} \quad \begin{array}{rccc} U: & S & \longrightarrow & S \\ & \{f_i(x)\} & \longmapsto & \{f'_i(x')\} \end{array} \tag{3.1.2}$$

[1]Though $f(t)$ is the position of an object rather than the size of some field, the mathematics is identical.

[2]Those configurations 'closest' (in some sense) to the classical solutions are though usually the most important in the quantum theory.

Note this definition allows us to consider two types of changes in the solutions. We can change the form of the fields but leave the space-time arguments unchanged, $f(x) \to f'(x)$, and these are called **internal symmetries**. We could also consider space-time transformations which relate fields at different space-time points, say $f(x) \to f(x')$. For instance, for our rolling ball problem if $F(t)$ is a solution then $F(t+c)$ is also a solution, it does not matter when we start our experiment or the precise setting of the clock at the start of the experiment. This translation in time is a symmetry. However, such transformations tell us more about the symmetry of space-time rather than the fields and particles. This is an important and interesting subject, which can be tackled with many of the same tools such as group theory. The nature of the symmetry of space-time is fundamentally different from symmetries which do change the form of the fields[3] and so we will only consider the internal symmetries.

If we are not altering the space-time arguments, then since the action S is just the space-time integral of the Lagrangian, we can express our definition of internal symmetry as follows‘

Definition 2 An internal symmetry transformation U leaves the Lagrangian invariant, and relates values of transformed fields only to the values of the original fields at the same space-time point

$$\mathcal{L}[\{f_i\}] = \mathcal{L}[\{f'_i\}] \quad \text{where} \quad \begin{array}{cccc} U: & \mathcal{S} & \longrightarrow & \mathcal{S} \\ & \{f_i(x)\} & \longmapsto & \{f'_i(x')\} = U\left(\{f_i(x)\}\right) \end{array} \tag{3.1.3}$$

Note though that we do allow the transformation U to be either a constant *or* to vary from space-time point to space-time point, $U = U(x)$. So it always the field at point x to a the old field value at the same point x (no space-time symmetries here) but the change may be different at each space time-point.

For our example of a frictionless ball in one dimension, the Lagrangian is $\mathcal{L} = (df/dt)^2$ which is indeed invariant under $f(t) \to f'(t) = -f(t)$ where the new field f' at each time is related to the old field at the same time. There is also a simple time-reversal symmetry where we relate a position $f(t)$ to a field $f(-t)$ but thats a change in the argument of the field and these are what we are not considering here. Also note that again that irreversible transformations such as $f \to 0f = 0$ can relate different solutions but they are not useful to us and are again excluded by the reversibility demand.

From this form, it is obvious from the Euler-Lagrange equations that the equation of motion for $\mathcal{L}[\{f_i\}]$ and $\mathcal{L}[\{f'_i\}]$ must contain the same physics, i.e. the set of all possible solutions, $\{\boldsymbol{F}\}$ and $\{\boldsymbol{F}'\}$ must be the same set $\mathcal{S}$ and the transformation, U must relate each of the solutions $\boldsymbol{F}$ to another unique (but not always different) solution. For the rolling ball and its symmetry under multiplication by -1, any moving solution is related to the solution moving with the opposite velocity, while the stationary solution is related to itself. In both cases, the transformation linked one solution to one and only one other solution[4]. What this means is that we can always undo the symmetry transformation and move from the complete set of solutions $\{\boldsymbol{F}\}$ to $\{\boldsymbol{F}'\} = \{\boldsymbol{F}\}$, the same set but shuffled. This reverse transformation is then also a symmetry transformation and is denoted by U^{-1}.

A final simplifying limitation we are going to make, but one rarely relaxed, is that we limit ourselves to **linear transformations** only. That is we will always consider transformations of the form

$$\boldsymbol{f}' = \mathsf{U}\boldsymbol{f}, \Leftrightarrow f'_i(x) = U_{ij}(x) f_j(x) \quad i,j = 1,\ldots d \tag{3.1.4}$$

Thus the maps U can always be represented in terms of matrices U and it makes sense now to think of the d fields as components of a d-component vector $\boldsymbol{f}$. We will try to use mid-latin alphabet letters, $i, j, \ldots$, to denote the components of such field vectors throughout this book, and U for the matrices representing these symmetries.

[3]The relevant symmetry groups are **non-compact** for space-time symmetries and **compact** for internal symmetries.

[4]The symmetry transformations are one to one and onto maps from the set of solutions $\mathcal{S}$ onto itself.

Using this matrix notation, we can check that internal symmetries do indeed give us the same equation of motion in the transformed fields. Suppose U is an internal symmetry for a Lagrangian of fields $f_i(x)$ then the equation of motion is given in (3.1.1) but it can also be written as

$$\partial_\mu \frac{\partial \mathcal{L}[\{f_i'\}]}{\partial\left(\partial_\mu f_i\right)} - \frac{\partial \mathcal{L}[\{f_i'\}]}{\partial f_i} = 0 \tag{3.1.5}$$

Using the fact that we have a linear transformation (3.1.4) and restricting to space-time constant transformations U we have that

$$\begin{aligned} 0 &= U_{ij}\partial_\mu \frac{\partial \mathcal{L}[\boldsymbol{f}']}{\partial\left(\partial_\mu f_i'\right)} - U_{ij}\frac{\partial \mathcal{L}[\boldsymbol{f}']}{\partial f_j'} && (3.1.6) \\ \Rightarrow \qquad 0 &= U_{ij}\left[\partial_\mu \frac{\partial \mathcal{L}[\boldsymbol{f}']}{\partial\left(\partial_\mu f_i'\right)} - \frac{\partial \mathcal{L}[\boldsymbol{f}']}{\partial f_j'}\right] && (3.1.7) \end{aligned}$$

Since this must be true for any symmetry transformation U then we see that the term in the square bracket must itself be zero. We do indeed have the same equation of motion of (3.1.1) and we have merely relabelled the fields. It is often useful to think of symmetry transformations as a relabelling of the fields, exactly in the same way that in many physics problems we can change our spatial coordinates from one orthogonal set of aces to another, say $(x, y, z) \to (x', y', z')$, without any problem. however, in the spatial coordinate case, changing to a set of rotating coordinates, is non-trivial, and in the same way, we can not simply allow our internal transformations to become space-time dependent. How to include this effect, and why it is relevant is the subject of chapter 6. Equally, changing from cartesian to spherical coordinates is a non-trivial and non-linear transformation and to do it one must use Jacobians etc. and in the context of field theories it is equally complicated. We will almost always restrict ourselves to the linear transformations encoded by matrices (3.1.4) but the use of the unitary gauge in chapter 7 is one point where we are breaking this limitation.

Symmetry transformations and the group axioms

The next step is to note that the set G of all possible invertible linear symmetry transformations[5] for any given theory, the set of invertible matrices $G = \{\mathsf{U}\}$, obey the four axioms required for them to form a faithful **representation** of a **group**. Let us derive these four axioms in our context of classical fields and Lagrangians, but also refer to the appendix B on group theory for a brief summary of groups.

Proposition 3.1
The set of all possible invertible matrices, $\{\mathsf{U}\}$, encoding the invertible linear symmetries of a classical field theory, form a representation group, G under matrix multiplication.

Its convenient to distinguish the different matrices of the group representation, $G = \{\mathsf{U}\}$, by labelling the g-th element as $\mathsf{U}(g)$. We drop any explicit reference to the space-time point in the transformation (3.1.4) as we are not making space-time transformations here, only field transformations[6]. We then find that the symmetry transformations satisfy the group axioms as follows:

Closure Consider two symmetries $\mathsf{U}(g_1), \mathsf{U}(g_2)$. We can transform any fields $\boldsymbol{f}$, into another set of fields $\boldsymbol{f}'$ such that the Lagrangian is invariant

$$\mathcal{L}[\boldsymbol{f}] = \mathcal{L}[\boldsymbol{f}' = \mathsf{U}(g_1)\boldsymbol{f}] \tag{3.1.8}$$

[5]We will tend not to add the adjectives invertible and linear from now on. They should be taken to be implicit.

[6]If we want to make a different field transformation at each space time point, we can make the label g a function $g(x)$. This does not effect our arguments. The labels are elements of a **faithful representation** of the group. See the appendix B for more details.

However if we have another symmetry $\mathsf{U}(g_2)$ we can also do this to the perfectly good $\boldsymbol{f}$ fields so

$$\mathcal{L}[\boldsymbol{f}'] = \mathcal{L}[\boldsymbol{f}'' = \mathsf{U}(g_2)\boldsymbol{f}'] \tag{3.1.9}$$

However combining these two results we see that

$$\mathcal{L}[\boldsymbol{f}] = \mathcal{L}[\boldsymbol{f}' = \mathsf{U}(g_1)\boldsymbol{f}] = \mathcal{L}[\boldsymbol{f}'' = \mathsf{U}(g_2)\mathsf{U}(g_1)\boldsymbol{f}] \tag{3.1.10}$$

This means though that we have a single map, $\mathsf{U}(g_2)\mathsf{U}(g_1)$, from $\boldsymbol{f}$ to $\boldsymbol{f}''$ which leaves the Lagrangian invariant. This map is therefore another symmetry, say $\mathsf{U}(g_3) = \mathsf{U}(g_2)\mathsf{U}(g_1)$ where the usual matrix multiplication is being used here. Hence the product of two matrices of symmetries must by definition produce another matrix of symmetry, i.e. closure

$$\mathsf{U}(g_3) := \mathsf{U}(g_2)\mathsf{U}(g_1) \in G \quad \forall \quad \mathsf{U}(g_2), \mathsf{U}(g_1) \in G \tag{3.1.11}$$

ssociativity Matrices under multiplication obey the associativity property, so the symmetry matrices of G automatically have this property

$$\mathsf{U}(g_3)\left(\mathsf{U}(g_2)\mathsf{U}(g_1)\right) = \left(\mathsf{U}(g_3)\mathsf{U}(g_2)\right)\mathsf{U}(g_1) \quad \forall\, \mathsf{U}(g_3), \mathsf{U}(g_2), \mathsf{U}(g_1) \in G \tag{3.1.12}$$

Identity The identity matrix $\mathbb{1}$ is a symmetry (if a trivial one) and so is always in G. It represents what is called the **identity element** of the group, usually labelled e. This element has the property that the transformation leaves the field unchanged:

$$\mathsf{U}(e) := \mathbb{1}, \quad \boldsymbol{f}' = \mathsf{U}(e)\boldsymbol{f} = \mathbb{1}\boldsymbol{f} = \boldsymbol{f} \;\Rightarrow \mathsf{U}(e) \in G \tag{3.1.13}$$

Inverse The inverse matrix $[\mathsf{U}(g)]^{-1}$ of any of our symmetry transformations $\mathsf{U}(g) \in G$ always exists as we demanded a reversible transformation for our symmetries. The inverse of the matrix is this reverse transformation. It is always a symmetry element in its own right, since U^{-1} takes us from fields $\{f'_i\}$ to fields $\{f_i\}$ which leave the Lagrangian invariant. Thus this reverse transformation U^{-1} is also in our set of all symmetries G

$$\boldsymbol{f}' = [\mathsf{U}(g)]^{-1}\boldsymbol{f} \;\Rightarrow [\mathsf{U}(g)]^{-1} \in G \tag{3.1.14}$$

These four properties of symmetry transformations of classical fields are the axioms of a group.

One very important thing to note is that the elements of a group do not in general **commute**, i.e. the order of the group elements is important, $\mathsf{U}(g_1)\mathsf{U}(g_2) \neq \mathsf{U}(g_2)\mathsf{U}(g_1)$ for at least some group elements in most groups. However there are special groups called **abelian groups** where this is true for all the elements of the group

Definition 3 A group is **abelian** if and only iff *all* elements commute

$$\mathsf{U}(g_1)\mathsf{U}(g_2) = \mathsf{U}(g_2)\mathsf{U}(g_1)\;\forall\;\mathsf{U}(g_2), \mathsf{U}(g_1) \in G \tag{3.1.15}$$

We will see later that there are important physical differences in models with abelian symmetries as compared to those containing non-abelian symmetries. For instance photons carry no electric charge because they are linked to an abelian symmetry. On the other hand gluons interact with themselves directly and so unlike the photon they do carry some type of charge (colour), and this is related to the properties of physics with non-abelian symmetry.

Let us look at this using the simplest examples of a group encountered in field theory is as follows

Example 4 Single real scalar field
$\phi(x) \in \mathbb{R}$ with a Lagrangian density of

$$\mathcal{L}[\Phi] = \frac{1}{2}\left(\partial^\mu \phi\right)\left(\partial_\mu \phi\right) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{3.1.16}$$

Note that while we want to eliminate linear terms in the field to interpret m as a mass parameter, there was nothing in principle to stop us including a $g\phi^3$ term is included. However, with only even powers of ϕ we have

$$\phi(x) \longmapsto \phi'(x) = -\phi(x) \qquad \Rightarrow \qquad \mathcal{L}[\phi'] = \mathcal{L}[\phi] \tag{3.1.17}$$

so $-1 \equiv \mathsf{U}(g)$ is a symmetry. Trivially we have also have $+1 \equiv \mathsf{U}(e)$ is another symmetry.

These two one-by-one matrices $\{+1, -1\}$ form what is called a faithful representation (see below) of a group of just two elements (see question 3.1), often denoted Z_2. In fact it is the only group of two elements, in the sense discussed in the appendix.

Also note that this symmetry is present only if there are even powers of ϕ in the Lagrangian. This is a clear indication that there is a strong limitation on the type of physical dynamics allowed mathematically or seen experimentally and the terms allowed in the Lagrangian used to describe the physics. □

Another important piece of terminology is as follows:

Definition 4 Two sets of matrices, $\{\mathsf{U}^{(1)}(g)\}$ and $\{\mathsf{U}^{(2)}(g)\}$, form two **representations** of the *same* group if they reproduce the same pattern of multiplication i.e.

$$\mathsf{U}^{(1)}(g_1)\mathsf{U}^{(1)}(g_2) = \mathsf{U}^{(1)}(g_3) \quad \Leftrightarrow \quad \mathsf{U}^{(2)}(g_1)\mathsf{U}^{(2)}(g_2) = \mathsf{U}^{(2)}(g_3) \quad \forall\, g_1, g_2 \in G \tag{3.1.18}$$

The real point for us here is that each group has infinitely many representations, and we will encounter several different representations of the same group, even in the same model. If you like there is some abstract group element, labelled by g_1, which we can encounter in real problems in many forms such as two different matrices $\mathsf{U}^{(1)}(g_1)$ and $\mathsf{U}^{(2)}(g_1)$. As long as these two matrices are different for at least one element, for at least one label g, then we have different representations. Because of this, we have been sloppy and switched in (3.1.18) to a definition of the set G as the set of possible labels rather than any one set of unitary matrices as we were doing above.

There are some important types of representation which we will encounter.

Definition 5 A **faithful representation** is one where every matrix is different

$$\mathsf{U}(g_1) \neq \mathsf{U}(g_2) \text{ if } g_1 \neq g_2, \quad \forall g_1, g_2 \in G \tag{3.1.19}$$

As this definition implies, there are representations where some of the matrices for different labels are the same:

Definition 6 A **unfaithful representation** is one where the same matrix element represents more than one group element

$$\exists g_1, g_2 \in G \;\; s.t. \quad \mathsf{U}(g_1) \neq \mathsf{U}(g_2) \tag{3.1.20}$$

We invariably define groups using some type of representation and they must be defined using a faithful representation[7]. Thus when we used a single field in example 4, we created a faithful representation.

Only when we have many fields do unfaithful representations have a role to play, and in our case there is only one which will appear

[7]Otherwise we would not have all the information on the whole structure of the group and then we could not use it as a definition

Definition 7 The **trivial** representation is one where all the matrices are simply the number 1

$$\mathsf{U}^{(\text{trivial})}(g) = 1 \quad \forall g \in G \tag{3.1.21}$$

It is possible to construct groups by combining other known groups. In particular

Definition 8 A representation of a **product group** $P = G \times H$ can be constructed as follows. Suppose that the $d_g \times d_g$ matrices G form a representation of G while $d_h \times d_h$ matrices H represent the group H. the group P is represented by the set of $(d_g d_h) \times (d_g d_h)$ matrices P, constructed for every combination of G and H matrices as follows

$$\mathsf{P} = \begin{pmatrix} G_{11}\mathsf{H} & G_{12}\mathsf{H} & \dots & G_{1d_g}\mathsf{H} \\ G_{21}\mathsf{H} & G_{22}\mathsf{H} & \dots & G_{1d_g}\mathsf{H} \\ \vdots & \vdots & \ddots & \vdots \\ G_{d_g1}\mathsf{H} & G_{d_g2}\mathsf{H} & \dots & G_{d_gd_g}\mathsf{H} \end{pmatrix} \tag{3.1.22}$$

where $G_{ij}\mathsf{H}$ indicates a $d_h \times d_h$ sized block equal to the matrix H with every entry multiplied by the same i, j-th element of G.

Thus there are as many matrices representing P as the product of the number of matrices needed for G and H, i.e. the dimension of P is $\dim P = \dim G \times \dim H$. The dimension of this representation of P is $(d_g d_h) \times (d_g d_h)$.

If a group can not be written as a product of two smaller groups it is called a **simple group**. All groups can be expressed as a product of simple groups.

One important use of such product groups is in reverse. That is many groups can be thought of as products of smaller groups. However one eventually can write a group as a product of groups. The properties of these product groups can be deduced from the properties of the smaller groups.

It is easy to spot product groups from the following property. All elements of a product group can be split into two parts belonging to subgroups of distinct elements except for the identity. That is any element$p \in P = G \times H$ can be written as $p = gh$ where $g \in G, h \in H, G \cap H = \{e\}$. In terms of our matrix elements we see that

$$\mathsf{P} = \mathsf{GH} \tag{3.1.23}$$

$$\mathsf{G} = \begin{pmatrix} G_{11}\mathbb{1} & G_{12}\mathbb{1} & \dots & G_{1d_g}\mathbb{1} \\ G_{21}\mathbb{1} & G_{22}\mathbb{1} & \dots & G_{1d_g}\mathbb{1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{d_g1}\mathbb{1} & G_{d_g2}\mathbb{1} & \dots & G_{d_gd_g}\mathbb{1} \end{pmatrix} \tag{3.1.24}$$

$$\mathsf{H} = \begin{pmatrix} 1\mathsf{H} & 0\mathsf{H} & \dots & 0\mathsf{H} \\ 0\mathsf{H} & 1\mathsf{H} & \dots & 0\mathsf{H} \\ \vdots & \vdots & \ddots & \vdots \\ 0\mathsf{H} & 0\mathsf{H} & \dots & 1\mathsf{H} \end{pmatrix} = \begin{pmatrix} \mathsf{H} & 0 & \dots & 0 \\ 0 & \mathsf{H} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathsf{H} \end{pmatrix} \tag{3.1.25}$$

We saw in example 4 a group of two elements but this is an example of just one of the two major types of group:

Definition 9 A group of a finite number of elements is called a **finite group**.
A group with an infinite number of elements is an **infinite group**

The group found above in example 4 has two elements so Z_2 is a finite group. Symmetries given by finite groups have interesting physical properties, e.g. defects in phase transitions leading to such phenomena as domain walls. The C, P and T transformations, when they are symmetries of space-time in a problem, are all discrete groups too. However we will not focus on these groups or their physical implications.

Not all the properties of finite groups are shared by infinite groups so these must be considered separately. We shall be focusing on one special type of infinite group, known as **continuous groups** or **Lie Groups**. These are groups of an infinite number of elements groups where the group elements, here labelled g_1, g_2 etc. so far, can be labelled by continuous parameters. For example, the simplest Lie group is defined by the set of phases

$$\exp\{i\theta\}, \theta \in \mathbb{R}, -\pi < \theta \leq \pi \tag{3.1.26}$$

called $U(1)$ (it is the group of one-by-one unitary matrices, see question 3.11 for details).

There is one final distinction to be made

Definition 10 If the matrices representing the Lie group elements are always finite, i.e. $|\mathsf{U}_{ij}(g)| < \infty \ \forall \ i,j = 1,2,\ldots,d, \ g \in G$, then the group is called a **compact** group. Otherwise the group is **non-compact**

The symmetries of space-time are often non-compact Lie groups, while we will focus on the symmetries of particles which are compact Lie groups.

Symmetry transformations and group irreducible representations

The compact Lie groups we are interested in for particle symmetries have a number of remarkable properties. We will state in our context without proof those we need. See the comments in the appendix B for references to more detailed texts.

Lemma 3.2

For compact groups, there exists am invertible matrix B which can create a **unitary representation**, $\{\bar{\mathsf{U}}(g)\}$ from any given representation $\{\mathsf{U}(g)\}$ as follows

$$\bar{\mathsf{U}} := \mathsf{B}\mathsf{U}\mathsf{B}^{-1} \quad \bar{\mathsf{U}}^\dagger(g)\bar{\mathsf{U}}(g) = \mathbb{1} \tag{3.1.27}$$

In general representations are not unitary, i.e. $\mathsf{U}^\dagger(g)\mathsf{U}(g)$ need not be the unit matrix, but this lemma tells us there is always a transformation which gives us another representation of the same group, one where the matrices are all unitary. It is convenient to use a unitary representation for several reasons but it implies that we must have taken new linear combinations $\bar{f}_i(x)$ of our original fields $f_j(x)$ where

$$\bar{\boldsymbol{f}} := \mathsf{B}\boldsymbol{f} \tag{3.1.28}$$

since the transformation (3.1.4) becomes

$$\boldsymbol{f}' := \mathsf{B}^{-1}\bar{\mathsf{U}}\mathsf{B}\boldsymbol{f} \ \Leftrightarrow \ \bar{\boldsymbol{f}}' := \bar{\mathsf{U}}\bar{\boldsymbol{f}} \tag{3.1.29}$$

From now on we will assume that we have made these transformations and our fields transform under unitary symmetry matrices.

For the next definition we first need a definition.

Definition 11 A **block diagonal** matrix is of the form

$$\mathsf{U}(g) = \begin{pmatrix} \mathsf{U}^{(1)}(g) & 0 & \cdots \\ 0 & \mathsf{U}^{(2)}(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \mathsf{U}^{(1)}(g) \oplus \mathsf{U}^{(2)}(g) \oplus \ldots \tag{3.1.30}$$

That is we divide the matrix into blocks given by the first d_1 rows, the next d_2 rows, etc, and the first d_1 columns, the next d_2 columns, etc. Each block along the diagonal is square, first $d_1 \times d_1$, the second $d_2 \times d_2$ and so on. These are the only non-zero blocks, all others contain only zeros. The $\oplus$ notation is a convenient shorthand. Pure diagonal matrices are special cases of block diagonal matrices where all the blocks are one-by-one, $d_1 = d_2 = \ldots = 1$.

This allows us to state the next lemma

Lemma 3.3
There are only two types of representation:
(A) An **irreducible** representation is one where there is no
(B) For **reducible** representations there is a transformation matrix B which can take any representation $\{\mathsf{U}(g)\}$ and gives a new *unitary representation* $\{\bar{\mathsf{U}}(g) := \mathsf{B}\mathsf{U}(g)\mathsf{B}^{-1}\}$ where *all* the matrices are block diagonal, with the smallest possible blocks. Further these blocks are always *irreducible* representations, though they need not all be distinct irreducible representations.

It is always the case that one matrix can be made pure diagonal, the most extreme form of block diagonal form, but the point of this lemma is that *all* the matrices of a reducible representation, for *all* labels g, can be made block diagonal (usually not pure diagonal) with the *same* matrix B. Further, once we know the irreducible representations, then we can construct *any* representation, or put another way, we can always express a representation in terms of the irreducible representations making up its block diagonal form. The group theory problem has been reduced to knowing the different irreducible representations of a given group. It turns out that the physics of a problem is reflected in both the symmetry group and in the choice of representations so to solve general problems one does have to know the machinery devoted to finding and using irreducible representations. However, in the rest of this book, we will be able to develop all the key ideas without requiring any more than an understanding of the simplest matrix algebra.

Fields, Particle Properties and Irreducible Representations

From the point of view of the fields, this change to a block diagonal representation $\{\bar{\mathsf{U}}(g)\}$ means we are switching to new linear combinations $\bar{f}_i(x)$ of our original fields $f_j(x)$ using (3.1.28) and just as before each set of field is equally good, has the same physical information encoded. However with the new fields $\bar{\boldsymbol{f}}$ the different components of these new fields $\bar{\boldsymbol{f}}$ are not completely mixed by the symmetry transformations $\bar{\mathsf{U}}$. The first d_1 components mixed only with each other through the matrices $\mathsf{U}^{(1)}$. The next d_2 components mix only with each themselves through the matrices $\mathsf{U}^{(2)}$ and so on.

$$\bar{\boldsymbol{f}}' = \bar{\mathsf{U}}\boldsymbol{f} \quad \Rightarrow \quad \bar{\boldsymbol{f}}^{(1)\prime} = \bar{\mathsf{U}}^{(1)}\bar{\boldsymbol{f}}^{(1)} \tag{3.1.31}$$

$$\bar{\boldsymbol{f}}^{(2)\prime} = \bar{\mathsf{U}}^{(2)}\bar{\boldsymbol{f}}^{(2)} \tag{3.1.32}$$

$$\tag{3.1.33}$$

etc. where

$$\bar{\boldsymbol{f}}^{(1)} := \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \bar{f}_{d_1} \end{pmatrix}, \qquad \bar{\boldsymbol{f}}^{(2)} := \begin{pmatrix} \bar{f}_{d_1+1} \\ \bar{f}_{d_1+2} \\ \vdots \\ \bar{f}_{d_1+d_2} \end{pmatrix} \tag{3.1.34}$$

and so forth. We say that each of these smaller vectors of fields, the $\bar{\boldsymbol{f}}^{(1)}, \bar{\boldsymbol{f}}^{(2)}, \ldots$ are **multiplets** and that each multiplet $\bar{\boldsymbol{f}}^{(P)}$ "lies in the irreducible representation $\mathsf{U}^{(P)}$". It is only the fields in these multiplets that are related by symmetry so only fields in multiplets can have similar properties as demanded by symmetry.

From now on, we will assume that we have always made the transformations to find these unitary block diagonal representations. We will not need to know the technology needed to find these transformations.

Thus we will always start with our fields of similar properties arranged in these small multiplets. Finding out what properties the particles of one multiplet have in common is the goal of the rest of the book. Typically it will be equal masses and simple relations between the interaction strengths. There will also be prescribed relationships between the conserved quantum numbers of the particles.

3.2 Global and Local Symmetries

A **global symmetry** is one where the fields are altered by the same amount at every space-time point: i.e.

$$f_j(x) \longmapsto f'_j(x) = U_{jk}(g) f_k(x) \qquad\qquad \forall x \quad j,k = 1,\ldots,d \tag{3.2.1}$$

where $\mathsf{U}(g)$ is the $d \times d$ matrix representing the abstract element g of the symmetry group G and it is independent of x. Thus, for a global theory the *same matrix is used for every point in space-time.* Equivalently, global symmetries satisfy

$$\partial_\mu \mathsf{U}(g) = 0 \quad \textbf{Global Symmetry} \tag{3.2.2}$$

We will study the consequences of such symmetries in the next couple of chapters.

In some circumstances though, we can find theories where we can change the fields by a different group element at every space-time point, i.e. we choose some $g(x), D_{ij}(x)$

$$\partial_\mu \mathsf{U}(g) \neq 0 \quad \textbf{Local Symmetry} \tag{3.2.3}$$

Such theories are more complicated as we will see that they require more fields to be present. However, these fields are the spin-one "gauge bosons", the force carrying particles of
QED, and the weak and strong nuclear forces. So such theories are common in the physical world. We will study them starting from chapter 6. Note that when we have a local symmetry, it includes as a special case a global symmetry. It makes sense therefore to restrict our studies first to the global symmetry case, as results apply to the local symmetry case with one major exception, namely in the local case identification of the physical particles becomes more complicated.

3.3 Questions

Q3.1. The finite group Z_2

Consider the set $\{a, b\}$ with a multiplication law where $ab = ba = a, a^2 = b$ and $b^2 = b$. Check that this set with this law forms satisfies all the axioms of a group. This means you must identify the identity and the inverse elements a^{-1}, b^{-1}, and they must be in the set. This is the group Z_2. What is the order of this group?

Q3.2. By contrast, consider the same set but with the multiplication rule $ab = ba = a, a^2 = b$ and $b^2 = a$. Show that this does not give a group.

Q3.3. The finite group Z_n

Consider two distinct elements e, and a which satisfy a multiplication rule such that $a^n = e$ and $eg = ge = g \;\forall\; g \in G$. Show that these rules **generate** a set G of only n distinct elements i.e. the set of distinct products of e and a in any number and in any combination contains only n distinct combinations.

Show that this set forms a group. It is known as Z_n or C_n and is called the **cyclic group**.

Q3.4. The finite groups of order four

There are two distinct groups of order four. One is Z_4 as defined in the previous question.

For the second consider three distinct elements e, a and b which satisfy a multiplication rule such that $a^2 = b^2 = e$, $ab = ba$ and $eg = ge = g \;\forall\; g \in G$. Show that these rules **generate** a set of only four distinct elements i.e. the set of distinct products of e, a, b in any number and in any combination contains only four distinct combinations.

Show that this set forms a group.

Show that this group is a product group and identify this group.

Q3.5. Verify that the matrices

$$\mathsf{D}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{D}(a) = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \tag{3.3.1}$$

form a representation of Z_2. Is this representation reducible or irreducible? If it is reducible determine the one-dimensional representations which form the direct sum of this representation.

Q3.6. A matrix representation $\{\mathsf{D}(g)\}$ $(g \in G)$ of the group G satisfies

$$\text{(abstract group)}\;\; c = a * b \quad\Rightarrow\quad \mathsf{D}(c) = \mathsf{D}(a).\mathsf{D}(b) \;\;\text{(matrix representation)}, \quad \forall\, a.b.c \in G. \tag{3.3.1}$$

Show that a set of matrices $\{\mathsf{D}'(g) = \mathsf{S}\mathsf{D}(g)\mathsf{S}^{-1}\}$, where S is any matrix with an inverse, is also a representation of the same group G.

Q3.7. Show that the matrices $\{\mathsf{D}''(g) := \mathsf{D}(g) \oplus \mathsf{D}'(g)\}$ $(g \in G)$ are a representation of a group G if they are formed by a direct sum of two representations of G, $\{\mathsf{D}(g)\}$ and $\{\mathsf{D}'(g)\}$.

Q3.8. The fundamental or defining representation of the group $O(2)$ is given by the set of orthogonal real two-by-two matrices $\{\mathsf{U}\}$ where

$$\mathsf{U}\mathsf{U}^T = \mathbb{1}, \quad U_{ij} \in \mathbb{R}, \quad \mathsf{U} \;:=\; \mathsf{Z}_\pm \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad -\pi \leq \theta < \pi, \tag{3.3.1}$$

$$\mathsf{Z}_+ \;:=\; \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}, \quad \mathsf{Z}_- := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3.2}$$

(i) Show these matrices form a group.

Show that this group is equivalent to a representation of the **product group** $Z_2 \times SO(2)$, i.e. every group element can be split into two mutually commuting parts, one part is a representation of Z_2 and the other a representation of elements of $SO(2)$. Show that the $SO(2)$ part is a one-dimensional compact Lie group.

(ii) Show that the generator of the Lie Algebra is

$$\mathsf{T} = \mathsf{T}^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{3.3.3}$$

Hint: Expand U up to $O(\theta)$ and compare with $\mathsf{U} = \exp\{i\epsilon^a \mathsf{T}^a\}$.

(iii) Show that this representation is *not* **reducible over the real numbers** but it *is* **reducible over the reals**. That is show that the unitary matrices that make the U block diagonal are complex, not pure real. In this case diagonalisation is equivalent to making U a diagonal matrix[8]. How is this related to the reality or complexity of the eigenvectors of U?

Q3.9. Consider the theory

$$\mathcal{L}_{O(2)} = \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - V(\frac{1}{2}\phi_i\phi_i) \tag{3.3.1}$$

where $V(x)$ can be any function[9].

Show that $\mathcal{L}_{O(2)}$ is invariant under a global $O(2)$ transformation, i.e. $\mathcal{L}_{O(2)}[\boldsymbol{\phi}] = \mathcal{L}_{O(2)}[\boldsymbol{\phi}']$ where $\boldsymbol{\phi}' = \mathsf{U}\boldsymbol{\phi}$ using U of (3.3.2).

Q3.10. The $O(N)$ Lie group and Lie Algebra

(i) Consider the number of independent real parameters needed to specify all elements of the group, $O(N)$, the group of real orthogonal $N \times N$ matrices. Prove that the dimension of $O(N)$ is $N(N-1)/2$.

(ii) Show that the determinant of such matrices is ± 1 and so deduce that the group $SO(N)$ also has dimension $N(N-1)/2$.

(iii) Show that $O(N)$ is a compact Lie group.

(iv) By thinking of simple explicit examples of allowable matrices $\mathsf{M} \in O(N)$, or otherwise, prove that the Lie groups $O(h)$ for $0 < h < N$ are all sub-groups of $O(N)$. You will need to prove they form a group, closure being crucial, as well as the fact that $O(h)$ is a subset of $O(N)$.

Q3.11. $U(1)$, the group of phase factors matrices

The simplest Lie group is U(1) and is defined by the set of phases

$$\exp\{i\theta\}, \theta \in \mathbb{R}, -\pi < \theta \leq \pi \tag{3.3.1}$$

Show this satisfies the four group axioms.

Q3.12. $U(N)$, the group of unitary matrices

[8] See my handout for undergraduate groups course on matrices if you need to revise properties of matrices.

[9] In four-space time dimensions for a well behaved renormalisable QFT, V can only be a quadratic polynomial at most since here $V(\phi^2)$ is then a quartic in the fields.

(i) Show that the set of all $d \times d$ unitary matrices, $\{\mathsf{U}\}$, forms a representation of a group. The group is called $U(N)$ and it is defined using this representation where the dimension of the representation $d = N$.

What is the dimension of this group?

(ii) Show that the determinant of U can be written as $\exp\{i\theta\}$ where θ is some real number.

(iii) Show that all $d \times d$ unitary matrices can be written as the product of two pieces, $\mathsf{U} = \mathsf{U}_Y.\mathsf{U}_S$ where U_S is a $d \times d$ unitary matrix with determinant one and U_Y is a product of the unit $d \times d$ matrix $\mathbb{1}$ and the determinant of U.

Show that the set of all U_S form a representation of a group. Is it an irreducible representation? They are the defining representation of the group $SU(N)$. What is the dimension of the group $SU(N)$?

(iv) Show the set of all possible $\{\mathsf{U}_Y\}$ also form a group and deduce it is a representation of $U(1)$. Is it irreducible?

(v) Show that $[\mathsf{U}_Y, \mathsf{U}_S] = 0$ always. Thus deduce that $U(N) \simeq U(1) \times SU(N)$.

(vi) Consider $N = d = 2$. Write down the generators of the $U(1)$ and $SU(2)$ parts of the group in this 2-dimensional representation (you need not derive the $SU(2)$ generators, look them up). What are their commutation relations?

Chapter 4

Global Symmetry and Conserved Charges

4.1 Noether's theorem and its consequences

Theorem 4.1 ***Noether's Theorem***

Any continuous global symmetry transformation implies the existence of a conserved current for every generator (T^a) of the associated Lie algebra.

We will only study this theorem in the context of internal particle symmetries of classical field theory. . The theorem works for any continuous symmetry transformation, including space-time symmetries but the derivation must be improved to cover these cases. It can also be generalised to QFT subject to one or two provisos which we will mention at the end of the chapter.

Proof

Consider[1]

$$\mathcal{L}[\{f_i(x)\}] \qquad f_i(x) \in \mathbb{R} \qquad i = 1, \ldots, d \tag{4.1.1}$$

The equations of motion are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu f_i)} - \frac{\partial \mathcal{L}}{\partial f_i} = 0 \tag{4.1.2}$$

Suppose

$$f_i(x) \longmapsto f_i'(x) = U_{ij} f_j(x) \tag{4.1.3}$$

is a continuous (global) symmetry, where U is finite-dimensional and unitary, i.e. $\mathsf{U}\mathsf{U}^\dagger = \mathbb{1}$. By a theorem of Lie groups, the set of all such U's is a representation of a compact Lie group, G. Thus, by another Lie group theorem, we can always write

$$\mathsf{U} = \exp\{i\varepsilon^a \mathsf{T}^a\} \qquad a = 1, \ldots, \dim(G) \tag{4.1.4}$$

where $\varepsilon^a \in \mathbb{R}$ and T^a are generators of Lie algebras.

Note: The T^a are always fixed, i.e. independent of x^μ, so, by (4.1.4), we conclude that a symmetry is global if and only if

$$\partial_\mu \varepsilon^a = 0 \tag{4.1.5}$$

For a small variation of $\boldsymbol{f}$, i.e. for $\boldsymbol{f} \mapsto \boldsymbol{f}' = \boldsymbol{f} + \delta \boldsymbol{f}$, the variation of $\mathcal{L}$ is

$$\delta L = \frac{\partial L}{\partial f_i} \cdot \delta f_i + \frac{\partial L}{\partial (\partial_\mu f_i)} \delta (\partial_\mu f_i) \tag{4.1.6}$$

[1] The case of complex fields is left as an exercise to the student.

A quick check shows here that $\delta\left(\partial_\mu \boldsymbol{f}\right) = \partial_\mu\left(\delta \boldsymbol{f}\right)$. Then for solutions to the equations of motion we can replace the variation of L with respect to the field with a term involving the variation of L with respect to the derivative of the field. This then gives

$$\begin{aligned} 0 &= \left(\partial_\mu \frac{\partial L}{\partial\left(\partial_\mu F_i\right)}\right)\delta F_i + \frac{\partial L}{\partial\left(\partial_\mu F_i\right)}\left(\partial_\mu \delta F_i\right) && (4.1.7)\\ &= \partial_\mu\left(\frac{\partial L}{\partial\left(\partial_\mu F_i\right)}\delta F_i\right) && (4.1.8)\end{aligned}$$

This we recognise of the form of a conserved four-current $\partial_\mu J^\mu = 0$.

If we now use (4.1.4) for the symmetry transformation we see that infinitesimal transformations in the field are of the form

$$\delta \boldsymbol{f} = \boldsymbol{f}' - \boldsymbol{f} = \left(\mathbb{1} + i\varepsilon^a \mathsf{T}^a\right)\boldsymbol{f} - \boldsymbol{f} + O\left(\varepsilon^2\right) \tag{4.1.9}$$

Thus, to first order,

$$\delta \boldsymbol{f} = i\varepsilon^a \mathsf{T}^a \boldsymbol{f} \tag{4.1.10}$$

Substituting into (4.1.8) we arrive at a form

$$i\varepsilon^a \partial_\mu J^{\mu a} = 0 \tag{4.1.11}$$

Given that the ε^a are arbitrary, we conclude that there are as many conserved currents as their are independent generators of continuous symmetries

$$\boxed{\partial_\mu J^{\mu a} = 0 \qquad\qquad J^{\mu a} = i\frac{\partial L}{\partial\left(\partial_\mu F_i\right)} T^a_{ij} F_j} \tag{4.1.12}$$

This is **Noether's theorem**.

□

Remarks

- The above equation tells us that the $J^{\mu a}$ are conserved currents with total charge

  $$Q^a = \int \mathrm{d}^3\boldsymbol{x}\; J^{0,a}(\boldsymbol{x}, t) \tag{4.1.13}$$

  where $\partial_t Q^a = 0$ by conservation.

- There are as many charges as the group g has dimensions, e.g. in EM, $J^\mu = (\rho, \boldsymbol{j})$, where ρ is the density, and $\boldsymbol{j}$ is the 3-current.

- **N.B.** Normalisation does not alter conservation, so if $\partial_\mu c = 0$ then $\overline{J}^{\mu a} = cJ^{\mu a}$ is also conserved, i.e. $\partial_\mu \overline{J}^{\mu a} = 0$. For most particle charges, e.g. strangeness, isospin, baryon number, we tend to work in integer units. This means we tend to work with $J^{\mu a}$ of (11.2.13) scaled to remove factors of $\sqrt{3}$ etc. that can appear in normalised generator definitions.

- In the eom, the current which appears, $\overline{J}^{\mu a}$, is scaled by a coupling constant, e.g. $\overline{J}^{\mu a} = eJ^{\mu a}$, so that $\overline{J}$ counts physical electromagnetic charge in EM and this appears in Maxwell's equations.

- We can also add a constant to our definition of the current, i.e. choose the zero charge state. In the quantum picture, this can be an infinite constant.

Quantum picture

Conserved charges in a quantum theory satisfy

$$\left[\hat{Q}^a, \hat{H}\right] = 0 \tag{4.1.14}$$

where $\widehat{H}$ is the Hamiltonian. (unless normalisation exists), where $\widehat{Q}^a$ is the same as Q_a, but quantised properly. We find that, in $U(1)$ symmetries,

$$\widehat{Q} \propto \int \mathrm{d}^3\boldsymbol{k}\ \left(\widehat{a}^\dagger_n \widehat{a}_n - \widehat{b}^\dagger_n \widehat{b}_n\right) \tag{4.1.15}$$

i.e. the number of particles less the number of anti-particles, or a similar form.

Lie Algebras

First note that the quickest way to check normalisation and orthogonality is to exploit these properties in the T^a given. Basically note that if $A, B \in \mathcal{A}$ then

$$\mathsf{A} = \sum_a \alpha_a \mathsf{T}^a, \quad \mathsf{B} = \sum_a \beta_a \mathsf{T}^a, \quad \Rightarrow \quad (\mathsf{A}|\mathsf{B}) \equiv \mathrm{Tr}\left\{\mathsf{AB}\right\} = \frac{1}{2}\sum_a \alpha_a \beta_a = \frac{1}{2}\boldsymbol{\alpha}.\boldsymbol{\beta} \tag{4.1.16}$$

The familiar notation on the RHS of a traditional dot or scalar product between two d dimensional vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)$ etc., reminds us that the Lie algebra elements, matrices like A, are indeed elements of a vector space.

4.2 Abelian charges

Let us look at the simplest scalar theory with a conserved charge.

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1\right)^2 + \frac{1}{2}\left(\partial_\mu \phi_2\right)^2 - V\left(\phi_1^2 + \phi_2^2\right) \tag{4.2.1}$$

where for the purposes of this proof, the potential V can be an arbitrary form[2]. This Lagrangian depends only on the 'lengths' $\phi_1^2 + \phi_2^2$. This suggests we define a two-dimensional real scalar field $\boldsymbol{\phi}$

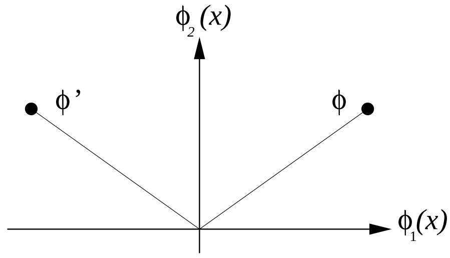
$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{4.2.2}$$

as we can then write the Lagrangian purely in terms of scalar dot products of this filed vector $\boldsymbol{\phi}$:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_i\right)\left(\partial^\mu \phi_i\right) - V(\boldsymbol{\phi}\cdot\boldsymbol{\phi}) \qquad\qquad i = 1, 2 \tag{4.2.3}$$

Clearly as rotation about the origin in the two dimensional internal space, that with field coordinates ϕ_1, ϕ_2, leaves the 'lengths' of the field vectors invariant, i.e. $|\boldsymbol{\phi}| = \sqrt{\phi_1\phi_1 + \phi_2\phi_2}$ is constant. $\mathcal{L}$ is clearly unchanged under such a transformation if its the same at all space-time points, i.e. global, as the derivative terms don't effect the issue.

[2]The typical form in four-dimensions would be $V(x) = \frac{1}{2}m^2 x + \frac{\lambda}{4!}x^4$.



The easiest way to see this is to define the transformed fields ϕ' through

$$\phi' = \mathsf{U}\phi, \qquad U_{ij} \in \mathbb{R} \qquad \mathsf{U}\mathsf{U}^{\mathsf{T}} = \mathbb{1} \qquad \partial_\mu \mathsf{U} = 0 \tag{4.2.4}$$

That is U is an orthogonal matrix which is a rotates in two-dimensions and which can also include a reflection. The properties given are necessary and sufficient to ensure that

$$\mathcal{L}\left[\phi'\right] = \mathcal{L}\left[\phi\right] \tag{4.2.5}$$

Thus we say this theory has an $O(2)$ symmetry as orthogonal 2×2 matrices form the two-dimensional representation of this group. While 2×2 matrices are just one representation of the group, it is the most obvious one, giving its name to the abstract group and because it is the **defining** or **fundamental** representation of $O(2)$. We also say the **fields lie in a two-dimensional representation**.

To extract the charges for this case, we either follow the derivation of Noether's current using this explicit form for the Lagrangian, or we can just use the formula (11.2.13). Top do the latter we need to know the generators and this is going to throw up a general problem when assigning definite charges to fields or particles.

4.2.1 Charge Eigenstates of O(2) model

We need to find the generators of this representation. The form of a general two-dimensional orthogonal matrix is

$$\mathsf{U} = \mathsf{Z}_\pm \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4.2.6}$$

where

$$-\pi < \theta \leq \pi \qquad \mathsf{Z}_+ = \mathbb{1} \qquad \mathsf{Z}_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.2.7}$$

Note that the group comes in two parts. The first Z factor is described by a discrete label. It distinguishes proper and improper rotations. It has no role to play in the conserved currents but it is not without physical significance. We defined Z by demanding that the second part is made of proper rotations, those with determinant +!. These form the subgroup $SO(2)$ and this is a simple Lie group.

To calculate the Noether current we need the generator T^a

$$\left.\frac{\mathrm{d}\mathsf{U}}{\mathrm{d}\epsilon^a}\right|_{\epsilon^b=0} = i\mathsf{T}^a \tag{4.2.8}$$

There is only one continuous parameter, $\theta = \varepsilon^a$, hence $a = 1$ and $\dim(G) = 1$. We find

$$\mathsf{T}^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.2.9}$$

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)} = \partial^\mu \phi_i \tag{4.2.10}$$

Hence we find the **Noether current for O(2) global scalar theory**:

$$\boxed{J^{\mu} = i\left(\partial^{\mu}\phi_i\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}_{ij}\phi_j = \phi_2 \boldsymbol{b}\partial^{\mu}\phi_1} \tag{4.2.11}$$

This is *not* a good form to identify fields/particles of definite charge – the **charge eigenstate**. For instance, the charge density J^0 is a mixture of terms, each a mixture of ϕ_1 and ϕ_2. If we were to substitute the classical fields with two free quantum fields

$$\hat{\phi}_j \sim \int \frac{\mathrm{d}^3\boldsymbol{k}}{2\omega_k}\left[\widehat{a}_j(k)e^{-ikx} + \widehat{a}_j^{\dagger}(k)e^{+ikx}\right] \qquad \left[\widehat{a}_j, \widehat{a}_l^{\dagger}\right] = \delta_{jl} \tag{4.2.12}$$

in (4.2.11) then we would find[3]

$$\widehat{Q} \sim \int \mathrm{d}^3\boldsymbol{x}\; J^0 \sim F\left(\widehat{a}_1^{\dagger}\widehat{a}_2, \widehat{a}_2^{\dagger}\widehat{a}_1\right) \tag{4.2.13}$$

This is *not* a function of the number operators, $\widehat{a}_j^{\dagger}\widehat{a}_j$, for either of the real scalar fields ϕ_1 and ϕ_2.

The problem is that the T^a here is not diagonal and this ensures that ϕ_1 and ϕ_2 fields are mixed in the current, preventing us from assigning to them definite charges. Since the generators T^a are always Hermitian matrices, $\mathsf{T}^a = (\mathsf{T}^a)^{\dagger}$ we can always find a matrix V which does this where

$$\mathsf{V}\mathsf{T}^a\mathsf{V}^{-1} = \mathsf{V}\mathsf{T}^a\mathsf{V}^{\dagger} = \overline{\mathsf{T}}^a = \mathrm{diag}\left(q_1, \ldots\right) \tag{4.2.14}$$

Returning to the current we see that

$$J^{\mu a} = i\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\boldsymbol{\phi}\right)}\right]^{\mathsf{T}}.\mathsf{T}^a.\boldsymbol{\phi} \tag{4.2.15}$$

$$= i\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\boldsymbol{\phi}\right)}\right]^{\mathsf{T}}.\left(\mathsf{V}^{-1}\overline{\mathsf{T}}^a\mathsf{V}\right).\boldsymbol{\phi} \tag{4.2.16}$$

This suggests that we need to work in terms of a new field vector $\overline{\boldsymbol{\phi}}$ where

$$\overline{\boldsymbol{\phi}} = \mathsf{V}\boldsymbol{\phi}, \quad \overline{\boldsymbol{\phi}}^{\dagger} = \boldsymbol{\phi}^{\mathsf{T}}\mathsf{V}^{\dagger} \tag{4.2.17}$$

Though the original fields are real so $\boldsymbol{\phi}^{\dagger} \equiv \boldsymbol{\phi}^{\mathsf{T}}$, V is unitary V ($\mathsf{V}^{-1} = \mathsf{V}^{\dagger}$) but in general complex. Thus the new field vector $\bar{\boldsymbol{\phi}}$ is also complex. Using this we have

$$J^{\mu a} = i\left[\left(\mathsf{V}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\boldsymbol{\phi}\right)}\right)^{\mathsf{T}}\mathsf{V}^{-1}\right]\overline{\mathsf{T}}^a\cdot\overline{\boldsymbol{\phi}} \tag{4.2.18}$$

– where we need the row vector in the partial derivative for the transpose.

To simplify matters, suppose that we have the usual simple kinetic term[4] so that

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi_i\right)} = \partial_{\mu}\phi_i \tag{4.2.19}$$

Since $\partial_{\mu}\mathsf{V} = 0$ to give

$$\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\boldsymbol{\phi}\right)}\right)^{\mathsf{T}}\mathsf{V}^{-1} = \partial_{\mu}(\mathsf{V}\boldsymbol{\phi})^{\dagger} \tag{4.2.20}$$

$$= \left(\partial_{\mu}\bar{\phi}^{\dagger}\right) \tag{4.2.21}$$

[3]See Question Sheet 1, question 5, equation (12).
[4]e.g. O(N) case on Handout 2, or O(2) case above.

Thus

$$J^{\mu a} = i\left(\partial_\mu \bar{\phi}\right)^\dagger \overline{\mathsf{T}}^a \bar{\phi} \tag{4.2.22}$$

so here $\boldsymbol{\phi}_1$ is *not* mixed with $\boldsymbol{\phi}_2$. In QFT

$$\widehat{Q}_0 \sim \int \frac{\mathrm{d}^3\boldsymbol{k}}{2\omega_k}\left(\widehat{A}_1^\dagger \widehat{A}_1 - \widehat{A}_2^\dagger \widehat{A}_2\right) \tag{4.2.23}$$

for $O(2)$, since there are only two fields, where we have

$$\widehat{\boldsymbol{\phi}}_j \sim \int \frac{\mathrm{d}^3\boldsymbol{k}}{2\omega_k}\left(\widehat{A}_j e^{-ikx} + \widehat{A}_j^\dagger e^{ikx}\right) \tag{4.2.24}$$

So, for a d-dimensional representation

$$J^{\mu a} = i\sum_{j=1}^{d} q_j\left(\partial_\mu \boldsymbol{\phi}_j\right)^\dagger \bar{\phi}_j \tag{4.2.25}$$

i.e. *there is no mixing.* Here the q_j are the eigenvalues of T^a, and are essentially the definite charges assigned to the j^{th} field.

To identify the charge eigenstates of charge Q^{0a}

(i) Diagonalise the generator: $\overline{\mathsf{T}}^a = \mathsf{V}\mathsf{T}^a\mathsf{V}^{-1}$.

(ii) Use U to choose new fields $\overline{\boldsymbol{\phi}} = \mathsf{V}\boldsymbol{\phi}$ so the $\bar{\phi}_j$ field components have definite charge q_j – the j^{th} eigenvalue of T^a where

$$(\mathsf{T}^a)_{ij} = q_j\delta_{ij} \tag{4.2.26}$$

Note that this works for non-Abelian symmetries where one can specify an normalisation for the generators without any problems. The different 'normalisations' allowed for the generators of abelian symmetries which correspond to the different physical charges and representations may require extra care . However the basic principles regarding diagonalisation of the generators apply in this case too.

The simplest and one of the most important examples is provided with the $O(2)$ scalar model, and its relationship with the $U(1)$ scalar field, i.e. real vs. complex representations of a simple complex scalar fields. Following the steps above we find that two real scalar fields with an $O(2)$ symmetry can be rewritten as follows. The generator of the symmetry of the real fields (11.2.6) can be diagonalised as follows

$$\overline{\mathsf{T}}^1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \mathsf{V}.\mathsf{T}^1\mathsf{V}^{-1} \tag{4.2.27}$$

$$\mathsf{V} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} \tag{4.2.28}$$

Note that the eigenvalues of the T^1 matrix are real but its eigenvectors are not. Thus we have to use complex combinations of real fields when working with this diagonalised basis, and the new vector $\overline{\boldsymbol{\phi}}$ is a complex doublet

$$\overline{\boldsymbol{\phi}} = \mathsf{V}\boldsymbol{\phi} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2\\ i\phi_1 + \phi_2\end{pmatrix} \tag{4.2.29}$$

Of course we can not start from two real fields and end up with two independent complex components of $\overline{\boldsymbol{\phi}}$ and indeed we see that $\overline{\boldsymbol{\phi}}_1 = (-i\overline{\boldsymbol{\phi}}_2)^*$. We could write our $O(2)$ Lagrangian (4.2.1) in this new complex doublet

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\overline{\boldsymbol{\phi}})^\dagger.(\partial_\mu\overline{\boldsymbol{\phi}}) - V(\overline{\boldsymbol{\phi}}^\dagger\overline{\boldsymbol{\phi}}) \tag{4.2.30}$$

noting that we have to upgrade the transposes of vectors to full hermitian conjugates. The Noether current would then be diagonal in this field basis

$$J^\mu \propto \left(\partial_\mu \overline{\phi}_1\right)^\dagger \overline{\phi}_1 - \left(\partial_\mu \overline{\phi}_2\right)^\dagger \overline{\phi}_2 + (\text{h.c.}) \tag{4.2.31}$$

This clearly shows that the particles represented by the first field component $\overline{\phi}_1$ have charge +1 relative to a -1 charge of particles of the second $\overline{\phi}_2$ field. These complex combinations are then called **charge eigenstates**. Generally we would expect to be able to measure particles of definite mass, mass eigenstates, and particles of definite charge. The fields which describe them must have no quadratic mixing terms in their Lagrangians for the first to be true, and no mixing terms in the Noether currents for the second. Thus while the real fields are mass eigenstates, they are not charge eigenstates so if we observes two scalar particles of equal mass and opposite charge, we should relate the physical particles directly to the two components of the complex doublet which are both mass and charge eigenstates[5].

However, while the general principle of diagonalising the generators and rotating the fields to find the mass and charge eigenstates, the normal physical observable fields, the simple $O(2)$ field is also illustrating another common idea. In this case we saw we had to take complex field combinations for the charge eigenstates. However, by exploiting the relationship between $\overline{\phi}^1$ and $\overline{\phi}^2$, we can represent everything in terms of a *single* complex field and its complex conjugate

$$\Phi := \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \qquad \Phi^\dagger := \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{4.2.32}$$

$$\Rightarrow \overline{\phi} = \begin{pmatrix} \Phi \\ i\Phi^\dagger \end{pmatrix} \tag{4.2.33}$$

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger.(\partial_\mu \Phi) - V(2\Phi^\dagger \Phi) \tag{4.2.34}$$

$$J^\mu \propto (\partial_\mu \Phi)^\dagger \Phi - (\partial_\mu \Phi)\Phi^\dagger \tag{4.2.35}$$

Note that the factors of 2 and $\sqrt{2}$ in the relationships between standard definitions of real and complex fields come from the demand that V was unitary. This in turn has led to factors of two difference between the standard mass and kinetic of standard real and complex field Lagrangians (4.2.1) and (4.2.30).

What has been illustrated for the simple $O(2)$ or $U(1)$ example is the idea that a group representation may not be reducible when it is limited to transforming real vectors, but it can become so if we alow it to act on complex fields. Note that the diagonal generator $\overline{\mathsf{T}}^1$ produces a diagonal two-by-two matrix for group elements, showing that in terms of the complex doublet $\overline{\phi}$, the doublet $O(2)$ representation is reducible. Of course we have no more and no less information in the single complex field than we had with two real fields we started from. In this case thought one representation displays the physics in a more obvious way, the complex Φ field and its conjugate are mass *and charge* eigenstates.

4.3 Charges in Non-Abelian groups

To find charge eigenstates, we must find *one* V, giving rise to one set of fields $\overline{\phi} = \mathsf{V}\phi$, which diagonalise the T^a. However, we cannot diagonalise all the T^a with one matrix V. The maximum number of T^a which can be diagonalised simultaneously is the **rank** of the group G, $r = r(G)$, and these diagonal generators form a **Cartan sub-algebra**:

$$\left\{\mathsf{C}^{\bar{a}}\right\} \subset \{\mathsf{T}^a\} \qquad a = 1, \ldots, \dim(G) \quad \bar{a} = 1, \ldots, r \tag{4.3.1}$$

[5]There are two important cases where despite the symmetry of the problem, the physical particles may not be simply be given by the mass and charge eigenstates. First is the case of symmetry breaking, to be discussed in chapter 5. The second is where we can not diagonalise both the mass matrix and the symmetry generators at the same time. This leads to particle oscillations such as are seen in the kaons and recently in the neutrinos. The meaning of 'particle' then becomes confused and can only really be tackled within QFT, see for example [?].

and

$$\left[\mathsf{C}^{\bar a}, \mathsf{C}^{\bar b}\right] = 0 \qquad \bar a, \bar b = 1, \ldots, r \tag{4.3.2}$$

These can be diagonalised simultaneously.

The $\left\{\mathsf{C}^{\bar a}\right\}$ generate a sub-algebra, and subgroup, of G. Let's assume we put the $\mathsf{C}^{\bar a}$ into diagonal form $\overline{\mathsf{C}}^{\bar a}$ using V as above:

$$\overline{\mathsf{C}}^{\bar a} = \mathsf{V}\mathsf{C}^{\bar a}\mathsf{V}^{-1} \qquad \bar a = 1, \ldots, r \tag{4.3.3}$$

We see that in a complex representation

$$\mathbf{\Phi} \longmapsto \mathbf{\Phi}' = e^{i\varepsilon^{\bar a}\overline{\mathsf{C}}^{\bar a}}\Phi = \begin{pmatrix} \left(e^{i\varepsilon^{\bar a}\overline{\mathsf{C}}^{\bar a}_{11}}\right)\Phi_1 \\ \left(e^{i\varepsilon^{\bar a}\overline{\mathsf{C}}^{\bar a}_{22}}\right)\Phi_2 \\ \vdots \end{pmatrix}$$

i.e. each component transforms as $e^{i\theta\overline{\mathsf{C}}^{\bar a}_{jj}}$ for a given $\bar a$ and $\overline{\mathsf{C}}^{\bar a}$, c.f. $\Phi \longmapsto e^{i\theta q}\Phi$. So, $\overline{\mathsf{C}}^{\bar a}_{jj}$ (which is *not* to be interpreted as a summation over the repeated index j), is the a^{th} charge of the j^{th} field and we say that the field, or particle, Φ_j is a charge eigenstate, with charges $\overline{\mathsf{C}}^{1}_{jj}$ and $\overline{\mathsf{C}}^{2}_{jj}$ (again, noting that the double j is a label, and not an implicit summation).

Example 5 $U(3) \cong U(1) \times SU(3)$ flavour symmetry: $u \leftrightarrow d \leftrightarrow s$ quark (approximate) symmetry.

An approximate global symmetry between the three lightest quarks (of QCD) is a global U(1) and a global SU(3). That is if the three lightest quark flavours (up, down, strange) were massless (they are approximately) and if you ignored the other (weaker) interactions, they have the same properties under the strong forces. Put more simply, for massless QCD with no other forces present, there are no processes which change the flavours of the quarks. Thus I would expect the number of up, down and strange quarks to be conserved separately.

Thinking of the u, d and s quarks as scalars,

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_u \\ \Phi_d \\ \Phi_s \end{pmatrix} \tag{4.3.4}$$

we would be working with a theory of the type

$$\mathcal{L} = \mathbf{\Phi}^\dagger\partial^2\mathbf{\Phi} - V(|\mathbf{\Phi}|) \tag{4.3.5}$$

The internal symmetries are unchanged by writing this using fermion fields (bosonic/fermionic nature of particles is linked to the space-time symmetry).

Let us look at this in the language of generators of the global symmetry. The maximum number of generators which can be diagonalised at any one time (the dimension of the Cartan subalgebra) is the number of conserved charges that EVERY particle carries. U(1) has one generator and this can always be chosen diagonal (it commutes with everything so its always part of the CSA). For reasons to be mentioned below its chosen to be $\overline{\mathsf{C}}^{\bar a=1}$

$$\frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathsf{T}^0 \tag{4.3.6}$$

The U(1) symmetry is just baryon number and is an exact symmetry as far as we know. Each quark has baryon number 1/3 rather than 1 because we found quarks after Baryons!

$SU(3)$ has dimension 8 and rank 2 so there are then two approximately conserved numbers for the SU(3) symmetry. Traditionally for the three-dimensional representation (we have three quarks) we use the **Gell-Mann matrices** in which the diagonal matrices are chosen to be:

$$\overline{\mathsf{C}}^{\bar{a}=1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} = \mathsf{T}^3 \qquad \overline{\mathsf{C}}^{\bar{a}=2} = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix} = \mathsf{T}^8 \tag{4.3.7}$$

T^3 gives you **isospin** component of the quarks, which is proportional to the number of up quarks minus the number of down quarks. T^8 is then proportional to the number of up plus down minus twice the number of strange. This is an odd combination which has no special name. We thus have

Particle	$3T^0$	$\sqrt{2}\mathsf{T}^3$	$2\sqrt{3}\mathsf{T}^8$
u	+1	+1	+1
d	+1	-1	+1
s	+1	0	-2

In this basis these three conserved numbers seem a bit strange. However any linear combination of the conserved numbers will also be conserved. It corresponds to working in a different basis in the Lie Group/Algebra. Then you can quickly show that the three conserved numbers are equivalent to conserving the number of each quark flavour individually, i.e. we can work with a basis in which we choose

$$T^u := \frac{3}{2}T^0 + \frac{1}{\sqrt{2}}\mathsf{T}^3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{4.3.8}$$

$$T^d := \frac{3}{2}T^0 - \frac{1}{\sqrt{2}}\mathsf{T}^3 = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{4.3.9}$$

$$T^s := \frac{3}{2}T^0 - \frac{\sqrt{3}}{\sqrt{2}}\mathsf{T}^8 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{4.3.10}$$

Thus when we talk about an up quark we are rally also saying it has plus one up charge under the T^u generator but also zero charge under the T^d and T^s generators and that we are discussing this in a basis where the diagonal generators were chosen to be T^u, T^d, T^s. That a strong force theory which approximately conserves quark flavours[6] can be represented by an $U(3)$ globally symmetric theory of the type (4.3.5). We can choose to talk about quark flavours, using the T^u, T^d, T^s basis. Thus an up quark would have charges $u = 1, d = s = 0$. One the other hand the standard Gell-Mann basis the up quark would be characterised by a baryon number 1/3, isospin (T^3) +1/2 and a T^8 charge of $1/[2\sqrt{(3)}]$.

In fact one would more normally encounter a third basis of Baryon number Isospin and Strangeness, T^0, T^3, T^s and this is what is given in the particle data book. Why? Historical reasons. Its much less sensible than the other two bases given here!

[6] This is appropriate for massless quarks in part because we can ignore mass differences between the quarks if we are looking at higher mass scale problems i.e. the quark masses and hence their differences are small. In practice the global flavour symmetry discussed here is broken in part by the generation of mass for the quarks using Electroweak Higgs physics. That mixes flavours, and allows quark flavour not to be conserved. As this is a weaker force, in some circumstances this is a good approximation. However the fact it is approximate does indicate that the idea that there are quantum states which are pure up with (approximate) charges u=1,d=0,s=0, and which is what we mean by an up quark, is fundamentally flawed. Such states do not really exist exactly but they are a very useful and reasonable approximation in many circumstances.

Overall however we look at it though, there are three observable charges and six irrelevant Noether currents (that is, we cannot see them).

□

Trivial Representation

$$\mathsf{U} = \mathbb{1} \qquad \Leftrightarrow \qquad \mathsf{T}^a = 0 \qquad \Leftrightarrow \qquad \boldsymbol{\Phi} = \boldsymbol{\Phi}' \tag{4.3.11}$$

(This makes sense only if the other fields in $\mathcal{L}$ change non-trivially.) Thus

$$\boxed{\mathsf{U} = \mathbb{1} \qquad \Leftrightarrow \qquad \text{particles/fields } \Phi \text{ have zero charge.}} \tag{4.3.12}$$

Abelian Symmetry $U(1)$

Here we encounter problems because the generators cannot be normalised relative to one another. There is nothing like $\mathrm{Tr}\left\{T^a T^b\right\} = \frac{1}{2}\delta^{ab}$ since, if we impose this on $\Phi \longmapsto e^{i\theta T}\Phi$ and choose $T = \frac{1}{\sqrt{2}}$, we would miss the possibility that another field was invariant under $\boldsymbol{\eta} \longmapsto e^{2i\theta T}\boldsymbol{\eta}$.

See examples in questions 1 and 2. The interactions force the relationship between phase factors. So here, $T_\phi = \frac{1}{\sqrt{2}}$, $T_\eta = 2 > \frac{1}{\sqrt{2}}$, and we have charges $\phi = +1$ and $\eta = +2$, under this $U(1)$ symmetry.

- We can only get extra factors of 2 with Abelian.
- Non-Abelian sets diagonal.
- Only ask free fields questions.

Chapter 5

Breaking Global Symmetry

If a theory has a continuous symmetry, then we see that this is related to certain patterns in particle properties, e.g. equal masses, coupling constants, patterns in charges. However, is the link between symmetry of a Lagrangian density $\mathcal{L}$ and particle properties always so simple, always true?

The answer is no! For example, the hydrogen atom in QM shows approximate rotational symmetry. The ground state and all other s-wave ($l = 0$) solutions are rotationally symmetric. However, most excited states, such as the $l = 1$ p-wave solutions are not. One might speculate that the lowest energy states always show full symmetry. However this statement is not generally true and this is the subject of symmetry breaking in QFT. A counter-example to this second statement is a ferromagnet.

Example 6 Symmetry breaking and ferromagnetism.

Consider a lattice in space, each site labelled by n, each with a spin which can point in any direction. These spins are given by a d-dimensional vector $\boldsymbol{S}(x_n)$ at each site x_n. The QMHamiltonian is

$$\widehat{H} = \sum_{\text{directions } \mu} \sum_{\text{sites } n} \sum_{i=1} d\widehat{S}_i(x_n) \cdot \widehat{S}_i(x_{n,\mu}) \tag{5.0.1}$$

where $x_{n,\mu}$ is the position of the site next to site n in the μ'th direction. If we have one-dimension and $|S| = 1/2$ we have the **Ising model**. If we rotate the vector of real spins by the same amount at every lattice site n, which is done using a d-dimensional orthogonal matrix U, then we see this is a symmetry

$$\widehat{H}[\widehat{\boldsymbol{S}}] = \widehat{H}[\mathsf{U}.\widehat{\boldsymbol{S}}] \qquad\qquad \mathsf{U} \in O(d) \tag{5.0.2}$$

In terms of the physics there are two regimes:

- $T > T_C$, at high temperatures there is no net magnetisation as all spins are randomly aligned as there is no special direction for the spins picked out by the Hamiltonian $\widehat{H}$. Equivalently, all directions are equivalent in the problem, we have $O(d)$ rotational symmetry;

- $T < T_C$, all spins line up to give a net magnetisation.

Thus at temperatures below a critical temperature T_C, the system settles into a state, and so the lowest energy state, which picks a direction. Yet all directions in the d-dimensional spin space are equal hence the $O(d)$ symmetry in the theory. □

This is an example of **symmetry breaking** — where a ground state does not show the full symmetry seen in the dynamics, as specified by a Lagrangian $\mathcal{L}$ or Hamiltonian $\mathcal{H}$.[1]

There are three types of symmetry breaking.

[1] In QFTinitial developments assume that like the high temperature Ising model $\left\langle 0 \middle| \widehat{\phi} \middle| 0 \right\rangle = 0 = \boldsymbol{\phi}_0$.

(i) *Explicit symmetry breaking*

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_{\text{small}} \tag{5.0.3}$$

$\mathcal{L}_1$ has a big symmetry by itself but $\mathcal{L}_{\text{small}}$ reduces this to a smaller symmetry (e.g. fewer generators) but this part has *small* coefficients. Thus the dynamics tends to show the consequences of the big symmetry with some small deviations due to the $\mathcal{L}_{\text{small}}$. This is only a useful approximation scheme. E.g. u, d quarks have nearly the same mass so

$$\mathcal{L}_{\text{small}} = \Delta m(\bar{u}u - \bar{d}d) \qquad \mathcal{L}_1 = m(\bar{u}u + \bar{d}d) = \boldsymbol{\psi}^\dagger \boldsymbol{\psi}, \qquad \boldsymbol{\psi} = \begin{pmatrix} u \\ d \end{pmatrix} \tag{5.0.4}$$

where $u = u^\alpha(x)$, $d = d^\alpha(x)$, are space-time fermions (i.e. they are fermionic fields similar to those which satisfy Dirac's equation, see $\psi^\alpha(x)$ in chapter 8.). The $\mathcal{L}_1$ has a $U(2) \sim U(1) \times SU(2)$ flavour symmetry in strong interactions. For instance the three pions are bound states of different combinations of u and d quarks and they all have approximately the same mass. However, $\mathcal{L}_{\text{small}}$ reduces the exact symmetry to the smaller $U(1) \times U(1)$ symmetry, under which each fermion field is transformed by a phase independent of the phase of the other field. With this term, we'd expect small deviations from the predictions made using the big $U(2)$ symmetry of order $\Delta m/m$. This small mass differences between the pions will in part be of this order.

(ii) *Dynamical symmetry breaking*

Here non-perturbative effects break symmetry seen in the Lagrangian $\mathcal{L}$; e.g. chiral symmetry breaking in simple QCD models such as the Nambu-Jona-Lasinio model.

(iii) *Spontaneous symmetry breaking (SSB)*

The shape of the classical potential terms suggests the ground state is not zero and is not invariant under all symmetries of $\mathcal{L}$.

5.1 Spontaneous symmetry breaking

How do we find the classical ground state — the lowest energy state? Let us work with a simple field theory of a d-dimensional vector of scalar fields ϕ_i $i = 1, \ldots, d$ where the Lagrangian is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_i)^2 - V(\phi_i) \tag{5.1.1}$$

First we have to find what field configurations minimise the kinetic energy terms, in $|\partial_\mu \phi_i(x)|^2$, in $\mathcal{L}$. However, $\mathcal{L}$ is not total energy but the difference between the kinetic and potential energy. Thus, the lowest energy state is best considered using the Hamiltonian, which is the total energy, i.e. (here $\pi_i = \dot{\phi}_i$),

$$H = (\pi_i)^2 + (\nabla \phi_i)^2 + V(\phi_i) \tag{5.1.2}$$

The derivative terms — the kinetic terms — are positive semi-definite in the Hamiltonian. Hence the lowest possible energy in the classical theory comes from a solution where the field is space-time independent, i.e. we demand that the ground state solution, $\boldsymbol{\phi}_0$, is fixed by

$$\partial_\mu \boldsymbol{\phi}_0 = 0 \tag{5.1.3}$$

This consideration applies to all fields[2].

[2] Of course most materials at zero temperature are found crystal form. Such a lowest energy solution is not invariant under space-time does not respect.

Having found the minimum of the kinetic terms, all that remains is to find the minimum of the potential terms $V(\phi_i)$ i.e.

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} = 0 \tag{5.1.4}$$

For definiteness let us consider a specific model with a global $O(d)$ symmetry, i.e. where the potential is a function of $|\boldsymbol{\phi}|$ only,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - V(|\boldsymbol{\phi}|) \qquad\qquad i = 1, \ldots, d \tag{5.1.5}$$

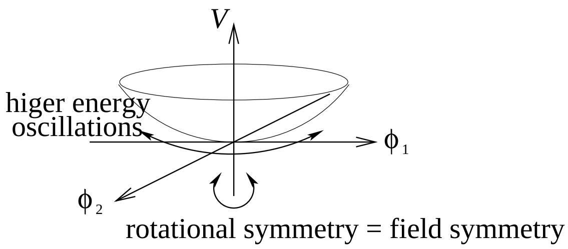
For instance a typical choice would be

$$V(x) = \frac{1}{2}m^2 x^2 + \lambda x^4 \tag{5.1.6}$$

There are then two cases to consider[3], a) $m^2 > 0$ and b) $m^2 < 0$

(a) $m^2 > 0$**,** $\lambda > 0$

In this case m is simply the mass of the d scalar particles. More generally the mass is related to the curvature of the potential near the vacuum solution i.e. the second derivative $\partial^2 V/\partial\phi_i\phi_j|_{\boldsymbol{\phi}_0}$. For the case of $d = 2$ we observe that the potential is approximately quadratic near the origin but for large $|\boldsymbol{\phi}|$ the $|\boldsymbol{\phi}|^4$ interaction dominates



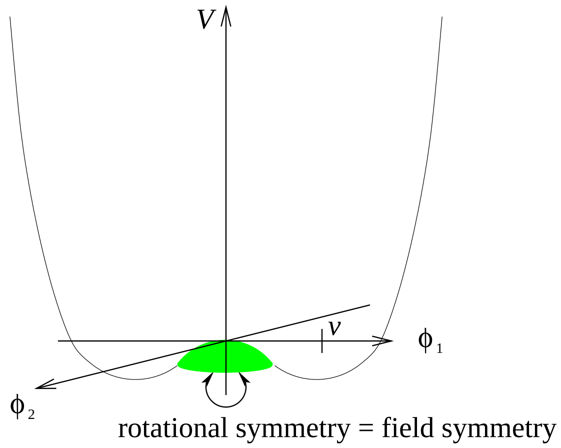
In general, we note

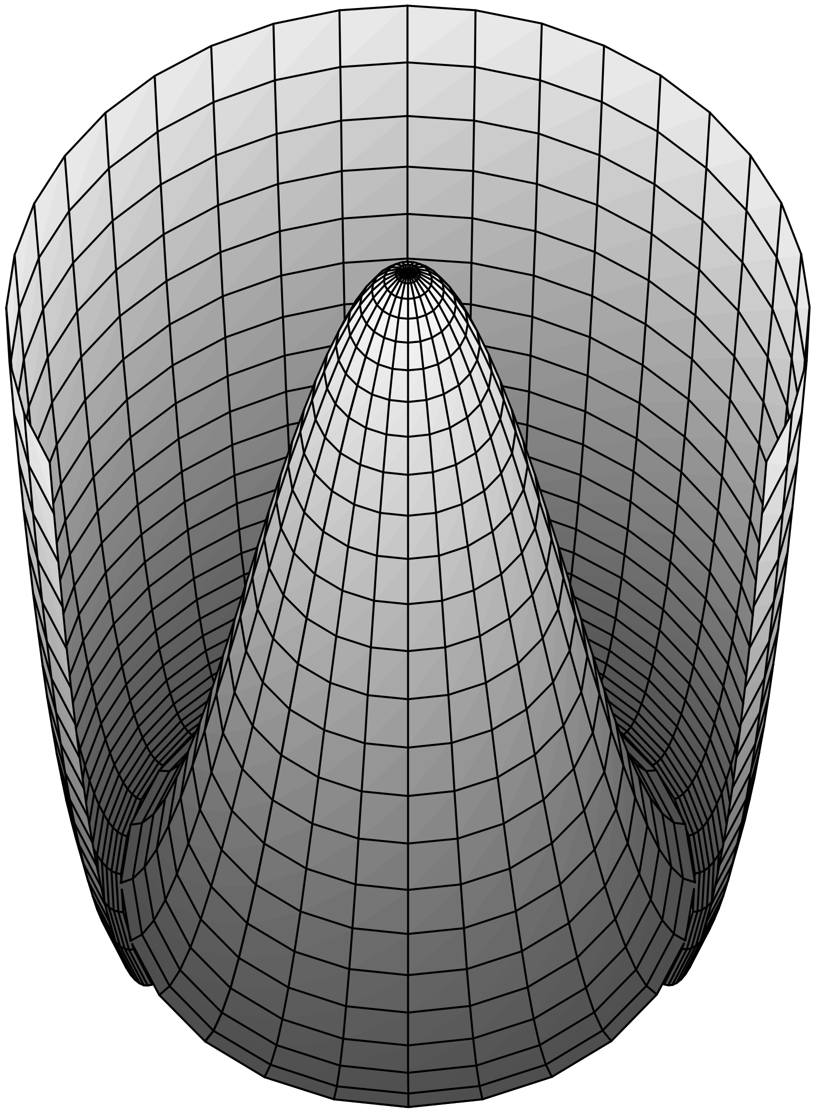
- the system exhibits rotational symmetry, which is equivalent to $O(d)$ field symmetry of $\mathcal{L}$.
- the ground state is $\phi_0 = 0$.

(b) $m^2 < 0$**,** $\lambda > 0$

In the $d = 2$ case, we note that, since $|\boldsymbol{\phi}|^4 \ll |\boldsymbol{\phi}|^2$ near the origin, then the potential has negative curvature in a neighbourhood about the origin, but turns up again for large $|\boldsymbol{\phi}|$ where the $|\boldsymbol{\phi}|^4$ interaction dominates.

[3] $\lambda > 0$ is necessary or the theory will not be stable.


V
v
ϕ1
ϕ2
rotational symmetry = field symmetry



Generalising to dimension d,

- $\boldsymbol{\phi}_0 = \mathbf{0}$ is now a maximum and an *unstable solution* which is related to the fact that $m^2 < 0$, i.e.

$$\sqrt{m^2} \in \mathrm{Im} \tag{5.1.7}$$

  m is imaginary and thus cannot be interpreted as the mass of a stable particle.
  In fact, in general, imaginary mass terms imply that we are we looking at either unstable solutions or perturbations about such solutions. since $\exp\{\pm i\omega t\} = \exp\{\pm\kappa t\}$ where $\kappa = \sqrt{-m^2 - \boldsymbol{k}^2} \in \mathbb{R}$ for $|\boldsymbol{k}| \ll |m|$.

- The minimum of V is then at

$$0 = \frac{\partial V}{\partial \phi_i} = m^2\phi_i + 4\lambda\phi_i\,|\boldsymbol{\phi}|^2 \tag{5.1.8}$$

  which gives either the previous solution, $\phi_i = 0$, a local maximum now, or a true minimum at

$$|\boldsymbol{\phi}|^2 = -\frac{m^2}{4\lambda} \tag{5.1.9}$$

  i.e.

$$v = |\boldsymbol{\phi}| = \sqrt{-\frac{m^2}{4\lambda}} \in \mathbb{R} \tag{5.1.10}$$

  Since it is only the modulus that has been constrained, we observe that

$$\boxed{\boldsymbol{\phi}_{\mathrm{min}} = \mathsf{U}\boldsymbol{e}_0 v} \tag{5.1.11}$$

  are all possible minimum energy $\boldsymbol{\phi}$ values, where $\boldsymbol{e}_0$ is any unit vector. That is, we can choose $\boldsymbol{e}_0 = (1, 0, \ldots)^{\mathsf{T}}$. Here, U is an $O(d)$ symmetry matrix, where $\mathsf{U}\mathsf{U}^{\mathsf{T}} = \mathbb{1}$.

- **Remark** The rotational symmetry is simply the rotations of in the d-dimensional internal space of the fields, an $O(d)$ symmetry. Furthermore, $m^2 < 0 \Rightarrow \phi = 0$ is unstable to small fluctuations, while for small fluctuations around $\boldsymbol{\phi}_{\mathrm{min}} = \mathsf{U}\boldsymbol{\phi}_0$, $\boldsymbol{\phi}_0 = \mathsf{U}\boldsymbol{e}_0$ is stable.

- We can see that there are many equally good vacua, all related by symmetric transformations U. So, if ϕ_0 is a vacuum, then $\mathsf{U}\boldsymbol{\phi}_0$ is also a good vacuum where

$$\mathcal{L}[\mathsf{U}\boldsymbol{\phi}_0] = \mathcal{L}[\boldsymbol{\phi}_0] \tag{5.1.12}$$

  In general, $\mathsf{U}\boldsymbol{\phi}_0 \neq \boldsymbol{\phi}_0$, for some $\mathsf{U} \in G$. In fact for the $O(d)$ case only the rotations about the axis through $\boldsymbol{e} = \mathsf{U}\boldsymbol{e}_0$ leave $\boldsymbol{\phi}_0$ invariant.

- An actual vacuum state in a real physical realisation of this theory picks *one* $\boldsymbol{\phi}_{\mathrm{min}}$ since $\dot{\boldsymbol{\phi}}_{\mathrm{min}} = \partial_i\boldsymbol{\phi} = 0$. We cannot choose a different ϕ_{min} at different $\boldsymbol{x}$ or t. (As temperature decreases with increasing time in the early universe, particles cannot all "line up" with each other.)

- Note here that the fluctuations about any vacuum satisfy

  a) Considering the radial components (i.e. $\phi = \phi_{\mathrm{r}} \equiv \phi_{\mathrm{radial}}$, $\mathcal{L} = \frac{1}{2}(\phi_{\mathrm{r}} - \phi_{\mathrm{min}})^2 + (\ldots)$):

$$\left.\frac{\partial^2 V}{\partial \phi_{\mathrm{r}}^2}\right|_{\phi=\phi_{\mathrm{min}}} > 0 \tag{5.1.13}$$

  implying a positive mass.

b) Angular fluctuation ($\phi \equiv \phi_\theta$)

$$\left.\frac{\partial^2 V}{\partial \phi_\theta^2}\right|_{\phi=\phi_{\min}} = 0 \tag{5.1.14}$$

if and only if the mass is zero.
The zero mass of fluctuations in the directions of symmetry transformations

$$\boldsymbol{\phi}_{\min} \longrightarrow \mathsf{U}\boldsymbol{\phi}_{\min} \simeq \boldsymbol{\phi}_{\min} + \delta\boldsymbol{\phi}_{\min} \tag{5.1.15}$$

must be the only flat direction in potential energy. Because of the continuous nature of the symmetry, this occurs if and only if the mass is zero.

The Vacuum Expectation Value in QFT

Finally, let us note on how these vacuum solutions appear in QFT. Our solutions $\boldsymbol{\phi}_{\min}$ appear as the **vacuum expectation energy**, **vev**, namely

$$\left\langle 0 \left| \widehat{\boldsymbol{\phi}} \right| 0 \right\rangle = \boldsymbol{\phi}_{\min} + O(\hbar) \tag{5.1.16}$$

Another way to view these vev's and a useful point of contact with condensed matter physics comes when we look at the nature of the vacuum solution in QFT. We have that $|\left\langle 0_B \left| \hat{\boldsymbol{\phi}} \right| 0_B \right\rangle| = v$ which tells us that the true vacuum $|0_B\rangle$ is full of charged $\boldsymbol{\phi}$ particles and anti-particles. For instance, try calculating the vacuum expectation value of a free scalar field (real or complex) using the vacuum annihilated by the annihilation operators used to construct the free fields and the vev is trivially zero. Thus the empty quantum vacuum state in the SSB case can not this empty vacuum state but has to be one full of particles so that the annihilation operators do not destroy them. Such a state has a non-zero vev even for free fields but we will not construct it here. Thus a vacuum for a theory with SSB is full of virtual fluctuations but it is now also contains a **condensate** of real physical particles. For instance **Bose-Einstein condensation** can be described using the language of SSB in QFT.

5.2 Goldstone's Theorem

To study SSB mathematically, we first realise that we wish to quantise small field fluctuations about the lowest energy solution.
Remark In QFT, we *assumed* without comment that the classical vacuum state satisfied

$$\phi_{\min} = \left\langle 0 \left| \hat{\phi} \right| 0 \right\rangle = 0 \tag{5.2.1}$$

We then see for the first time the key trick of QFT which is to study classical fluctuations about one vacuum solution and later quantise these fluctuations.

Consider the theory of $\phi_i(x) \in \mathbb{R}$, $i = 1, \ldots, d$, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_i\right)^2 - V(\phi_i) \tag{5.2.2}$$

and symmetry group $G = \{\mathsf{U}\}$ such that

$$V(\mathsf{U}\boldsymbol{\phi}) = V(\boldsymbol{\phi}) \qquad \mathsf{U}\mathsf{U}^{\mathsf{T}} = \mathbb{1} \qquad U_i \in \mathbb{R} \tag{5.2.3}$$

Consider a solitary vacuum solution $\boldsymbol{\phi}_0$, $\partial_\mu \boldsymbol{\phi}_0 = 0$. Performing a Taylor expansion of $V(\boldsymbol{\phi})$ about $\boldsymbol{\phi}_0$, i.e. studying the new field

$$\boldsymbol{\eta}(x) = \boldsymbol{\phi}(x) - \boldsymbol{\phi}_0 \tag{5.2.4}$$

yields

$$V(\boldsymbol{\phi}) \quad = \quad V(\boldsymbol{\phi}_0) + (\boldsymbol{\phi} - \boldsymbol{\phi}_0) \left.\frac{\partial V}{\partial \boldsymbol{\phi}}\right|_{\boldsymbol{\phi}_0} + \frac{1}{2}(\phi_i - \phi_{0i})(\phi_j - \phi_{0j}) \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} + O(\boldsymbol{\eta}^3) \tag{5.2.5}$$

where, since $\boldsymbol{\phi}_0$ is a vacuum, or minimum,

$$\left.\frac{\partial V}{\partial \boldsymbol{\phi}}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} = 0 \tag{5.2.6}$$

so that

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta_i)^2 - V(\boldsymbol{\phi}_0) - \frac{1}{2}\eta_i \left(\mathsf{M}^2\right)_{ij} \eta_j + O(\eta^3) \tag{5.2.7}$$

where

$$\left(\mathsf{M}^2\right)_{ij} = \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} = (\text{Mass})^2 \text{ matrix} \tag{5.2.8}$$

and the eigenvalues of M^2 tells us about masses of particles.

Symmetry then tells us that V is invariant under any symmetry transformation of a field, i.e. $\delta V = 0$ if $\boldsymbol{\phi} \longmapsto \boldsymbol{\phi} + \delta\boldsymbol{\phi}$ under symmetry. Looking at small transformations[4] we see that

$$0 = \delta V = \frac{\partial V}{\partial \phi_i} \cdot \delta\phi_i + O\left((\delta\phi_i)^2\right) \tag{5.2.9}$$

This is true for any $\boldsymbol{\phi}$ and any small symmetric transformation, not just for the minimum field solution. Thus we can differentiate with respect to the field ϕ_j,

$$0 = \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\delta\phi_i + \frac{\partial V}{\partial \phi_i}\frac{\partial(\delta\phi_i)}{\partial \phi_j} \qquad \forall \boldsymbol{\phi}(x) \tag{5.2.10}$$

Evaluating at $\boldsymbol{\phi} = \boldsymbol{\phi}_{\text{min}}$, we find

$$0 = \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_{\text{min}}} \cdot \delta^{(\text{min})}\phi_i \tag{5.2.11}$$

where $\delta^{(\text{min})}\phi_i$ is a small transformation of the minimum value. Under $\boldsymbol{\phi} \longmapsto \mathsf{U}\boldsymbol{\phi}$ then

$$\delta^{(\text{min})}\phi_i \quad = \quad \mathsf{U}\boldsymbol{\phi}_{\text{min}} - \boldsymbol{\phi}_{\text{min}} \tag{5.2.12}$$

where

$$\mathsf{U} = e^{i\mathsf{A}}, \quad \mathsf{A} = \varepsilon^a T^a \in \mathcal{A} \tag{5.2.13}$$

and if $|\varepsilon^a| << 1$ then

$$\delta^{(\text{min})}\phi_i \quad = i\mathsf{A}\boldsymbol{\phi}_{\text{min}} + O(\varepsilon^2) \tag{5.2.14}$$

Thus

$$(\mathsf{M}^2)(\mathsf{A}\boldsymbol{\phi}_{\text{min}}) = \mathbf{0} \qquad \forall\, |\varepsilon^a| << 1 \tag{5.2.15}$$

Since $\mathsf{A}\boldsymbol{\phi}_{\text{min}}$ is some vector, what we have is an eigenvalue equation for the mass matrix, $\mathsf{M}^2\boldsymbol{e} = \lambda\boldsymbol{e}$, with zero eigenvalue $\lambda = 0$ *provided* $\boldsymbol{e} = \mathsf{A}\boldsymbol{\phi}_{\text{min}}$ is a non-zero vector. So for certain algebra elements, we can deduce the existence of zero eigenvalues of the mass matrix and thus the existence of zero mass particles. Let us be more precise and quantify how many zero eigenvalues there are.

The key to this is to see that the elements of the symmetry group and algebra are split into two types,

$$\text{a)} \qquad \mathsf{U}\boldsymbol{\phi}_{\text{min}} = \boldsymbol{\phi}_{\text{min}}, \quad \mathsf{A}\boldsymbol{\phi}_{\text{min}} = 0 \qquad \text{Unbroken} \tag{5.2.16}$$

$$\text{b)} \qquad \mathsf{U}\boldsymbol{\phi}_{\text{min}} \neq \boldsymbol{\phi}_{\text{min}}, \quad \mathsf{A}\boldsymbol{\phi}_{\text{min}} \neq 0 \qquad \text{Broken} \tag{5.2.17}$$

[4]These are ones close to the identity element, so it can only be parts of the group generated by generators of a Lie Algebra. The following argument can not be applied to discrete symmetries.

5.2.1 Broken and Unbroken Generators

Unbroken symmetry

Let us start the algebra elements A' which act on the minimum field solution to give zero and denote them with a prime

$$\mathsf{A}'\boldsymbol{\phi}_{\min} = 0 \tag{5.2.18}$$

These are the algebra elements which give us no information about the mass matrix according to (5.2.15), but we must first learn how to remove these cases. It is best start by thinking of the corresponding Lie group elements, $\mathsf{U}' = \exp\{i\mathsf{A}'\}$, which leave the vacuum invariant i.e.

$$\mathsf{U}'\boldsymbol{\phi}_{\min} = \boldsymbol{\phi}_{\min} \tag{5.2.19}$$

We'll denote the full set of all symmetry elements satisfying (5.2.27) as H so this is a subset of the full symmetry group. It is quick to see that this set, H, is also a group in its own right. The key property required is closure within H.

$$\begin{aligned}
\text{Let} \qquad \mathsf{U'}_1, \mathsf{U'}_2 \in H \quad &\Rightarrow \quad \mathsf{U'}_1\phi_{\min} = \phi_{\min}, \qquad \mathsf{U'}_2\phi_{\min} = \phi_{\min} && (5.2.20)\\
&\Rightarrow \qquad \mathsf{U'}_2\mathsf{U'}_1\phi_{\min} = \phi_{\min} && (5.2.21)\\
&\Rightarrow \qquad \mathsf{U'}_2\mathsf{U}'_1 \in H && (5.2.22)
\end{aligned}$$

So H is a subgroup of G which, because of equation (5.2.27), is called the **stability group** or the **little group**. H is the group of **unbroken** symmetry, since both the Lagrangian and the lowest energy solution $\boldsymbol{\phi}_{\min}$ is invariant under these symmetry transformations.

Now let us return to the arguments above given for small group elements, which led to (5.2.15). Depending on the details of the problem, depending on the form of $\phi_{\min}$, G etc., it is quite possible that the stability subgroup H of the full symmetry group G may not be a Lie Group. In such a case there are no generators associated with it, no continuous parameters and (5.2.18) is never satisfied. In this case we can go no further with the unbroken symmetry case and (5.2.15) always non-trivial.

However suppose that the subgroup H is a Lie group. Then group elements connected to the identity of the the group H are generated by elements A' of an associated Lie Algebra[5] $\mathcal{A}_H$, and the algebra elements satisfy (5.2.18). In the usual way, all the Lie algebra elements can be represented as linear combinations of generators, which we will denote by $\mathsf{T'}^A$ with $A = 1, \ldots, \dim(H)$, a complete basis set for the sub-algebra H. The definition of the unbroken algebra elements can be summarized through their generators as

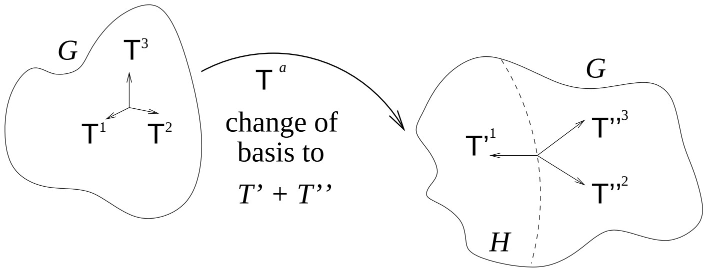
$$\boxed{\mathsf{T'}^A\boldsymbol{\phi}_{\min} = 0, \quad A = 1, \ldots, \dim(H)} \tag{5.2.23}$$

The dimension of H is just the number of basis elements of the algebra as with all Lie groups. We say that the $\left\{\mathsf{T'}^A\right\}$ and the A' they generate destroy (kill or annihilate) the vacuum and generate a subgroup of symmetry transformations which leave the vacuum invariant.

The generators we use for the unbroken algebra, $\mathsf{T'}^A$ need *not* be equal to any of the original generators used to define the original algebra $\mathcal{A}_G$ of the full symmetry group G, which we can continue to denote as $\mathsf{T}^a, a = 1, \ldots, \dim(G)$ without a prime and with a lower case index. and they may not be be part of a different one than the one we choose for G initially. One may need to perform a simple rotation and change of basis. We can express this in the usual way as

$$\mathsf{T'}^A = \sum_{a=1}^{\dim G} c^{Aa}\mathsf{T}^a \tag{5.2.24}$$

[5] It is straightforward to show directly from (5.2.16) that the unbroken algebra elements defined by (5.2.16) do form a Lie algebra.



Though we don't have to, it is normal to choose the unbroken generators to be orthogonal and we shall assume that this is done from now on, i.e. in addition to (5.2.27) we shall choose

$$\text{Tr}\left\{\mathsf{T}'^A\mathsf{T}'^B\right\} = \frac{1}{2}\delta_{AB} \tag{5.2.25}$$

Broken generators T''^Z

While the $\{\mathsf{T}'\}$ form *part* of a perfectly acceptable basis for the full algebra $\mathcal{A}_G$ of the full symmetry group G, they are usually incomplete. The broken generators, $\{\mathsf{T}''^Z\}$ are simply the remaining basis vectors needed to span the Lie algebra of the full symmetry group G, so they do not annihilate the quantum vacuum

$$\boxed{\mathsf{T}''^Z\boldsymbol{\phi}_{\text{min}} \neq 0} \tag{5.2.26}$$

By definition then, the $\{T''\}$ generate a subset of Lagrangian symmetry transformations which *change* the vacuum

$$\boxed{\mathsf{U}'' = \exp\{i\mathsf{A}''\}, \quad \mathsf{U}''\phi_{\text{min}} \neq \phi_{\text{min}}} \tag{5.2.27}$$

The subset U'' do not form a group! In particular the identity $\mathbb{1}$ is not a broken element. The number of broken generators though is just that needed to complete the basis for the full algebra. So there are

$$b = g - h = \dim(G) - \dim(H) \tag{5.2.28}$$

broken generators. It is usually best to choose an orthonormal basis for the algebra generating the whole symmetry group G so we will add orthonormality to the requirements

$$\text{Tr}\left\{\mathsf{T}'^A\mathsf{T}''^Z\right\} = 0 \qquad\qquad \text{Tr}\left\{\mathsf{T}''^Y\mathsf{T}''^Z\right\} = \frac{1}{2}\delta^{YZ} \tag{5.2.29}$$

The first condition ensures the broken generators are independent of the unbroken ones, the latter is simple orthogonality within the broken generators.

We can exponentiate the broken generators to get group elements $U'' = e^{i\varepsilon^Z\mathsf{T}''^Z} \in G$ which satisfy

$$\boxed{\mathsf{U}''\boldsymbol{\phi}_{\text{min}} \neq \boldsymbol{\phi}_{\text{min}}} \tag{5.2.30}$$

However, unlike in the unbroken sector, the broken generators need not form a subgroup of G.

Masses and Goldstone's Theorem

Having split our group and algebra into the two relevant parts, we can now return to our analysis of the masses using (5.2.15). We now see that we have

$$\mathsf{M}^2\mathsf{T}^a\boldsymbol{\phi}_{\text{min}} = 0 \tag{5.2.31}$$

in the original basis for G. However the full information only comes in the broken/unbroken basis

a) Unbroken case

$$\mathsf{M}^2(\mathsf{T}'^A\boldsymbol{\phi}_{\min}) = \mathsf{M}^2 \cdot 0 = 0, \quad A = 1, \ldots, h = \dim(H) \tag{5.2.32}$$

no information about eigenvalues

b) Broken case

$$\mathsf{M}^2(\mathsf{T}''^Z\boldsymbol{\phi}_{\min}) = 0, \quad Z = h+1, \ldots, h+b = \dim(G) \tag{5.2.33}$$

zero eigenvalues for every T''.

Note how it is important to work in terms of a complete basis of independent generators to ensure we don't over count equations. We can now quote Goldstone's theorem

$$\boxed{\begin{array}{c}\textbf{Goldstone's Theorem}\\ \text{For each broken generator, there is one distinct massless scalar mode.}\end{array}} \tag{5.2.34}$$

These massless scalar modes are called **Goldstone bosons**. Their number equals the number of broken generators, b, which is the difference between the dimensions of the full and unbroken symmetry groups. The proof given here is purely classical. However, the real power of the theorem is that it holds *exactly* in QFT (no perturbative approximation). There are not many results in QFT which are exact.

To summarise, to identify the masses of scalar particles coming from a scalar field involved in the breaking of a global symmetry involved in one must first find the unbroken generators. From this one deduces the number of broken generators and hence the number of Goldstone bosons. Finally, if the field was real and of d dimensions, this tells us that the remaining $d - b = d - g + h$ scalar particles will remain massive. These are the **Higgs particles**.

5.2.2 Examples of global symmetry breaking

Example 7 Global symmetry breaking in $O(3)$

Consider a theory of three real scalar fields, $\phi_i(x) \in \mathbb{R}$, $i = 1, 2, 3 = d = N$, described by the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_i)^2 - V(\phi_i\phi_i) \tag{5.2.35}$$

This is invariant under transformations by orthogonal 3×3 real matrices $\mathsf{U}.\mu^T = \mathbb{1}$, $\mathsf{U}_{ij} \in bbR$, i.e. $G = O(3)$. This group comes in two pieces depending on whether a reflection is included in the matrices or equivalently on the sign of the determinant $\det(\mathsf{U}) = \pm 1$. The special group elements, those with $\det(\mu) = +1$ form the Lie Group $SO(3)$, all of whose elements can be described in terms of the generators of the associated Lie algebra. A suitable basis is given by $T^a_{ij} = \frac{i}{2}\varepsilon^{aij}$

$$\mathsf{T}^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +i \\ 0 & -i & 0 \end{pmatrix} \qquad \mathsf{T}^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix} \qquad \mathsf{T}^3 = \frac{1}{2}\begin{pmatrix} 0 & +i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.36}$$

These ensure the group elements are real as required. We can also see that each generator has one row and column zero and the remaining is the generator of $SO(2)$, (an infinitesimal rotation about one axis while mixing the other co-ordinates). We have also chosen them to be orthonormal in the usual sense

$$\mathrm{Tr}\left\{\mathsf{T}^a\mathsf{T}^b\right\} = \frac{1}{2}\delta_{ab} \tag{5.2.37}$$

Suppose the potential V in (5.2.35) is such that the minimum is at

$$\boldsymbol{\phi}_{\min} = v\boldsymbol{e} \qquad\qquad \boldsymbol{e}\cdot\boldsymbol{e} = 1 \tag{5.2.38}$$

For instance, if we had

$$V = \frac{1}{2}m^2\boldsymbol{\phi}^2 + \lambda\boldsymbol{\phi}^4 \tag{5.2.39}$$

with $m^2 < 0$, $\lambda > 0$, we would have the vacuum solutions at $v = \sqrt{-\frac{1}{4}\frac{m^2}{\lambda}}$.

Any three-dimensional unit vector $\boldsymbol{e}$ will give a good lowest energy solution so suppose we study the case of

$$\boldsymbol{e} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{5.2.40}$$

The task is to find the masses of the physical scalar modes.

1. *Finding the unbroken generators.*
   Consider a general element A of the Lie algebra $\mathcal{A}$ of the full symmetry group G

$$\mathsf{A} = c_1\mathsf{T}_1 + c_2\mathsf{T}_2 + c_3\mathsf{T}_3 \in \mathcal{A}, \tag{5.2.41}$$

   with real coefficients $c_a \in \mathbb{R}$.[6] For unbroken generators we require

$$\mathbf{0} = \mathsf{A}(v\boldsymbol{e}) = \frac{v}{2}\begin{pmatrix} 0 & +ic_3 & -ic_2 \\ -ic_3 & 0 & +ic_1 \\ +ic_2 & -ic_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{5.2.42}$$

   This yields three equations for three unknowns. In some cases, these equations may not be linearly independent, which would indicate there is more than one unbroken generator. Here, we find just one unique solution

$$c^3 = 0 \qquad\qquad c^1 = c^2 \tag{5.2.43}$$

   Thus there is a *single unique unbroken generator*, which we define as

$$\mathsf{T}'^1 = \frac{1}{\sqrt{2}}(\mathsf{T}_1 + \mathsf{T}_2) \tag{5.2.44}$$

   normalized in the usual way (5.2.25)[7].

2. *Finding the remaining, broken, generators.*
   The Lie algebra A is three-dimensional and we have defined one basis generator, T'^1. Counting now tells us that to complete the basis two more generators, the broken generators, are needed. Thus there are two broken generators and so there are two massless scalar modes/particles.
   Often we don't need to find them but lets do so here. We have to search for two new generators, and it is easiest to stay in an orthonormal basis so we search for T''^Z. By analogy with simple vectors in (x, y, z) space, if $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are unit orthogonal vectors then

$$\frac{1}{\sqrt{2}}(\boldsymbol{e}_x + \boldsymbol{e}_y) \qquad\qquad \frac{1}{\sqrt{2}}(\boldsymbol{e}_x - \boldsymbol{e}_y) \qquad\qquad \boldsymbol{e}_z \tag{5.2.45}$$

   are also orthogonal unit vectors. Thus we can choose

$$\mathsf{T}''^2 = \frac{1}{\sqrt{2}}(\mathsf{T}_1 - \mathsf{T}_2) \qquad\qquad \mathsf{T}''^3 = \mathsf{T}^3 \tag{5.2.46}$$

[6] Note that $c_a = c^a$, that is we have a flat metric.
[7] Check using known the algebraic properties of the original T^a's (5.2.37).

for orthonormal basis generators in the broken part of the algebra. We can see these satisfy the orthogonality conditions (5.2.25) and (5.2.29) by using the properties of the original generators (5.2.37) and we do not need to multiply out three-by-three matrices.

Thus for our theory with a global $G = SO(3)$ symmetry we find that symmetry breaking leaves us with one unbroken generator. This can only generate the Lie group $SO(2)$ as it is the only group with one generator. So the stability group $H \cong SO(2)$.[8].

3. There were three degrees of freedom in the original model, the three scalar particles associated with the original scalar field. Thus after symmetry breaking the system must still have three distinct spin-0 modes for transferring energy, momentum etc. through the system. We have identified two massless scalar particles, the Goldstone particles demanded by Goldstone's theorem as we have two broken generators. The third mode will just be some massive mode ($\boldsymbol{e}.\boldsymbol{\phi}$ in fact), a precise value can be calculated from the potential, but we'll postpone this for a moment.

We have thus determined the number of unbroken symmetries to determine the number of broken symmetries, and therefore the number of massless particles. We did this choosing the vacuum (5.2.40). However, the choice of (5.2.40) was given to illustrate the way one can change basis in the Lie algebra.

However, there are better ways to extract the physics. First we could have chosen a different vacuum as all unit vectors $\boldsymbol{e}$ are equally good. It would make sense to have chosen one which avoided the need to change basis in the Lie algebra. For instance by inspection we see that $\boldsymbol{e} = (1,0,0)$ ensures that rotations about the number one axis leave this invariant, i.e. $\mathsf{T}^1 = \mathsf{T}'^1$ is the unbroken generator, and we can then choose $\mathsf{T}^2 = \mathsf{T}'^2, \mathsf{T}^3 = \mathsf{T}'^3$

Perhaps I perversely insist on using the vacuum given. In some physical problems there may be other issues that require us to stay with this choice. Then we can still simplify the analysis by exploiting the symmetries of $\mathcal{L}$ and working in terms of new transformed fields $\boldsymbol{\phi}'$ defined as

$$\boldsymbol{\phi} \longmapsto \boldsymbol{\phi}' = \mathsf{U}\boldsymbol{\phi} \qquad\qquad \mathsf{U} \in O(3) \tag{5.2.47}$$

and we *choose* U such that $\boldsymbol{\phi}'_{\mathrm{min}}$ is easier to consider than $\boldsymbol{\phi}_{\mathrm{min}}$

$$\boldsymbol{\phi}'_{\mathrm{min}} = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{5.2.48}$$

Its a good idea in general to see if the symmetries of a problem can be exploited to make the analysis simpler. One loses *nothing* in making a change of field variables using symmetry transformations.

□

5.3 Charges and broken symmetry

In a theory with broken symmetry, the consequences of having some unbroken symmetry are just as before. The usual symmetry considerations given in section (4) now apply but just for the stability group H. Thus particles with the same mass and related interaction strengths must fall into irreps of H, and they will have charges associated with the diagonal generators of the unbroken algebra. In particular the Goldstone's and Higgs, since they have different masses, must fit into different representations of H.

[8]We are ignoring the structure of the little group a long way from the identity. For instance it is not clear from this analysis if any of the original $O(3)$ group elements containing reflections, $\det(\mathsf{U}) = -1$, are broken or unbroken. However, such questions are not relevant to the question of the masses as the central result (5.2.15) is only concerned with Group elements near the identity.

Example 8 Irreps in broken global $O(3)$

We saw above that we got one massive Higgs particle. The only one-dimensional irrep of the unbroken $H \cong SO(2)$ is the trivial representation. Thus this particle is unrelated to any others by symmetry and must have SO(2) zero charge. The two Goldstones though must fit into the usual two dimensional irrep of SO(2). Thus we can find that if we write them as a complex field and its conjugate we have a pair of charged Goldstones, a particle/anti-particle pair. □

5.4 Scalar fields as matter or Higgs fields

Note that scalar fields can play two roles. The scalar fields with vevs are centre of SSB. However any scalar field with zero vev is just a simple matter field, a spectator to the symmetry breaking. That is not to say that simple matter fields are no effected by the SSB. We start with them in representations of the full symmetry group G when we write down the Lagrangian $\mathcal{L}$ but the physical modes must lie in the smaller representations of H after SSB.

Example 9 A Matter field in broken global $O(3)$

Consider a theory where the Higgs field involved in SSB is the three-dimensional $\phi_i(x) \in \mathbb{R}$, $i = 1, \ldots, 3$, as before. Suppose we add a second scalar triplet $\eta_i(x)$ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_i\right)^2 + \frac{1}{2}\left(\partial_\mu \eta_i\right)^2 - V(|\boldsymbol{\phi}|) - V_\eta(|\boldsymbol{\eta}|) - g(\boldsymbol{\phi}.\boldsymbol{\eta})^2 \tag{5.4.1}$$

A careful study show that the last term forces the symmetry group to be $G \cong O(3)$ as before where the symmetry transformations are $\boldsymbol{\phi} \to \mathsf{U}\boldsymbol{\phi}$ and $\boldsymbol{\eta} \to \mathsf{U}\boldsymbol{\eta}$ with the *same* three-dimensional group element U. If we keep the $\boldsymbol{\Phi}$ mass squared negative but the $\boldsymbol{\eta}$ fields mass squared positive, we can find solutions where only the $\boldsymbol{\phi}$ has a vev. Suppose for simplicity we choose this to be equal to $(1, 0, 0)v$. Then we note that the g interaction forces a correction to the η_1 mass term of

$$-g(\boldsymbol{\phi}.\boldsymbol{\eta})^2 = gv^2(\eta_1)^2 + O(\text{cubic,quartic}) \tag{5.4.2}$$

That is the three η fields also splits up into a singlet, η_1 of mass squared $m_\eta^2 + 2gv^2$ and a doublet, η_2 and η_3 of mass squared m_η^2. They are in two distinct irreps of $H \cong SO(2)$. Without detailed calculation we can guess that in terms of observable charges, η_1 must be zero charge, while the $\eta_2 \pm i\eta_3$ fields are the ± 1 charge eigenstates.

□

Chapter 6

Local (Gauge) Symmetry

In chapter 4, we studied global field symmetries where the same transformation was done to the field at every point in space. If we allow the transformation to differ from point to point and it is still a symmetry of the Lagrangian, then it is a **local** or **gauge** symmetry

$$\mathsf{F} \longmapsto \mathsf{F}'(x) = \mathsf{U}(x).\mathsf{F}(x), \qquad \partial^\mu \mathsf{U}(x) \neq 0. \tag{6.0.1}$$

It will turn out that this is required to describe photons and other "force carrying" particles — the $W^\pm$ and Z^0 of EWmodel, the eight gluons of QCD*et cetera*. More precisely, when this symmetry is present we have fundamental particles called **gauge bosons**. However when the relevant symmetry is a non-abelian groups the gauge bosons mediate the force but now also feel the force themselves like any other particle of matter. The gluons carry charges associated with their non-abelian symmetry while the photon of electromagnetism has no electric charge, related to an abelian local symmetry, and interacts only with charged matter fields, not with other photons. Thus local symmetry leads us to new types of interaction and ones that are vital to understand the standard model of particle physics. The gauge bosons have spin one and this is related to their space-time symmetry properties where the corresponding field transform like a Lorentz vector. Thus they are also called **vector bosons**.

6.1 Local Abelian Symmetry

We have already seen theories with abelian global symmetry such as

$$\mathcal{L}_{\text{global}} = |\partial_\mu \Phi|^2 - V(|\Phi|) \tag{6.1.1}$$

where the single complex field $\Phi(x) \in \mathbb{C}$, has global $U(1)$ symmetry (a phase symmetry). How do we change $\mathcal{L}_{\text{global}}$ to make it invariant under local $U(1)$ transformations, i.e. under

$$\Phi \longmapsto \Phi'(x) = e^{i\theta(x)}\Phi(x) \qquad ? \tag{6.1.2}$$

The potential V remains invariant, since $|\Phi'| = |\Phi|$ under local or global phase transformations, but the kinetic terms are now a problem. In local case

$$(\partial_\mu \Phi) \longmapsto \left(\partial_\mu \Phi'\right) = e^{i\theta}\left[(\partial_\mu \Phi) + i\left(\partial_\mu \theta\right)\cdot \Phi\right] \tag{6.1.3}$$

Thus

$$\begin{aligned} (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) \longmapsto \quad & \left[\partial_\mu \Phi^\dagger - i\left(\partial_\mu \theta\right)\cdot \Phi^\dagger\right] \cdot \left[\partial^\mu \Phi + i\left(\partial^\mu \theta\right)\Phi\right] & (6.1.4)\\ = \quad & (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) + (\partial_\mu \theta)\left[-i\Phi^\dagger(\partial^\mu \Phi) + i(\partial^\mu \Phi^\dagger)\Phi\right] & \\ & + ((\partial^\mu \theta)(\partial_\mu \theta))\,|\Phi|^2 & (6.1.5) \end{aligned}$$

Only for a global transformation where $\partial^\mu U(x) = 0$, i.e. $\partial^\mu \theta(x) = 0$ do we get invariance.

To find a Lagrangian invariant under local phase transformations, we will try a very pragmatic approach and will just try replacing ∂_μ in the Lagrangian with a new operator $\mathrm{D}_\mu \equiv \partial_\mu - iB_\mu(x)$ where $B_\mu(x)$ is to be a new real field.[12] Thus we want $|D^\mu \Phi|^2$ to be invariant. The idea is that the new field B_μ will transform under local symmetry transformations in such a way as to compensate for the extra $\partial^\mu\theta$ terms in (6.1.5). To determine a suitable transformation rule for B_μ, and hence D_μ, suppose $B_\mu \longmapsto B'_\mu$, then

$$(D_\mu \Phi) \longmapsto \left(D'_\mu \Phi'\right) = \left(\partial_\mu - iB'_\mu\right)\left(e^{i\theta}\Phi\right) \tag{6.1.6}$$

We could then stare at the proposed kinetic term and its gauge transformed form, taking care to note the presence of B_μ in the D_μ field, and demand[3]

$$\left(D'_\mu \Phi'\right)^\dagger \left(D'^\mu \Phi'\right) = \left[\left(\partial_\mu + iB'_\mu\right)\left(\Phi^\dagger e^{-i\theta}\right)\right]\left[\left(\partial^\mu - iB'^\mu\right)\left(e^{i\theta}\Phi\right)\right] = \left(D_\mu\Phi\right)^\dagger \left(D^\mu \Phi\right) \tag{6.1.7}$$

This is a mess to expand and compute[4].

Rather than stare at the expression (6.1.7), we will anticipate a more general result and note that if

$$[D_\mu \Phi] \longmapsto \left[D'_\mu \Phi'\right] = e^{i\theta}\left[D_\mu \Phi\right] \tag{6.1.8}$$

then $|D_\mu \Phi|^2$ is invariant. The ordinary field derivative in the global symmetry case behaved in exactly this way so it is not unreasonable to demand this form to be true here. We will comment further about the significance of this form at the end of the analysis, but for now let us just take it as an inspired guess and see how it produces a suitable result.

Substituting for D_μ and D'_μ in (6.1.8), $D'_\mu \Phi'$ becomes[5]

$$\left[D'_\mu \Phi'\right] = e^{i\theta}\left(\partial_\mu \Phi\right) + e^{i\theta} i \left(\partial_\mu \theta\right)\Phi - iB'_\mu e^{i\theta}\Phi = e^{i\theta}\left[\partial_\mu \Phi - iB_\mu \Phi\right] \tag{6.1.9}$$

from which we deduce that we require

$$\begin{aligned} e^{i\theta}\left[-B_\mu\right]\Phi &= e^{i\theta}\left[i\partial_\mu\theta - iB'_\mu\right]\Phi & (6.1.10)\\ &= e^{i\theta}\left[-i\left(B'_\mu - \partial_\mu\theta\right)\right]\Phi & (6.1.11)\end{aligned}$$

So demanding local invariance using a $|D^\mu\Phi|^2$ term to replace the $|\partial^\mu\Phi|^2$ kinetic term of the global theory requires the new B^μ field to transform as

$$B'_\mu = B_\mu + \partial_\mu \theta \tag{6.1.12}$$

This looks rather odd when compared with the transformations of scalar fields[6] Φ under local or global theories. Instead of multiplying by the group element, $\exp\{i\theta(x)\}$, we are adding the derivative of the parameter of the group to find the transformed field. However, we could discuss the differential operator D^μ rather than the field B^μ as knowing either one is sufficient. Then a quick calculation shows that the transformed D^μ has a much more familiar form, namely

$$D_\mu \longmapsto D'_\mu = e^{i\theta}\overrightarrow{D_\mu} e^{-i\theta}. \tag{6.1.13}$$

[1] We could let B^μ be complex but for simplicity we anticipate the fact that it must be real. The fact that B^μ has a Lorentz index, μ, indicates that it is not a simple scalar field like Φ, but this is not important just yet.

[2] There are other ways to approach this issue, e.g. from a geometric point of view which gives rise to fibre bundles.

[3] As an exercise, consider what would happen if $B_\mu \in \mathbb{C}$.

[4] Had we started with fermion fields, the result would be simple to see, as chapter 8 will note.

[5] B'_μ is just a number and so commutes.

[6] Fermion fields, for instance, also behave in the same way as the scalars see chapter 8.

Be careful with this form. The order is important as D^μ contains differentials and the arrow on top of the D^μ, usually omitted, is to remind you that the differential in D^μ acts on everything to the right, not just the exponential factor[7]. The transformation for D^μ is expressed purely in terms of group elements, $\exp\{i\theta\}$ or its group inverse element $\exp\{i\theta\}$.

We therefore have a prototype locally invariant QFTof the form

$$\mathcal{L}_{\text{local},1} = |D_\mu \Phi|^2 - V(|\Phi|) \tag{6.1.14}$$

In terms of mathematical symmetry, this is a perfectly reasonable theory with a $U(1)$ symmetry and we can investigate its physical predictions to see if it corresponds to anything familiar. However what one quickly realises is that there are no kinetic terms for the $B_\mu(x)$ field, i.e. no terms of the form $\frac{1}{2}(\partial_\mu B_\nu)^2$. As was stated in section 2.3.1, the field B_μ cannot represent a physical mode or particle, one able to transport (propagate) energy, momentum, charge *et cetera* through the system, without the correct kinetic terms. Put another way, as it stands B_μ is an auxiliary field which can be eliminated from the problem by using its equation of motion.

To make further progress we need to study Maxwell's equations of EMand identify the physics behind the B^μ field which so far has just been a mathematical construct.

What is a gauge field?

The field $B_\mu(x)$ introduced above must transform in the same way as ∂_μ under space-time symmetries if it is not to spoil the space-time properties of the action and Lagrangian, namely that they are scalars, appear to the same to all observers in space-time. It must therefore be a Lorentz four-vectors, such as ∂_μ, as its index μ suggests. Luckily, we have already seen a four-vector field such as this, namely the electromagnetic four-vector field $A_\mu(x)$, so we will show how to link this with $B_\mu(x)$ and use Maxwell's equations to find a kinetic term for the B_μ field.

Working in a relativistic notation we have, the four-current is

$$J^\mu = (\rho, \boldsymbol{j}) \tag{6.1.15}$$

where ρ and $\boldsymbol{j}$ are the electric charge density and three-current respectively. The electric and magnetic fields, $\boldsymbol{E}, \boldsymbol{B}$ respectively, have a complicated behaviour under Lorentz boosts, so it is convenient to work in terms of four-vector potential, $A^\mu(x)$

$$A^\mu = (\phi, \boldsymbol{A}) \qquad\qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\nabla \cdot \phi - \dot{\boldsymbol{A}} \tag{6.1.16}$$

Remember $A^\mu(x)$ contains redundant information. Two four-vector potentials, $A_\mu(x)$ and $A'_\mu(x)$ represent the same physics if they are related by a **gauge transformation**. For instance adding a constant to the zero-th component, ϕ the electric potential, corresponds to changing ones convention for the zero of voltage in a problem but leaves the electric and magnetic fields and so all the physics of electromagnetism unchanged.

Maxwell's equations have a simple form if write them in terms of a third quantity, the **field tensor**[8]

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{6.1.17}$$

which, as its name and two Lorentz indices suggest, transforms as a tensor under Lorentz symmetry. We then find two Maxwell equations are given by

$$\partial_\mu F^{\mu\nu} = -J^\nu \qquad \Leftrightarrow \qquad \begin{matrix} \nabla \cdot \boldsymbol{E} = \rho \\ \nabla \times \boldsymbol{B} = \boldsymbol{J} + \dot{\boldsymbol{E}} \end{matrix} \tag{6.1.18}$$

[7] e.g. $D_\mu \exp\{-i\theta\} f(x) = \exp\{-i\theta\} [-i(\partial_\mu \theta) f(x) + (\partial_\mu f(x)) - B_\mu f(x)]$.

[8] Care is needed here. This is *not* a differential operator. Rather the partial derivatives act only on the gauge fields and we should really indicate this with brackets $F^{\mu\nu} = (\partial^\mu A^\nu) - (\partial^\nu A^\mu)$. This is important later when we look at the square of this term and even more so when we look at non-abelian generalisations and other equivalent forms for $F^{\mu\nu}$. See the comments on Universality section 6.1.

One can then note that these Maxwell's equations are equations of motion of

$$\mathcal{L}_{\text{Max}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_L^\mu A_\mu \tag{6.1.19}$$

provided the current-like term in the Lagrangian, J_L^μ, is independent of A_μ

$$\frac{\partial J_L^\lambda}{\partial \left(\partial_\mu A_\nu\right)} = \frac{\partial J_L^\lambda}{\partial A_\nu} = 0. \tag{6.1.20}$$

If J_L^μ satisfies (6.1.20) then the equation of motion for the $A^\mu(x)$ field from the Lagrangian (6.1.19) is just Maxwell's equations (6.1.18) where the physical current, J^μ in Maxwell's equations is identical to the coefficient, J_L^μ, of the single A^μ term in the Lagrangian (6.1.19)

$$J_L^\mu = J^\mu \tag{6.1.21}$$

i.e. the physical current.

Let us ignore the currents for the time being, i.e. imagine free electric and magnetic field, or equivalently, just free photons, but no electric charges or currents, $J_L^\mu = J^\mu = 0$. This matter-free EMtheory would be described by the Lagrangian

$$\mathcal{L}_{\text{no-matter}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \left(\partial_\mu A_0(x)\right)^2 + \frac{1}{2} \left(\partial_\mu A_i(x)\right)^2 . \tag{6.1.22}$$

Writing this out we see that each of the three real fields, the components A_1, A_2 and A_3 of the vector potential, have exactly the form required if each is to represent a massless non-interacting particle (2.3.8). The problem is that the A_0 component has the wrong sign for a kinetic term. The solution is complicated but this is indeed a suitable kinetic term for a field transforming as a Lorentz vector. Roughly speaking the A_0 component with its unphysical kinetic term cancels out exactly one of the A_i modes, to leave just two physical modes for the transport of energy and momentum. This corresponds to the fact that in vacuo (equivalent to having no current or charges), Maxwell's equations allow electromagnetic waves to propagate in two independent ways, the two polarisations e.g. the electric field in one of two directions perpendicular to the direction of energy propagation. The conclusion of this discussion is that the F^2 term of (6.1.19) provides a suitable kinetic term for a vector field but that such vector fields represent only two physical degrees of freedom.

The last piece of information we need from the EMcase is the relationship between gauge freedom to local symmetry. Inspired by the discussion of local symmetry and the B^μ field above, or by recalling properties of Maxwell's EMequations, we note that the physics, i.e. the equation of motion, Maxwell's equations, are invariant under gauge transformations of the form

$$A^\mu \longmapsto A^\mu + \partial^\mu \bar{\theta} \tag{6.1.23}$$

where $\bar{\theta}(x)$ an arbitrary real function. This is another way of noting that not all the information of the four real functions in A^μ can related to physical particles. However, we want to return to construction of a kinetic term for our B^μ in our local symmetric Lagrangian (6.1.14). We see that the field tensor, F is invariant

$$F_{\mu\nu} \longmapsto F'_{\mu\nu} = F_{\mu\nu} \tag{6.1.24}$$

so that the F^2 term is invariant of (6.1.22) is also under such transformations.

Lagrangian with Abelian local symmetry

We then produce a second prototype for a locally invariant theory

$$\mathcal{L}_{\text{local},2} = -\frac{c}{4} F_B^{\mu\nu} F_{B\mu\nu} + |D_\mu \Phi|^2 - V(|\Phi|) \tag{6.1.25}$$

with $F_B^{\mu\nu}$ is defined as in (6.1.17) with B^μ replacing A^μ, and c is some constant. This is invariant under local transformations given by (6.1.2) and (6.1.12) whatever value of $c > 0$ we choose, as each term is separately invariant. However, we know from the EMexample that the coefficient of the F^2 term should be 1 to get the coefficient of the B^μ kinetic terms correct. Rather than set $c = 1$, we can absorb this into the normalisation of the B^μ field by defining $c = 1/g^2$, $B^\mu = gA^\mu$. This does not effect the scalar field's derivative term.

So finally, we are left with a field theory invariant under a U(1) local or gauge symmetry, with a single scalar field. The model often called **scalar QED** (SQED) and has the Lagrangian

$$\mathcal{L} \quad = \quad -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi^\dagger \Phi) \tag{6.1.26}$$

where we define the **covariant derivative**

$$D_\mu = \partial_\mu - igA_\mu \tag{6.1.27}$$

This $\mathcal{L}$ is locally, or gauge, invariant under

$$\Phi(x) \longmapsto \Phi'(x) = e^{i\theta(x)} \Phi(x) \qquad \Phi^\dagger(x) \longmapsto e^{-i\theta(x)} \Phi^\dagger(x) \tag{6.1.28}$$

$$A^\mu(x) \longmapsto A'^\mu(x) = A^\mu(x) + \frac{1}{g} \partial^\mu \theta \qquad D^\mu(x) \longmapsto D'^\mu(x) = e^{i\theta(x)} D^\mu(x) e^{-i\theta(x)} \tag{6.1.29}$$

Let us summarise the physics described by this Lagrangian:

Scalar Particles

A Lagrangian of a single scalar field describes the propagation and interaction of two scalar particles of equal mass, each one degree of freedom. The mass term for the scalars is in the scalar potential $m^2 = (\partial^2 V / \partial \Phi^\dagger \partial \Phi)$ and the symmetry demands that both particles have the same mass m. The two scalar particles differ in their U(1) charges, one has the opposite charge of the other. Note that this is identical to our earlier conclusions of a pure complex scalar field, with global U(1) symmetry. It should not be surprising as the global symmetry transformations are just a special subset of the local U(1) symmetries, and there are a whole class of $A^\mu = 0$ solutions which are just those we can find in the pure scalar field case with its global symmetry.

Gauge Particles

The vector or gauge field A^μ, by analogy with EM, describes a single massless photon-like particle, with two physical two degrees of freedom (two-polarisations) and spin one. This particle *must be massless by symmetry.* No $A^\mu A_\mu$ term is present in (6.1.26) or in Maxwell's Lagrangian (6.1.19). If we tried to add a term $m_\gamma^2 A_\mu A^\mu$ we would quickly find it is not gauge invariant and contradicts the fundamental principle behind the construction.

Interactions

Pure scalar particle interactions are encoded in this Lagrangian in the scalar potential $V(\Phi)$ e.g. as a $-\lambda|\Phi|^4$ term, just as in the global case.

Expanding out the $|D\Phi|^2$ term shows us

$$|D^\mu\Phi|^2 = |\partial_\mu\Phi|^2 - ig(\partial_\mu\Phi^\dagger)\Phi A^\mu + igA_\mu\Phi^\dagger(\partial^\mu\Phi) + g^2|\Phi|^2 A^\mu A_\mu \tag{6.1.30}$$

It contains the correct kinetic term for a complex field Φ (2.3.19) but it also has extra terms, cubic and quartic in the fields, mixing the vector field A^μ and the complex scalar Φ. These are photon-scalar particle interaction terms. The strength of these interactions is set by the real constant g in the covariant derivative (6.1.27). Thus g is another coupling constant called the **gauge coupling**. While symmetry fixes the coefficients of the relative terms in (6.1.30), the pure scalar interaction strengths (λ in the example given) and the gauge coupling g are not related by local symmetry.

Charges and Currents

Careful. The *physical* current is in the A^μ equation of motion

$$\partial_\mu F^{\mu\nu} = -gJ_N^\mu = -J^\mu \tag{6.1.31}$$

where J_N is the Noether current, e.g. as obtained by looking at global symmetry (set $\partial_\mu\theta = 0$ to study global transformation, a subset of the set of all local transformations)

$$J_N^\mu = -i(\partial^\mu\Phi)^\dagger\Phi + i\Phi^\dagger(\partial\Phi) + 2gA_\mu\,|\Phi|^2 \tag{6.1.32}$$

Of course the overall normalisation of Noether currents is arbitrary and we have chosen it so that J_N^μ counts +1 (-1) for each Φ particle (for each $\Phi^\dagger$ anti-particle). The first two terms are exactly as we had before in the global case (its the $A^\mu = 0$ limit of our theory). The point here is that in the local case the physical current, the one which appears in the equation of motion where the electric four-current does in Maxwell's equations, is this Noether current *scaled* by g. That is for every scalar particle we would count $+g$, and $-g$ for every scalar anti-particle. Thus in a gauge theory we do have an absolute size for our physical charges and it is always given in units of g the gauge coupling of the covariant derivative (6.1.27). For instance if Φ describes a

- $\pi^\dagger$, then $g = e$, as this is the electric charge on the π^+;
- Cooper pair in a superconductor (a two-electron bound state), then $g = -2e$.

In general, though the Lagrangian has a local $U(1)$ symmetry, this need not be the EM symmetry and so the charge need not be the electric charge so g may have no relation to e.

Also note, this $\mathcal{L}$ has $A_\mu A^\mu\,|\Phi|^2$ terms, i.e. we cannot write it as $A_\mu J_L^\mu$ without J_L^μ depending on A_μ. Equivalently $J_N^\mu \neq J_L^\mu$ here; we are not able to read off the Noether current as the coefficient of the A_μ term and must be solved for.

Universality

Note how ∂_μ in the global theory can be replaced *everywhere* by the covariant derivative D_μ in the local theory, e.g.

- in the kinetic terms of the Lagrangian

$$|\partial_\mu\Phi|^2 \longmapsto |D_\mu\Phi|^2 \tag{6.1.33}$$

- in the Noether current, where the simple global result of (**??**) becomes (6.1.32), but this can be rewritten in terms of D^μ, as

$$J_\mu = i\Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi \quad \longmapsto \quad i\Phi^\dagger \left(D_\mu \Phi\right) - i\left(D_\mu \Phi\right)^\dagger \Phi \tag{6.1.34}$$

i.e. the usual global current plus a scalar field term (so, for example, this does not occur with fermions): $J_N = J_N(A^\mu)$ and $A^\mu \neq 0$.

- in the field tensor, which can be rewritten as

$$F^{\mu\nu} = (\partial^\mu A^\nu) - (\partial^\nu A^\mu) = (D^\mu A^\nu) - (D^\nu A^\mu) = \frac{i}{g}[D^\mu, D^\nu] \tag{6.1.35}$$

Care must be taken with the last form. It is to be treated as a differential operator and the differential in the left hand covariant derivative always acts on both the second covariant derivative and on a test function. Conversely, the first two forms are not differential operators with the partial or covariant derivatives acting only on the gauge fields as indicated by the brackets. Note that the brackets are often dropped in the first two forms which when one first encounters them in the kinetic terms $F^{\mu\nu}F_{\mu\nu}$ can be confusing.

Geometry and Gauge Fields

Note from the last form of the field tensor, and from the form of the kinetic term we can replace all reference to the field A^μ in our field theory with the covariant derivative D^μ. This hints at a deeper and more natural structure to gauge theories. This can be obtained by working with differential geometry, see for example [6]. General Relativity, the classical theory of gravity, also uses covariant derivatives. It can be regarded as a gauge theory where the local symmetry acts on space-time coordinates (physics is invariant under choice of space-time coordinates at each space-time point), rather than mixing fields in some internal space (physics is invariant under the choice of fields coordinates at each space-time point).

The Representation of Gauge Fields

The symmetry transformation for $D^\mu(x)$ given in (6.1.29) is in terms of group elements, here $e^{i\theta}$, and this, rather than the transformation for the vector field $A^\mu(x)$ which is similar to the symmetry transformation for the matter field, the scalar field. The fact that two group elements are needed for D^μ rather than one indicates that D^μ transforms under a different representation of the group than that of the field. In fact, it transforms like a field would in the **adjoint** representation. Thus we say that the gauge field lies in the adjoint representation.

6.2 Generalisation to Non-Abelian gauge theories

These are also known as **Yang-Mills theories**[9]. Consider a $\mathcal{L}$ of a vector of complex scalar fields with global non-Abelian symmetry

$$\mathcal{L} = \left(\partial_\mu \mathbf{\Phi}\right)^\dagger \left(\partial^\mu \mathbf{\Phi}\right) - V(\mathbf{\Phi}^\dagger, \mathbf{\Phi}) \tag{6.2.1}$$

which is invariant under some global group of matrices $G = \{\mathsf{U}\}$

$$\mathbf{\Phi}' \longmapsto \mathbf{\Phi}' = \mathsf{U}\mathbf{\Phi} \tag{6.2.2}$$

[9] Yang and Mills, 1954.

where

$$\mathsf{U} \in G, \qquad \partial_\mu \mathsf{U} = 0, \qquad \mathsf{U}^\dagger \cdot \mathsf{U} = \mathbb{1} \tag{6.2.3}$$

or in terms of the generators of the Lie Algebra

$$\mathsf{U} = \exp\{i\varepsilon^a \mathsf{T}^a\} \qquad \partial_\mu \varepsilon^a = 0 \tag{6.2.4}$$

for global transformations.

Now if we look at the local symmetry transformation of the same group G, then we just allow $\partial_\mu \epsilon^a \neq 0$. With no derivatives in the potential, this remains invariant but once again there is a problem with the kinetic term

$$\partial_\mu \mathbf{\Phi} \to \partial_\mu \mathbf{\Phi}' = \mathsf{U}(\partial_\mu \mathbf{\Phi}) + (\partial_\mu \mathsf{U})\,\mathbf{\Phi} \tag{6.2.5}$$

so that

$$|\partial_\mu \mathbf{\Phi}|^2 \to (\partial_\mu \mathbf{\Phi})^\dagger\, \mathsf{U}^\dagger \cdot \mathsf{U}\,(\partial^\mu \mathbf{\Phi}) + O(\partial_\mu \mathsf{U}) \tag{6.2.6}$$

Since $\partial_\mu \mathsf{U} \neq 0$, this is *not* equal to $|\partial_\mu \mathbf{\Phi}|^2$.

Following the central role of the covariant derivative, let us follow the idea of universality and try replacing ordinary derivatives with a covariant derivative

$$\partial_\mu \to \mathsf{D}_\mu = \partial_\mu \mathbb{1} - i\mathsf{B}_\mu(x) \tag{6.2.7}$$

Note that since the covariant derivative now acts on a vector of fields, we have to allow the new field B^μ to be a matrix. We demand

$$(\mathsf{D}_\mu \mathbf{\Phi}) \to (\mathsf{D}'_\mu \mathbf{\Phi}') = \mathsf{U}(\mathsf{D}_\mu \mathbf{\Phi}) \tag{6.2.8}$$

so that a suitable kinetic term for the scalar field is then $|\mathsf{D}_\mu \Phi|^2$ since

$$|\mathsf{D}_\mu \mathbf{\Phi}|^2 \to \left|\mathsf{D}'_\mu \mathbf{\Phi}'\right|^2 = (\mathsf{D}_\mu \mathbf{\Phi})^\dagger \mathsf{U}^\dagger \cdot \mathsf{U}(D^\mu \mathbf{\Phi}) = |\mathsf{D}_\mu \mathbf{\Phi}|^2 \tag{6.2.9}$$

As before, we can now determine how B^μ transforms to B'_μ.

$$\begin{aligned}
\mathsf{D}_\mu(\mathbf{\Phi}) \to (\mathsf{D}'_\mu \mathbf{\Phi}') &= (\partial_\mu \mathbb{1} - i\mathsf{B}'_\mu)(\mathsf{U}\mathbf{\Phi}) && (6.2.10)\\
& && (6.2.11)\\
&= (\partial_\mu \mathsf{U})\mathbf{\Phi} + \mathsf{U}(\partial_\mu \mathbf{\Phi}) - i\mathsf{B}'_\mu \mathsf{U}\mathbf{\Phi} && (6.2.12)\\
& && (6.2.13)\\
&= \mathsf{U}[\partial_\mu \mathbf{\Phi} - i(\mathsf{U}^\dagger \mathsf{B}' \mathsf{U})\mathbf{\Phi} + \mathsf{U}^\dagger(\partial_\mu \mathsf{U})\mathbf{\Phi}] && (6.2.14)
\end{aligned}$$

for which, following (6.2.8), we require

$$\begin{aligned}
\mathsf{D}_\mu(\mathbf{\Phi}) \to (\mathsf{D}'_\mu \mathbf{\Phi}') &= \mathsf{U}[\partial_\mu \mathbf{\Phi} - i\mathsf{B}_\mu \mathbf{\Phi}] && (6.2.15)\\
&= \mathsf{U}(\mathsf{D}_\mu \mathbf{\Phi}) && (6.2.16)
\end{aligned}$$

We thus have[10]

$$\mathsf{B}_\mu = \mathsf{U}^\dagger \mathsf{B}'_\mu \mathsf{U} + i\mathsf{U}^\dagger(\partial_\mu \mathsf{U}) \tag{6.2.17}$$

Thus

$$\mathsf{B}'_\mu = \mathsf{U}\mathsf{B}_\mu \mathsf{U}^\dagger - i\mathsf{U}^\dagger \partial_\mu \mathsf{U} \tag{6.2.18}$$

Thus, under this transform, $|\mathsf{D}_\mu \mathbf{\Phi}|^2$ is invariant.

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[10]There may be a sign error here, check the handout.

What type of object is B_μ?

It turns out that B_μ is an element of the Lie algebra $\mathcal{A}$. To check this, consider an infinitesimal U; i.e.

$$\mathsf{U} \simeq \mathbb{1} + i\varepsilon^a \mathsf{T}^a \qquad |\varepsilon^a| \ll 1 \tag{6.2.19}$$

Then

$$\mathsf{B}'_\mu = \mathsf{B}_\mu + i\varepsilon^a[\mathsf{T}^a, \mathsf{B}_\mu] + \underbrace{(\partial_\mu \varepsilon^a)\,\mathsf{T}^a}_{c^a\mathsf{T}^a \in \mathcal{A}} \tag{6.2.20}$$

The last term is clearly an element of the algebra as its a sum of generators multiplied by real coefficients. If we assume that $\mathsf{B} \in \mathcal{A}$ then closure under the Lie bracket multiplication, i.e. the fact that $\mathsf{A}_1, \mathsf{A}_2 \in \mathcal{A}$ implies $i[\mathsf{A}_1, \mathsf{A}_2] \in \mathcal{A}$, shows that the second term is in the algebra. Thus

$$B \in \mathcal{A} \qquad \Rightarrow \qquad B' \in \mathcal{A} \tag{6.2.21}$$

It therefore is consistent to assume the field B is in the algebra. Not a proof but we will now assume our new field is in the Lie algebra of the symmetry group.

Note that for an abelian group, the commutator term of (6.2.20) is not present. Commutators such as this complicate calculations involving non-abelian symmetry, but these are essential for local non-abelian symmetry.

Just as in the Abelian case we can scale the gauge field without changing the symmetry[11] then write $\mathsf{B}_\mu(x)$ as sum of $\dim(G)$ real coefficients multiplying the generators

$$\mathsf{B}^\mu(x) = g\mathsf{W}^\mu(x) = g\mathsf{W}^{\mu a}(x)\mathsf{T}^a \tag{6.2.22}$$

c.f. $c^aT^a \in \mathcal{A}$, $c^a \in \mathbb{R}$. Without loss of generality, we have introduce the real **gauge coupling constant** g by analogy with the abelian case. This shows that the $\dim(G)$ vector in the internal space[12] fields $W^{\mu a}(x)$, the **non-Abelian gauge bosons**, are the natural extensions of an abelian vector field A^μ when the symmetry is non-abelian. The $\mathsf{W}^a(x)$ are matrices lying in the Lie Algebra, and it is often more convenient to study symmetry in terms of these *Lie Algebra valued* gauge fields. Use orthogonality of generators to relate the two forms for the non-abelian gauge fields,

$$2\mathrm{Tr}\{\mathsf{T}^a\mathsf{W}_\mu(x)\} = W^a_\mu(x). \tag{6.2.23}$$

Our covariant derivative, (6.2.7) is then

$$\mathsf{D}_\mu(x) = \partial_\mu \mathbb{1} - ig\mathsf{W}^\mu(x) = \partial_\mu \mathbb{1} - ig\mathsf{T}^a W^a_\mu(x) \tag{6.2.24}$$

and a prototype theory with a local non-abelian symmetry and a complex field vector $\mathbf{\Phi}$ is

$$\mathcal{L}_{\text{local 1}} = (\mathsf{D}^\mu \mathbf{\Phi})^\dagger \cdot (\mathsf{D}_\mu \mathbf{\Phi}) - V(\mathbf{\Phi}^\dagger, \mathbf{\Phi}) \tag{6.2.25}$$

Just as in the abelian case (6.1.14), this form is a perfectly good Lagrangian but the gauge fields have no kinetic term, no terms of the form $\frac{1}{2}\left(\partial_\mu W^{\mu a}\right)^2$ in (6.2.25). The gauge fields in (6.2.25) do not represent propagating particles. They are actually auxiliary fields which can be removed by using their equation of motion[13]. Let us turn to find a kinetic term for the non-abelian gauge fields.

[11] This anticipates that this is equivalent to choosing the correct normalisation for the gauge field kinetic term.

[12] Again these fields transform like Lorentz four-vectors under space-time symmetry transformations so each has another four components.

[13] In fact parts of the scalar field are also removed, see the discussion on Unitary Gauge in section 7.1 as the same type of procedure is needed here

6.2.1 Kinetic terms for the $\mathsf{W}^{\mu a}(x)$

To find a kinetic term, let us imitate the abelian case and use the covariant derivative[14] found above to construct a field tensor, i.e. try

$$\begin{aligned}
\mathsf{F}^{\mu\nu} &:= \frac{i}{g}[\mathsf{D}_\mu, \mathsf{D}_\nu] && (6.2.26)\\
&= (\mathsf{D}^\mu \mathsf{W}^\nu) - (\mathsf{D}^\nu \mathsf{W}^\mu) && (6.2.27)\\
&= (\partial^\mu \mathsf{W}^\nu) - (\partial^\nu \mathsf{W}^\mu) - ig[\mathsf{W}^\mu, \mathsf{W}^\nu] && (6.2.28)
\end{aligned}$$

where the commutator term in the last form is not present in the case of abelian symmetry. The field tensor like the covariant derivative and gauge fields is a matrix. Again care is needed to see that the partial and covariant derivatives in the second two terms act only on the gauge fields while we must treat the first form as a differential operator with it acting on a test function on the right.

The first form (6.3.6) for the field tensor allows us to find its transformation properties easily as from those of the covariant derivative (6.2.8)

$$\begin{aligned}
\mathsf{F}_{\mu\nu} \to \mathsf{F}'_{\mu\nu} &= \frac{i}{g}[\mathsf{D}'_\mu, \mathsf{D}'_\nu] = \frac{i}{g}[\mathsf{U}\cdot\mathsf{D}_\mu\mathsf{U}^\dagger, \mathsf{U}\cdot\mathsf{D}_\nu\mathsf{U}^\dagger] && (6.2.29)\\
&= \frac{i}{g}\mathsf{U}[\mathsf{D}_\mu, \mathsf{D}_\nu]\mathsf{U}^\dagger = \mathsf{U}\cdot\mathsf{F}\cdot\mathsf{U}^\dagger && (6.2.30)
\end{aligned}$$

Unfortunately, we see that unlike the abelian case, the non-abelian field tensor $\mathsf{F}_{\mu\nu}$ is not invariant but transforms exactly like the covariant derivative. This is not necessarily a problem as to find suitable kinetic terms for the gauge fields $\frac{1}{2}(\partial^\mu W^a_\nu(x))^2$, we need a Lorentz invariant pair, $\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}$. However we can quickly see that

$$\begin{aligned}
\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu} \to \mathsf{F}'_{\mu\nu}\mathsf{F}'^{\mu\nu} &= \mathsf{U}\cdot\mathsf{F}_{\mu\nu}\cdot\mathsf{U}^\dagger.\mathsf{U}\cdot\mathsf{F}^{\mu\nu}\cdot\mathsf{U}^\dagger && (6.2.31)\\
&= \mathsf{U}\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\mathsf{U}^\dagger && (6.2.32)
\end{aligned}$$

This should have been obvious as $\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}$ is a matrix with free indices in an internal symmetry representation space, so it must transform non-trivially under the internal symmetry transformations.

We therefore repeat the trick we used with the Lorentz indices and try to contract out the internal symmetry indices, i.e. since $\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu} \equiv F^{\mu\nu}_{ij} F_{\mu\nu,jk}$ we should try setting $i = k$, i.e. we take the trace. So we look at

$$\mathrm{Tr}\{\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\} \to \mathrm{Tr}\{\mathsf{F}'_{\mu\nu}\mathsf{F}'^{\mu\nu}\} = \mathrm{Tr}\left\{\mathsf{U}\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\mathsf{U}^\dagger\right\} = \mathrm{Tr}\left\{\mathsf{U}^\dagger\mathsf{U}\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\right\} = \mathrm{Tr}\{\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\} \quad (6.2.33)$$

since all the representations are unitary. Thus $\mathrm{Tr}\{\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\}$ is invariant under Lorentz and internal symmetries and is a suitable term for a Lagrangian. Its renormalisable too. It is also a promising candidate for the kinetic $\mathsf{W}^{\mu a}(x)$ term since it contains the square of the derivative of each component of the $W^{\mu a}(x)$ field

$$-\frac{1}{2}\mathrm{Tr}\{\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\} = -\frac{1}{2}(\partial_\mu \mathsf{W}^a_\nu)^2 + \ldots \quad (6.2.34)$$

where we used $\mathrm{Tr}\{\mathsf{T}^a\mathsf{T}^b\} = \frac{1}{2}\delta_{ab}$. Like the Abelian case, the $\mu = 0$ components have the wrong sign indicating that there are unphysical degrees of freedom in the W^μ_a fields. Again, while there are four real fields $\mu = 0, 1, 2, 3$ for each value of a, there are only two physical gauge boson modes for each value of a allowed. Removing the unphysical modes in a non-abelian gauge theory in the full quantum field theory is more difficult than in the abelian gauge theory case, and additional tricks such as Fadeev-Popov ghosts are used. There are also terms cubic and quartic in the non-abelian gauge field present in this kinetic term, that is there are interactions, but we will comment on this below.

[14] In fact we can use *any* covariant derivative associated with any field, or even one we invent using any non-trivial representation for the generators. For simplicity, we can stick with the one used for the matter field $\mathbf{\Phi}$ so far but see the comments on multiple matter fields in section 6.5 below.

Infinitesimal transformations and $\mathsf{F}^{\mu\nu}$.

To illustrate the general level of complications faced when calculating in a non-abelian gauge theory, let us look at the infinitesimal transformation of the kinetic term of the gauge fields.

$$\mathsf{U} = \mathbb{1} + i\varepsilon^a \mathsf{T}^a + O(\varepsilon^2) \tag{6.2.35}$$

To simplify the notation, it is convenient to define an algebra element $\mathbf{\Lambda}$

$$\mathbf{\Lambda} := \varepsilon^a \mathsf{T}^a \in \mathcal{A} \tag{6.2.36}$$

as this contains all the information we need about the infinitesimal transformations. Using (6.2.22) to substitute for B_μ in (6.2.20)

$$\delta \mathsf{W}^\mu = i[\mathbf{\Lambda}, \mathsf{W}^\mu] + \frac{1}{g}(\partial^\mu \mathbf{\Lambda}) \qquad\qquad \mathbf{\Lambda} = \epsilon^a \mathsf{T}^a \tag{6.2.37}$$

We then have

$$\begin{aligned}
\delta \mathsf{F}^{\mu\nu} &= \partial^\mu(\delta \mathsf{W}^\nu) - ig[\delta \mathsf{W}^\mu, \mathsf{W}^\nu] - (\mu \leftrightarrow \nu) & (6.2.38)\\
&= i\partial^\mu[\mathbf{\Lambda}, \mathsf{W}^\nu] + \frac{1}{g}\partial^\mu\partial^\nu \mathbf{\Lambda} + g\,[[\mathbf{\Lambda}, \mathsf{W}^\mu], \mathsf{W}^\nu] - i\,[(\partial^\mu \mathbf{\Lambda}), \mathsf{W}^\nu] - (\mu \leftrightarrow \nu) & (6.2.39)
\end{aligned}$$

The second order derivatives cancel under μ/ν symmetry, while we can combine the commutator factors of i to yield:

$$\delta \mathsf{F}^{\mu\nu} = i[\mathbf{\Lambda}, \partial^\mu \mathsf{W}^\nu] + g\,[[\mathbf{\Lambda}, \mathsf{W}^\nu], \mathsf{W}^\nu] - (\mu \leftrightarrow \nu) \tag{6.2.40}$$

Now

$$[[\mathbf{\Lambda}, \mathsf{W}^\mu], \mathsf{W}^\nu] = \mathbf{\Lambda}\mathsf{W}^\mu\mathsf{W}^\nu - \mathsf{W}^\nu\mathbf{\Lambda}\mathsf{W}^\mu - \mathsf{W}^\nu\mathbf{\Lambda}\mathsf{W}^\mu + \mathsf{W}^\nu\mathsf{W}^\mu\mathbf{\Lambda} \tag{6.2.41}$$

where we note the symmetry of the second and third terms. Thus

$$\begin{aligned}
\delta \mathsf{F}^{\mu\nu} &= i[\mathbf{\Lambda}, \partial^\mu \mathsf{W}^\nu] + g[\mathbf{\Lambda}, \mathsf{W}^\mu\mathsf{W}^\nu] - (\mu \leftrightarrow \nu) & (6.2.42)\\
&= i[\mathbf{\Lambda}, \partial^\mu \mathsf{W}^\nu - ig\mathsf{W}^\mu\mathsf{W}^\nu] - (\mu \leftrightarrow \nu) & (6.2.43)\\
&= i\,[\mathbf{\Lambda}, \partial^\mu \mathsf{W}^\nu - \partial^\nu \mathsf{W}^\mu - ig[\mathsf{W}^\mu, \mathsf{W}^\nu]] & (6.2.44)
\end{aligned}$$

That is

$$\delta \mathsf{F}^{\mu\nu} = i[\mathbf{\Lambda}, \mathsf{F}^{\mu\nu}] \tag{6.2.45}$$

This is completely different to the Abelian case since $\mathsf{F}^{\mu\nu} \neq 0$ in general. In fact $\delta \mathsf{F}^{\mu\nu} = 0 \Rightarrow \left[\mathsf{T}^a, \mathsf{T}^b\right] = 0 \forall a, b$, i.e. $\mathsf{F}^{\mu\nu}$ is gauge invariant if and only if we have an Abelian gauge group.

However, we know that $\mathsf{F}^{\mu\nu}$ is not a suitable kinetic term on its own as it is not invariant under Lorentz symmetry (the two free indices remind us its transforms as a tensor). A kinetic term also needs to be quadratic in field derivatives, $(\partial_\mu W_\nu^a)^2$ and $\mathsf{F}_{\mu\nu}$ is only linear in derivatives. So it is more logical to construct a Lorentz scalar by contracting Lorentz indices between a pair of $\mathsf{F}^{\mu\nu}$. While this is invariant under Lorentz transformations we must still check its behaviour under internal symmetry transformations and we find

$$\begin{aligned}
\delta(\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}) &\propto (\delta \mathsf{F}^{\mu\nu})\mathsf{F}_{\mu\nu} + \mathsf{F}^{\mu\nu}(\delta \mathsf{F}_{\mu\nu}) & (6.2.46)\\
&= \mathbf{\Lambda}\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu} - \mathsf{F}^{\mu\nu}\mathbf{\Lambda}\mathsf{F}_{\mu\nu} + \mathsf{F}^{\mu\nu}\mathbf{\Lambda}\mathsf{F}_{\mu\nu} - \mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\mathbf{\Lambda} & (6.2.47)\\
&= [\mathbf{\Lambda}, \mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}] \neq 0 & (6.2.48)
\end{aligned}$$

Thus $\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}$ is not going to be invariant under internal symmetry transformations unless it is something like a unit matrix or part of the CSA[15]. However, it is in general just some matrix and the commutator with $\boldsymbol{\Lambda}$ is not zero. Again, the fact that $\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}$ is a matrix with two free indices (say i, j) associated with a representation space of the Lie Algebra reminds us that it must have a non-trivial transformation under the internal symmetry. We should have guessed F^2 would have a non-trivial symmetry transformation. Again it suggests we contract the free indices, i.e. take the trace $\mathrm{Tr}\{\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\}$, and we now find its transforms as

$$\begin{aligned} \delta\left(\mathrm{Tr}\{\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\}\right) &= \mathrm{Tr}\{\delta\left(\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\right)\} && (6.2.49)\\ &= \mathrm{Tr}\{\boldsymbol{\Lambda}\mathsf{F}\mathsf{F} - \mathsf{F}\mathsf{F}\boldsymbol{\Lambda}\} && (6.2.50)\\ &= 0 && (6.2.51) \end{aligned}$$

Thus $\mathrm{Tr}\{\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\}$ is invariant under Lorentz and infinitesimal internal symmetries and confirms our more general proof above. The point to note here is how much harder non-abelian fields with their non-commuting algebra elements are to deal with from a calculational point of view. This is the same reason why the kinetic term contains terms cubic and quartic in the non-abelian gauge field as they are such terms are proportional to the commutators, so they were not present in the abelian case. Their physical significance will be noted below.

6.3 Physical content of the Lagrangian

Summary

Our Lagrangian is[16]

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left\{\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\right\} + \left(\mathsf{D}_\mu\Phi\right)^\dagger\left(\mathsf{D}^\mu\Phi\right) - V(\boldsymbol{\Phi}, \boldsymbol{\Phi}^\dagger) \qquad (6.3.1)$$

where the field $\boldsymbol{\Phi}$ is a d-dimensional vector Φ_j so $i, j = 1, 2, \ldots, d$. The various terms are defined to be

$$\begin{aligned} \mathsf{D}_\mu(x) &:= \partial_\mu \mathbb{1} - ig\mathsf{W}_\mu(x) && (6.3.2)\\ \mathsf{F}_{\mu\nu} &:= (\partial_\mu\mathsf{W}_\nu) - (\partial_\nu\mathsf{W}_\mu) - ig[\mathsf{W}_\mu, \mathsf{W}_\nu] && (6.3.3)\\ &= (\mathsf{D}_\mu\mathsf{W}_\nu) - (\mathsf{D}_\nu\mathsf{W}_\mu) && (6.3.4)\\ &= \frac{i}{g}[\mathsf{D}^\mu, \mathsf{D}^\nu] \quad \text{(for simple group only)} && (6.3.5) \end{aligned}$$

Note that *all* the indices (a, i, μ etc.) are contracted in each term in the Lagrangian. This indicates that the Lagrangian is invariant, is a scalar, with respect to the symmetry transformations (internal for a, i, or space-time for μ) associated with these indices. Also the last form for the field tensor is very useful for proving its symmetry properties but note the comments on non-abelian product groups below in section 6.4.

You can write any algebra valued object in two forms: (a) as real functions of space-time with an $a = 1, 2, \ldots, \dim(G)$ index as well as any space-time indices, and (b) as an algebra element, a matrix, with space-time indices and space-time argument:

$$\mathsf{W}_\mu(x) = W^a_\mu(x)\mathsf{T}^a, \qquad \mathsf{F}_{\mu\nu} = F^a_{\mu\nu}\mathsf{T}^a, \qquad -\frac{1}{2}\mathrm{Tr}\{\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \qquad (6.3.6)$$

[15] If the internal symmetry is abelian then all generators are in the Cartan subalgebra, i.e. everything commutes.

[16] Note we could have chosen a different coefficient for the F^2 term but we would just have to absorb this in a redefinition of $g \in \mathbb{R}$, $\partial_\mu g = 0$, is a coupling constant. At this stage, it reflects our ability to rescale kinetic terms for $\mathsf{W}^{\mu a}(x)$ and $\Phi(x)$ relative to one another without affecting gauge invariance.

The local symmetry transformations are the unitary matrices $\mathsf{U} = \mathsf{U}(x) \in G$ (the potential $V(\mathbf{\Phi}, \mathbf{\Phi}^\dagger)$ is assumed to be invariant), and the fields transform as

$$\mathbf{\Phi} \rightarrow \mathbf{\Phi}' = \mathsf{U}.\mathbf{\Phi} \tag{6.3.7}$$

$$\mathsf{D}_\mu \rightarrow \mathsf{D}'_\mu = \mathsf{U}.\mathsf{D}_\mu.\mathsf{U}^{-1} \tag{6.3.8}$$

$$\mathsf{W}_\mu \rightarrow \mathsf{W}'_\mu = \mathsf{U}.\mathsf{W}_\mu.\mathsf{U}^{-1} - \frac{i}{g}(\partial_\mu \mathsf{U}).\mathsf{U}^{-1} \tag{6.3.9}$$

$$\mathsf{F}_{\mu\nu} \rightarrow \mathsf{F}'_{\mu\nu} = \mathsf{U}.\mathsf{F}_{\mu\nu}.\mathsf{U}^{-1} \tag{6.3.10}$$

Since the field $\mathbf{\Phi}$ is in a d-dimensional representation the matrices here are d by d matrices, T^a_{jk}, $(j,k = 1,2,\ldots,d)$. There are dim(G) generators labelled by $a = 1,2,\ldots,\dim(G)$.

Gauge Fields

It is the real valued fields $W^a_\mu(x)$ which are the direct non-abelian versions of the gauge field $A^\mu(x)$ (both are also called vector fields). Thus there is *one photon-like field per local symmetry generator*. Each of these $\dim(G)$ gauge fields transforms as Lorentz vector, just like the abelian gauge field used for the photon. Thus they represent spin 1 particles but with two physical particles or modes corresponding to the two polarisations in 3+1 space-time dimensions. So two of the four real functions of space-time for each value of a, are unphysical and must be dealt with by gauge fixing.

Note that $\partial_\mu \mathsf{U} = 0$ is a special case of local symmetry so all the conclusions drawn about global symmetry transformations still apply in the local case. In particular the observable conserved currents are fixed by the diagonal generators, those in the Cartan subalgebra. The conserved currents pick up contributions from every field that has a non-trivial global transformation, as equation (4.1.8) shows.

Recalling that the Noether current gets a contribution from any field with a non-zero global transformation, we note that this means such fields have non-zero charge. If we look at the transformation for the W^a_μ field (6.3.9) we see that unlike the abelian case, $\mathsf{W}' \neq \mathsf{W}$ even for global transformations because of the $\mathsf{U}.\mathsf{W}_\mu.\mathsf{U}^{-1}$ term. Thus non-abelian gauge fields have quadratic contributions to currents just like a scalar field, i.e. non-abelian gauge bosons carry non-abelian charges. This is to be contrasted with abelian gauge fields where because all the group elements commute the charge on an abelian gauge boson is always zero, that is $\mathsf{W}' = \mathsf{U}.\mathsf{W}_\mu.\mathsf{U}^{-1} = e^{i\theta}\mathsf{W}_\mu.e^{-i\theta} = \mathsf{W}$. Global transformations produce no change in an abelian gauge field so produce no quadratic contribution to the current. This is the symmetry reason behind the physical observation that a photon has no electric charge. One can follow this trough further. Namely the global infinitesimal change in a non-abelian gauge field is

$$\partial W^b_\mu = i\mathsf{A}_{bc} W^c_\mu = i\epsilon^a \left(\mathsf{T}^a_{(W)}\right)_{bc} W^c_\mu \tag{6.3.11}$$

with the generators for the W fields being given in terms of the structure constants for the symmetry group

$$\mathsf{T}^a_{bc} = -if^{abc} \tag{6.3.12}$$

Compare this to what we would have for a scalar field

$$\partial\phi_i = i\mathsf{A}_{ij}\phi_j = i\epsilon^a \left(\mathsf{T}^a_{(phi)}\right)_{ij} \phi_j \tag{6.3.13}$$

and we see that the gauge fields lie in the adjoint representation, defined by (6.3.12) and which exists for all Lie groups. The gauge fields have the charges associated with this representation too.

Another effect of the non-commuting generators in the non-abelian case is seen in the $-\frac{1}{2}\mathrm{Tr}\left\{(\mathsf{F}_{\mu\nu})^2\right\}$ kinetic term which contains

- Quadratic field terms: $\mathsf{W}^{\mu a}$ kinetic terms
- Cubic and higher field terms: (W^3) and (W^4) pure gauge field interactions present if

$$\left[\mathsf{T}^a, \mathsf{T}^b\right] \neq 0 \tag{6.3.14}$$

These describe the direct interaction of force carry particles. The picture is that the force carrying fields, the gauge bosons, interact only with particles carry the relevant charges. A photon interacts with objects carrying electric charge. The photon does not interact with itself because it has no electric charge. However, a non-abelian gauge boson, such as the gluons of QCD, will have the cubic and quartic terms from the F^2 kinetic term. We can again see this as the force carrying particles of a non-abelian local symmetry having non-abelian charges so they can interact with themselves directly. The gluons of QCDcarry 'colour' charges just as the matter fields, the quarks, do. Non-abelian gauge bosons by themselves with no other matter fields provide a non-trivial interacting system. To be more precise, the gauge bosons lie in the adjoint representation of the gauge group.

The final observation for the gauge bosons is that there is again no mass term allowed by local symmetry, no $m^2 \mathsf{W}_{\mu a} \mathsf{W}^{\mu}_{a}$. So local symmetry does not allow a mass term for any gauge boson, abelian or non-abelian since such a term is not invariant under G.

Scalar Fields

In this simple model with a complex scalar field, there are $2d$ scalar fields/particles (d particles and d anti-particles) of spin zero where $\mathbf{\Phi}$ is a d dimensional representation of group G.

Real fields can also be involved in non-abelian theories and in exactly the same way we would have

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left\{(\mathsf{F}_{\mu\nu})^2\right\} + \frac{1}{2}\left|\mathsf{D}_\mu \boldsymbol{\phi}\right|^2 - V(\boldsymbol{\phi}) \tag{6.3.15}$$

If $\boldsymbol{\phi}$ was d-dimensional and V again depended only on the length of the field vector $|\boldsymbol{\phi}|$, then we would have $O(d)$ symmetry. The number of physical gauge degrees of freedom fields would still be twice the dimension of the group ($d(d-1)/2$ for $O(d)$) but there would now be just d scalar particles.

The scalar field kinetic terms, $|\mathsf{D}_\mu \mathbf{\Phi}|^2$, contain the quadratic scalar field derivatives which are the $\mathbf{\Phi}$ kinetic terms. Just like the abelian case, there are also cubic and quartic terms in the fields: $g\Phi_i^\dagger \Phi_j \mathsf{W}_{\mu a}$ and $g^2 \Phi_i \Phi_j^\dagger \mathsf{W}_{\mu a} \mathsf{W}_{\nu b}$ type interactions.

The scalar potential $V(|\mathbf{\Phi}|)$ is not altered by the local or global nature of the problem[17]. So here is the mass term for $\mathbf{\Phi}$ field, and self-interactions $\lambda(\mathbf{\Phi})^4$.

6.4 Product Groups

The example above was for a simple Lie group. For product groups every element can be written as a product of two parts which mutually commute, for simple groups this is not so. In terms of the Lie algebra of a product group one can split the generators T^a into mutually commuting sets. We will consider here a product of just two simple groups as the principles are easily generalised to larger product groups. So we have

$$G_1 \times G_2 \qquad \Leftrightarrow \qquad [\mathsf{T}^{a_1}, \mathsf{T}^{a_2}] = 0 \qquad \begin{array}{c} a_1 = 1, \ldots, \dim(G_1) \\ a_2 = \dim(G_1) + 1, \ldots \dim(G_1) + \dim(G_2) \end{array} \tag{6.4.1}$$

[17] This is if it contains no derivatives. In theories renormalisable in 3+1 dimensions this is always true.

The fact that the group has two essentially independent parts means that the gauge bosons, each of which is intimately linked to one of the generators of the local symmetry, also split into two sets, each of which acts completely independently from the other. For instance one finds that each part of the product group may be associated with its own independent gauge coupling constant

$$\mathsf{D}_\mu = \partial_\mu - ig_1 \mathsf{T}^{a_1}\mathsf{W}^{\mu a_1}(x) - ig_2 \mathsf{T}^{a_2}\mathsf{W}^{\mu a_2}(x) \tag{6.4.2}$$

In the same way, we find that the field tensor splits into two mutually commuting parts as

$$\begin{aligned}
\mathsf{F}_{\mu\nu} &= (\partial_\mu W^a_\nu \mathsf{T}^a) - \left(\partial_\nu \mathsf{W}^a_\mu \mathsf{T}^a\right) - ig[\mathsf{T}^a \mathsf{T}^b] W^a_\mu, W^b_\nu && (6.4.3)\\
&= \mathsf{F}^{(1)}_{\mu\nu} + \mathsf{F}^{(2)}_{\mu\nu} && (6.4.4)\\
\mathsf{F}^{(1)}_{\mu\nu} &:= (\partial_\mu W^{a_1}_\nu \mathsf{T}^{a_1}) - \left(\partial_\nu \mathsf{W}^{a_1}_\mu \mathsf{T}^{a_1}\right) - ig[\mathsf{T}^{a_1} \mathsf{T}^{b_1}] W^{a_1}_\mu, W^{b_1}_\nu && (6.4.5)\\
\mathsf{F}^{(2)}_{\mu\nu} &= +(\partial_\mu W^{a_2}_\nu \mathsf{T}^{a_2}) - \left(\partial_\nu \mathsf{W}^{a_2}_\mu \mathsf{T}^{a_2}\right) - ig[\mathsf{T}^{a_2} \mathsf{T}^{b_2}] W^{a_2}_\mu, W^{b_2}_\nu && (6.4.6)
\end{aligned}$$

The kinetic term for the gauge fields is then written as

$$-\frac{1}{2}\mathrm{Tr}\left\{\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\right\} = -\frac{1}{2}\mathrm{Tr}\left\{\mathsf{F}^{(1)}_{\mu\nu}\mathsf{F}^{(1)\mu\nu}\right\} - \frac{1}{2}\mathrm{Tr}\left\{\mathsf{F}^{(2)}_{\mu\nu}\mathsf{F}^{(2)\mu\nu}\right\} \tag{6.4.7}$$

Example 10 $\mathbf{SU(2)} \times \mathbf{U(1)}$

Suppose we had a complex doublet $\Phi_i(x) \in \mathbb{C}$, $(i = 1,2)$ and the scalar potential V was a function of $|\boldsymbol{\Phi}|$ only, then we would have a global $U(2) \cong SU(2) \times U(1)$ symmetry with four generators, three for SU(2) T^{a_1} $(a_1 = 1,2,3)$ which commute with the one for U(1) $\mathsf{T}^{a_2=4}$.

In two-dimensional representation used for the scalar doublet we have

$$\mathsf{T}^{a_1} = \frac{1}{2}\boldsymbol{\tau}^{a_1} \qquad (a_1 = 1,2,3) \qquad\qquad \mathsf{T}^{a_2=4} = \frac{1}{2}q\mathbb{1} \tag{6.4.8}$$

where the $\boldsymbol{\tau}^{a_1}$ are the three Pauli matrices. The group elements for the scalar symmetry transformations are

$$\exp\{i\theta\}\exp\{\frac{i}{2}\varepsilon^{a_1}\tau^{a_1}\} \tag{6.4.9}$$

a combination of a phase factor from $U(1)$ and the usual 2×2 special unitary matrix (i.e. from $SU(2)$).

If we made this theory locally invariant under the full $U(2) \cong U(1) \times SU(2)$ symmetry then there would be eight gauge boson modes usually described as four photon-like particles (one per local symmetry generator) each with two degrees of freedom. There would also be four scalar particles, two particle/anti-particle pairs. The (unbroken) local symmetry tells us that the gauge bosons are all massless while the scalars must all have the same mass.

Finally there are two charges, one for the $U(1)$ symmetry $\mathsf{T}^{a_2=4}$ and one for the diagonal $SU(2)$, say $\mathsf{T}^{a_1=3}$. The four scalars would have all possible combinations of $(\pm 1, \pm 1)$ for the two charges.

The real gauge bosons fall into two sets. One is linked with the $U(1)$ generator so is just like the photon and has no charge under the $U(1)$. However it has nothing to do with the SU(2) symmetry (the U(1) generators commute with the SU(2) ones) so the U(1) gauge boson must lie in the trivial representation of SU(2), $\mathsf{T}^{a_1} = 0$. A quick check will indeed show it does not transform under $SU(2)$ transformations. Remember *all* particles, whatever their origin, lie in irreps of the symmetry group, gauge bosons are no exception.

Conversely the $SU(2)$ gauge bosons do not transform under the U(1) so they have zero $U(1)$ charge. However they are all mixed by the SU(2) transformations so they must lie in a three-dimensional irrep of $SU(2)$. This is the adjoint representation. Further looking at the three dimensional irrep of SU(2) we see that if we make $\mathsf{T}^{a_1=3}$ diagonal it tells us the SU(2) gauge bosons have three SU(2) charges, $+1, 0$ and

-1. Since two of the SU(2) fields have the same mass and opposite charge you can guess that the charge eigenstates are complex gauge fields, a particle/anti-particle pair. We would find just as in the case of moving from two real scalar fields to their one complex field, we the gauge field charge eigenstates are built out of combinations such as $(W^(a_1 = 1)_\mu \pm iW^(a_1 = 3)_\mu)$ with the third gauge field $W^(a_1 = 2)_\mu$ being massless and zero charge.

This simple $U(2)$ model is in fact the simplest possible Higgs sector of the EW model.

□

6.5 Multiple matter fields

So far, we have had *one* matter field $\mathbf{\Phi}$. In reality we have many more, e.g. $\Theta(x) \in \mathbb{C}$; *but* these need not be in the same representation:

$$\begin{array}{cccc} \Phi_i & i = 1, \ldots, d_\Phi & \Leftrightarrow & \mathsf{T}_{(\mathbf{\Phi})} \\ \Theta_j & j = 1, \ldots, d_\Theta & \Leftrightarrow & \mathsf{T}_{(\Theta)} \end{array}$$

However, they still have *one gauge field* for each generator. We must therefore construct a D_μ for each scalar field out of the same gauge boson (i.e. vector) field $\mathsf{W}^{\mu a}(x)$ and *relevant* generators:

$$\mathsf{D}^\mu_{(\theta)} = \partial^\mu - igW^{\mu a}(x)\mathsf{T}^a_{(\theta)} \tag{6.5.1}$$

so $\left|\mathsf{D}^\mu_{(\theta)}\mathbf{\Theta}\right|^2$ is the kinetic term. There is only one set of gauge fields $\mathsf{W}^{\mu a}(x)$ as there is one for every independent symmetry transformation, i.e. one per generator. The number of distinct matter fields and the nature of the different irreps in which they lie make no difference, the gauge bosons are completely fixed buy the local continuous symmetry. This now suggests that our use of the matrix valued fields $\mathsf{W}_\mu = W^a_\mu \mathsf{T}^a$ is fine for a covariant derivative as there the generator needed is specifically linked to relevant the matter field, each matter field needing its own covariant derivative. However when in section 6.2.1 we used covariant derivatives or matrix valued fields to build the field tensor, as in (6.3.6), we were being a little too slick in some senses. The field tensor is a pure gauge field term and has nothing to do with any matter field. Indeed it gives a non-trivial interacting theory all by itself. We could have used *any* non-trivial representation, any non-trivial covariant derivative (even one not associated with any physical particles) in our definitions of the field tensor, and the derivations in section 6.2.1 would still have worked.

6.6 Anomalies

The approach of book has been to assume that most classical features remain qualitatively true in the full quantum theory. While some arguments were put forward as to why this might be so, it is clearly a statement that can only be verified by checking the quantum field theory. Since this is not a text on QFT, we can not do more than flag when the classical analysis does not tell us the true story.

When quantising a classical theory, one should always wonder if all the symmetries seen in the classical theory are still symmetries in the full interacting quantum theory. When a classical symmetry is not present in the quantum theory then we say that the theory has an **anomaly**. We are not talking about a symmetry which is unbroken in a classical theory but broken in the quantum, the symmetry is still present in both cases. Rather it is genuine situation where a symmetry is destroyed by quantum effects. The abelian U(1) factors are usually the problem

6.7 Questions

Q6.1. Trying to combine local gauge fields and matter: Abelian case

Unless otherwise asked, consider arbitrary and *not* infinitesimal local $U(1)$ transformations in this question.

Consider the theoretical models known as *Scalar QED* or the*Abelian Higgs model*, for which the Lagrangian contains two fields, a complex Lorentz scalar, $\Phi(x)$, and a real Lorentz vector, $A^\mu(x)$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger\Phi) \tag{6.7.1}$$

(i) Write down the definitions of $F_{\mu\nu}$ and D_μ in terms of the gauge field $A_\mu(x)$ associated with local $U(1)$ invariance, the charge e, and in terms of ∂_μ. Hence show that $F_{\mu\nu} = \frac{-i}{e}[D_\mu, D_\nu]$.

(ii) Write down the how the fields A and Φ transform under an arbitrary (not infinitesimal) local $U(1)$ group element g.

(iii) Calculate how $D_\mu\Phi$ transforms, and hence find how the operator D_μ transforms. (Note that in my lectures on non-abelian local symmetry, I used these properties as a starting point. I started by demanding that the non-abelian generalisation of D_μ transformed in the same way as the local case because this ensures that $|D_\mu\Phi|^2$ is invariant).

(iv) Prove that $F_{\mu\nu}$, $(D_\mu\Phi)^\dagger(D^\mu\Phi)$ and $V(\Phi^\dagger\Phi)$ are each separately invariant under these local $U(1)$ transformations.

(v) Prove that $m^2A^\mu A_\mu$ is not an invariant, and hence local symmetry does not allow an explicit mass term of this type to appear.

(vi) Calculate the classical equations of motion for the fields A^μ, Φ and $\Phi^\dagger$. What is the contribution coming from the F^2 term?

(vii) Calculate the Noether current, J_N^ν, associated with the global $U(1)$ invariance present in the Lagrangian.

(viii) Hence show that $\partial_\mu F^{\mu\nu} = -eJ_N^\nu$ is one of the classical equations of motion.

(ix) Remember that some of Maxwell's equations can be obtained as the equation of motion for A with the Lagrangian

$$\mathcal{L}_{Maxwell} = -(1/4)F^{\mu\nu}F_{\mu\nu} - J_L^\mu A_\mu \tag{6.7.2}$$

What conditions must be put on J_L if this is to be the physical current which appears in the Maxwell equations?

(x) Try to rewrite the scalar QED Lagrangian as

$$\mathcal{L} = -(1/4)F^{\mu\nu}F_{\mu\nu} - J_L^\mu A_\mu + \text{ (pure matter terms)} \tag{6.7.3}$$

Identify J_L and show it is not the Noether current. Based on the answer to (viii), why does J_L not appear as the physical current in Maxwell's equations in scalar QED? What physics should lead us to expect J_N to appear in the Maxwell equations?

Q6.2. Gauge fields and matter: Non-Abelian case

Unless otherwise asked, consider arbitrary and *not* infinitesimal transformations of an arbitrary Lie group.

Due to printing limitations and a desire to avoid lengthy $i, j = 1, 2, \ldots, d$ indices (the ones associated with the representation of the group or algebra), I will usually drop these indices. I will denote the presence of such indices by $\Phi_i \equiv \mathbf{\Phi}$ for vectors, and $g_{ij} \equiv \mathsf{U}$ for matrices, whereas in my lectures I used a single and double underline symbol respectively.

Consider a theoretical model with d complex Lorentz scalar fields, $\Phi_i(x)$ $(i = 1, 2, \ldots, d)$, transforming under a d dimensional representation of some Lie group. Start from the Lagrangian $\mathcal{L}_{\Phi,\Phi W}$ which is

$$\mathcal{L}_{\Phi,\Phi W} = (\mathsf{D}_\mu \mathbf{\Phi})^\dagger (\mathsf{D}^\mu \mathbf{\Phi}) - V(\mathbf{\Phi}^\dagger \mathbf{\Phi}) \tag{6.7.1}$$

Assume that the potential V is globally invariant under some arbitrary global Lie group transformation

$$\mathbf{\Phi}(x) \to \mathbf{\Phi}'(x) = \mathsf{U}.\mathbf{\Phi}(x) \tag{6.7.2}$$

(i) Show that $\mathcal{L}_{\Phi,\Phi W}$ is invariant under local transformations, $\mathsf{U} = \mathsf{U}(x)$ if the covariant derivative, D, transforms as

$$\mathsf{D} \to \mathsf{D}' = \mathsf{U}.\mathsf{D}.\mathsf{U}^{-1} \tag{6.7.3}$$

(ii) The non-abelian gauge fields, the Lie algebra valued functions and Lorentz vectors, $\mathsf{W}_\mu(x)$, are related to the covariant derivative through

$$\mathsf{D}_\mu(x) = \partial_\mu \mathbb{1} - ie\mathsf{W}_\mu(x) \tag{6.7.4}$$

where e is the charge associated with this symmetry. How does $\mathsf{W}_\mu(x)$ transform?

(iii) Hence derive the transformation law for infinitesimal group elements for the gauge field written as a real number valued function and Lorentz vector, $W^a_\mu(x)$, defined through

$$\mathsf{W}_\mu(x) = W^a_\mu(x).\mathsf{T}^a \tag{6.7.5}$$

These $W^a_\mu(x)$ fields are the ones most like the non-abelian generalisation of a photon.

(iv) Hence show that a simple mass term, $m^2 W^a_\mu(x) W^{\mu,a}(x)$ is not invariant.

(v) The non-abelian field strength tensor, its Lie algebra valued version being $\mathsf{F}^{\mu\nu}$, can be defined to be

$$\mathsf{F}^{\mu\nu} = \frac{i}{e}[\mathsf{D}^\mu, \mathsf{D}^\nu] \tag{6.7.6}$$

Find the transformation law for $\mathsf{F}^{\mu\nu}$ and hence show that $\mathrm{Tr}\{\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\}$ is invariant under these non-abelian transformations.

(vi) Expand out the definition (6.7.6) in terms of partial derivatives and $\mathsf{W}_\mu(x)$ to show that it is not a differential operator.

(vii) Consider the field strength tensor written as a real valued function $F^{\mu\nu,a}(x)$ where $\mathsf{F}^{\mu\nu}(x) = F^{\mu\nu,a}(x).\mathsf{T}^a$. By considering the case of an infinitesimal transformation, show that $F^{\mu\nu,a}(x)$ transforms in the adjoint representation, in which the generators $T^a_{ij} \equiv if^a_{bc}$, $i = b, j = c$ and $f^{abc} = f^a_{bc}$ are the structure constants for the non-abelian Lie algebra.

(viii) The kinetic terms for the $W^{\mu,a}(x)$ are contained in the term

$$\mathcal{L}_W = -\frac{1}{4}F^{\mu\nu,a}(x)F^a_{\mu\nu} \tag{6.7.7}$$

By expanding this in terms of the gauge fields, $W^a_\mu(x)$, show that $\mathcal{L}_W$ contains an appropriate kinetic term for the field but also cubic and quartic interaction terms. Prove that these cubic and quartic interaction terms are zero only if the symmetry group is abelian.

(ix) Derive the classical equations of motion for the fields, $\Phi_i(x)$ and $W^a_\mu(x)$ if the total Lagrangian , $\mathcal{L}$, is given by

$$\mathcal{L} = \mathcal{L}_W + \mathcal{L}_{\Phi,\Phi W} \tag{6.7.8}$$

noting exactly what contributions come from each of the two parts of $\mathcal{L}$. Leave the equations in as simple a notation as possible, leaving F, D in if this helps.

(x) Interpreting the equation of motion for the W field as the non-abelian generalisation of Maxwell's equations, $\partial_\mu F^{\mu\nu} = -J^\nu$, is J a conserved current? Write $J = J_W + J_{\Phi,\Phi W}$ where J_W comes from the $\mathcal{L}_W$ term in L and $J_{\Phi,\Phi W}$ is the $\mathcal{L}_{\Phi,\Phi W}$ contribution. Assuming J is the Noether current (as it was in the abelian case) what does this tell you about the charges on non-abelian gauge fields? Compare this with the abelian case.

(xi) Calculate the Noether current directly. It should be consistent with the answer to the previous part, but you may have to think carefully about the definition/derivation of the Noether current formula. Thinking about the pure gauge field case (dropping Φ completely and just using $\mathcal{L}_W$ as the globally invariant Lagrangian) may help.

Q6.3. Non-Simple Groups and local charges

Up to now all groups I have considered have been simple Lie groups except for the global $U(1)_1 \times U(1)_2$ considered in another question. When one has a non simple group $G_1 \times G_2 \times \ldots \times G_n$ where each G_i is a simple group (so that generators commute with all others except those generating the same simple subgroup). In this case there is some extra freedom in constructing the covariant derivative, which takes the form

$$D_\mu = \partial_\mu - ie_1 T^{a_1} W^{a_1}_\mu - ie_2 T^{a_2} W^{a_2}_\mu \ldots - ie_n T^{a_n} W^{a_n}_\mu \tag{6.7.1}$$

where the a_i run through generators generating the i-th simple subgroup G_i $(i = 1, \ldots n)$. Thus $[T^{a_i}, T^{a_j}] = 0$ unless $i = j$. Note that there are now n *independent coupling constants* associated with the gauge fields, $e_1, e_2, \ldots, e_n$. Show that (in the notation from the previous question)

$$\mathcal{L} = \mathcal{L}_W + \mathcal{L}_{\Phi,\Phi W} \tag{6.7.2}$$

is invariant under arbitrary (not infinitesimal) gauge transformations.

Chapter 7

Local symmetry breaking (*Higgs-Kibble mechanism*)

7.1 SSB of local Abelian symmetry

Consider our Lagrangian of a complex scalar field with Abelian U(1) local symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger (D^\mu\Phi) - V(\Phi^\dagger\Phi) \tag{7.1.1}$$

The fields are $A^\mu(x) \in \mathbb{R}$ and $\Phi(x) \in \mathbb{C}$. In the context of SSB, the model often called the **Abelian Higgs model**[1].

Using the same reasoning as for the global case in section 5, the lowest energy field configurations are still space-time independent ones, $\partial^\mu\Phi_0 = 0$. So for spontaneous symmetry breaking, the scalar potential, $V(|\Phi|)$, is minimised at[2]

$$\Phi_0 = e^{i\theta}\frac{v}{\sqrt{2}}, \quad \partial^\mu v = 0. \tag{7.1.2}$$

For instance, the usual form for V with negative mass gives a non-zero v

$$V(\Phi^\dagger\Phi) = m^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \qquad (m^2 < 0, \lambda > 0) \qquad \Rightarrow \qquad v = \sqrt{-\frac{m^2}{2\lambda}} \tag{7.1.3}$$

We now have to consider the lowest energy of the gauge field. We will take the minimum energy of the gauge field to be at $A^\mu(x) = 0$ i.e. we assume that in QFT we have that

$$\left\langle 0_B \left| \hat{A}^\mu \right| 0_B \right\rangle = 0^\mu. \tag{7.1.4}$$

If the quantum vacuum is Lorentz invariant[3] then the right hand side must be a Lorentz vector. Since we (almost) always want the physical results to respect Lorentz invariant, the only way we can leave Lorentz

[1] The model is identical to SQED of (6.1.26) and the change of name merely indicates that we are looking at SSB in the model.

[2] In the quantum theory $\Phi_0 = \left\langle O_B \left| \hat{\Phi} \right| O_B \right\rangle = e^{i\theta}\frac{v}{\sqrt{2}}$. See section 5.1.

[3] The Lorentz invariant nature of the quantum vacuum of relativistic QFT, is one of the key assumptions of QFT. If this expectation value was equal to some non-zero four vector, say n^μ, then the quantum vacuum is picking a preferred frame of reference, the one where n^μ has no spatial components. In such a case the application of special relativity is much more complicated and quantities such as $E^2 - p^2 = p^\mu p_\mu = m^2$ would no longer be a constant but would depend on $p^\mu n_\nu$, or equivalently, on velocity. Of course 'mass' does depend on velocity in condensed matter physics, they do not have simple dispersion relations, but in condensed matter the rest frame of the sample is a special frame. Likewise, adding gravity and upgrading to general relativity causes new interpretational problems.

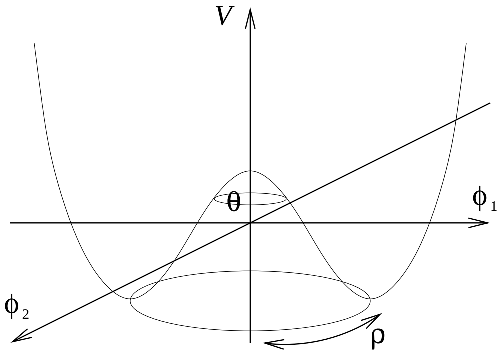
symmetry intact is to have the right-hand side zero. Note a similar argument applies to any field *except* a scalar filed since only that field is a Lorentz invariant. Thus in relativistic QFT, it is *only* scalar fields which can acquire non-zero expectation values and *only* scalar fields which are responsible for SSB. We can proceed as before with the scalar field and see how SSB effects the abelian gauge field.

Shifting the scalar field

First we switch to field variables, $\sigma(x)$ and $\rho(x)$, representing perturbations around the true vacuum

$$\Phi = \frac{e^{i\theta}}{\sqrt{2}} \left(v + \sigma(x) + i\rho(x) \right) \qquad \sigma(x), \rho(x) \in \mathbb{R} \qquad \partial_\mu \theta = 0 \tag{7.1.5}$$

The scalar potential V has not changed from the case of global symmetry breaking so the σ field still represents the massive Higgs perturbations while the ρ represents oscillations in directions given by broken symmetries which in turn ensures they are flat directions, i.e. massless perturbations. The v is the vacuum solution given in (7.1.2). Note the presence of an overall global phase $e^{i\theta}$ in our definitions. It does not appear in the new Lagrangian since that is invariant under global phase shifts.



Substituting into (7.1.1) we find

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} \tag{7.1.6}$$

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \left(\partial_\mu \sigma \right)^2 + \frac{1}{2} \left(\partial_\mu \rho \right)^2 - evA_\mu \left(\partial^\mu \rho \right) + \frac{1}{2} e^2 v^2 A_\mu A^\mu + \frac{1}{2} m_\sigma^2 \sigma^2 \tag{7.1.7}$$

The cubic and quartic terms are in $\mathcal{L}_{\rm int}$ and are not important here but to illustrate their nature, we use the form (7.1.3) for the scalar potential to see

$$\mathcal{L}_{\rm int} = eA_\mu \rho \left(\partial^\mu \sigma \right) - eA_\mu \sigma \left(\partial^\mu \rho \right) + gv\sigma A_\mu A^\mu - 4\lambda v \sigma \rho^2 \tag{7.1.8}$$

$$+ \frac{1}{2} g^2 \sigma^2 \sigma A_\mu A^\mu + \frac{1}{2} g^2 \rho^2 \sigma A_\mu A^\mu - \lambda (\sigma^2 + \rho^2)^2 \tag{7.1.9}$$

The pure scalar terms coming from $V(|\Phi|)$ are exactly as we found before in section 5.1. Thus *independent* of the form of the scalar potential, by choosing v correctly so we are perturbing around the minimum value of the scalar potential, terms linear in the scalar fields are zero, and the σ field has a non-zero mass ($m_\sigma^2 = -2m^2$ for V of (7.1.3)). There is no mass term for the ρ field which was the Goldstone boson in the global case but is now called the **would-be Goldstone boson**. The gauge field, A^μ, adds two new features. First it has a mass term with $m_\gamma = ev$, but secondly there is ρ-A^μ quadratic mixing term.

The Mass problem

Since the A^μ and ρ fields (but *not* Higgs field σ) mix at the quadratic level, so we cannot interpret these may not represent physical modes. So the mass terms for these fields, ev for the photon and zero for the would-be-Goldstone ρ, may not have any physical meaning.

What then are the physical particles and what are their masses? To see what these are, it is better to work in terms of a new set of fields which will then show us a way forward. Consider the scalar fields written in polar field 'coordinates', i.e.

$$\Phi = \frac{v + r(x)}{\sqrt{2}} e^{i\chi(x)} \qquad r(x), \chi(x) \in \mathbb{R} \tag{7.1.10}$$

where r and χ are real scalar fields. These two real functions are sufficient for a complete description of the complex field so we are losing no information[4] and we see that

$$D_\mu \Phi := (\partial_\mu - ieA_\mu)\,\Phi = \frac{1}{\sqrt{2}} \left[(\partial_\mu r)\, e^{i\chi} + i\,(\partial_\mu \chi)\,(v + r) e^{i\chi} - ieA_\mu (v + r) e^{i\chi} \right] \tag{7.1.11}$$

$$= \frac{1}{\sqrt{2}} (\partial_\mu r)\, e^{i\chi} - \frac{ie(v + r)}{\sqrt{2}} \left(A_\mu - \frac{1}{e} (\partial_\mu \chi) \right) e^{i\chi} \tag{7.1.12}$$

The term in parentheses is reminiscent of a gauge transformation and this suggests that we make a gauge transformation.

Unitary gauge

Define gauge transformed fields

$$\Phi' = U\Phi \qquad B^\mu = A'^\mu = A^\mu - \frac{1}{e} (\partial_\mu \chi) \tag{7.1.13}$$

where we choose the U(1) group element U to be

$$U = e^{-i\chi(x)} \qquad \theta(x) = \chi(x) \tag{7.1.14}$$

The transformed scalar field is pure real

$$\Phi' = \frac{v + r(x)}{\sqrt{2}} \in \mathbb{R} \tag{7.1.15}$$

However we know that this is a local symmetry transformation and that the Lagrangian is invariant so it has the identical form in the $B_\mu = A^p_\mu rime$ and Φ' fields

$$\mathcal{L} = -\frac{1}{4} \left(F' \right)^2 + \left| D'_\mu \Phi' \right|^2 - V(\Phi'^\dagger \Phi') \qquad \Phi' \in \mathbb{R} \tag{7.1.16}$$

but now Φ' is pure real. Writing this in terms of the shifted field $\Phi' = \frac{1}{\sqrt{2}}(v + r(x))$ gives

$$\mathcal{L} = -\frac{1}{4} \left(F' \right)^2 + \frac{1}{2} \left(\partial_\mu r \right)^2 + \frac{1}{2} (v + r)^2 B_\mu B^\mu + V \left(\frac{(v + r)^2}{2} \right) \tag{7.1.17}$$

That is, by making the gauge transformation to fields in the **unitary gauge** (7.1.13) we have eliminated the quadratic mixing term so we can now interpret these fields in terms of physical particles

[4] We will comment later about hidden difficulties in QFT with this trick.

Physics of the Unitary Gauge

The Abelian Higgs model with SSB, and after making the transformation to the Unitary gauge fields, so working in polar scalar field coordinates, shows us that the particle content is

- One *real* scalar field $r(x) = 1$ so *one* scalar particle (its own anti-particle) of mass given by a Taylor expansion of the potential

- One vector (or gauge) field B_μ with mass ev.

One apparently bizarre point of the form (7.1.17) is that it is mathematically *identical* to our original form (7.1.1) yet we seem to have lost a scalar mode. We can not possibly lose a physical degree of freedom just by using mathematical identities. The same physics, in this case degrees of freedom or number of particles, is encoded in either form, its just that some mathematical variables may show the physics more clearly than others. The resolution is that a *massive* gauge boson field represents three physical modes, which together with the last scalar mode $r(x)$ gives four degrees of freedom. This is the same as in the unbroken case which had two scalar modes $\Phi(x) \in \mathbb{C}$ and two massless physical gauge modes in A^μ. SSB in with local symmetry mixes scalar and gauge modes.

One way to see that massive gauge bosons have three physical modes, while in vacuo they have two, is to look at Maxwell's equations. Maxwell's equations then allow two transverse modes for EM waves in vacuo. When we consider EM fields in media, where we have a lower speed for these waves, *three* polarisations are allowed for EM waves: two transverse and one longitudinal. This is linked to our discussion about mass as in a relativistic context only massive particles have speeds less than the speed of light which is case for EM waves in media[5].

Also note we do *not* change physics by choosing different fields: we simply repackage it. While one field set may make the physics clear, another on may be better for calculations. Here a unitary gauge is good for physics but bad for calculations for two reasons. First, the polar fields do not range from $-\infty$ to $+\infty$ and the standard approach to quantisation must take this into account. Second in changing from cartesian to spatial coordinates in ordinary integrals, there is a non-trivial Jacobian needed. The same occurs in QFT where a non-trivial determinant of fields is needed in path integrals for instance. It is possible to work with polar fields and deal with these complications, but it is usually simpler to do calculations in the original $\Phi(x) \in \mathbb{C}$ and A^μ coordinates, and then deal with the fact that the unphysical modes are then a mixture of gauge and scalar modes, so identifying requires more effort (see [**?**] for a nice example of this in practice).

7.2 Non-Abelian SSB

We can now study SSB (spontaneous symmetry breaking) in a theory with a local non-abelian symmetry. The aim is to try and relate the particle content to the symmetry in the model, both before and after SSB, just Goldstone's theorem encapsulates for the case of global symmetry.

Consider

$$\mathcal{L} = (D_\mu \boldsymbol{\phi})^\dagger (D^\mu \boldsymbol{\phi}) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - V(\boldsymbol{\phi}) \tag{7.2.1}$$

where, from (6.3.6) $F^a_{\mu\nu} F^{a\mu\nu} = \frac{1}{2}\mathrm{Tr}\left\{\mathsf{F}^{\mu\nu}\right\}$. This theory is characterised by:

[5]While our basic conclusion that massive photons with speeds less than one have three polarisation modes, the situation is more complicated. Only in superconductors is the $U(1)$ gauge symmetry of electro-magnetism broken in the sense discussed here. In most materials, e.g. glass, the speed of EM waves is still less than in vacuo but this is due to the material being at rest in some special frame of reference. The application of special relativity symmetries is more complicated when such a frame exists. Understanding the implications of Lorentz symmetry in many-body field theory problems is the subject of **Thermal Field Theory** (or Finite Temperature Field Theory).

- d-dimensional representation real scalar field $\phi_i(x) \in \mathbb{R}$ $(i = 1, \ldots, d)$
  (a complex field is a straightforward generalisation)
- A local symmetry (or gauge) group G, generated by $\dim(G)$ generators T^a, $(a = 1, 2, \ldots, \dim(g))$.
- One photon-like particle per generator, $W^{\mu a}$ the non-abelian gauge bosons
- a scalar potential with a minimum at $\boldsymbol{\phi}(x) = \boldsymbol{\phi}_0$

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} = 0, \qquad \partial_\mu \boldsymbol{\phi}_0 = 0 \tag{7.2.2}$$

The first step is to find the broken and unbroken generators and hence discover the little group H. This is done exactly as in the global SSB case, namely by solving[6] for the largest set of orthogonal generators, $\{\mathsf{T}'^A\}$ which annihilate the vacuum. So we will suppose we have done this and the unbroken group elements and generators are

$$\mathsf{U}'\boldsymbol{\phi}_0 = \boldsymbol{\phi}_0, \qquad \mathsf{T}'^A\boldsymbol{\phi}_0 = 0 \qquad A = 1, \ldots, \dim(H) \tag{7.2.3}$$

These generate H, the little (sub-)group of G arising from SSB, i.e. $G \xrightarrow{\text{SSB}} H$.

The broken generators are the orthogonal set of generators which, when added to the unbroken generators $\{\mathsf{T}'^A\}$, complete the generation of the full symmetry group G. We choose this basis to be orthogonal and by definition, the broken generators are the ones which do not annihilate the vacuum. Thus broken generators generate group elements which alter the vacuum i.e.

$$\mathsf{U}''\boldsymbol{\phi}_0 \neq \boldsymbol{\phi}_0, \qquad \mathsf{T}''^Z\boldsymbol{\phi}_0 \neq 0, \qquad Z = \dim(H) + 1, \ldots, \dim(G) \tag{7.2.4}$$

For simplicity we will assume that we have already chosen to work in the $\{\mathsf{T}'\} \cup \{\mathsf{T}''\}$ basis so that

$$\mathsf{T}^a = \begin{cases} \mathsf{T}'^A, & a = A = 1, \ldots, \dim(H), \\ \mathsf{T}''^Z, & a = Z = (\dim(H) + 1), \ldots, \dim(G). \end{cases} \tag{7.2.5}$$

In the case of SSB of a global symmetry, section 5.1, we found that there was one massless Goldstone boson per broken generator T''^Z. Try $\boldsymbol{\phi}(x) = \boldsymbol{\phi}_0 + \boldsymbol{\eta}(x)$, $\eta_i(x) \in \mathbb{R}$. Thus

$$\mathsf{D}_\mu \boldsymbol{\phi} = \partial_\mu \boldsymbol{\eta} - ig\mathsf{W}_\mu \boldsymbol{\eta} - ig\mathsf{W}_\mu \boldsymbol{\phi}_0 \tag{7.2.6}$$

and

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu \boldsymbol{\eta}\right)\left(\partial^\mu \boldsymbol{\eta}\right) - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} & (7.2.7) \\ &\quad -\eta_i \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} \eta_j + ig\boldsymbol{\phi}_0 \mathsf{W}_\mu \left(\partial^\mu \boldsymbol{\eta}\right) - ig\left(\partial_\mu \boldsymbol{\eta}\right)^\dagger \mathsf{W}^\mu \boldsymbol{\phi}_0 & (7.2.8) \\ &\quad + \frac{1}{2}g^2 \boldsymbol{\phi}_0^\dagger \mathsf{W}_\mu \mathsf{W}^\mu \boldsymbol{\phi}_0 & (7.2.9) \\ &\quad + (\text{cubic, quartic terms}) & (7.2.10) \end{aligned}$$

where we have performed a Taylor expansion around $\boldsymbol{\phi}_0$. Note that all indices contract to give terms which are scalars under both space-time and local symmetries, i.e. they are invariant under symmetry transformations.

[6]Again note that this split into broken and unbroken generators, or equivalently the possible H little groups, depend in a non-trivial way on both the group G and the representation of the Higgs field $\boldsymbol{\phi}$. Not all subgroups of G may be allowed H for a given representation.

In the term on line (7.2.9) we see we have a mass matrix for the gauge bosons

$$g^2\boldsymbol{\phi}_0^\dagger \mathsf{W}_\mu \mathsf{W}^\mu \boldsymbol{\phi}_0 = W^{\mu a}(x)(\mathsf{M}^2)^{ab}W^{\mu b}(x), \quad (\mathsf{M}^2)^{ab} := [g^2\boldsymbol{\phi}_0^\dagger \mathsf{T}^a\mathsf{T}^b\boldsymbol{\phi}_0] \tag{7.2.11}$$

The mass matrix is seen in a mixing of the $a, b = 1, 2, \ldots, \dim(G)$ labels, Lorentz indices are not involved in the matrix structure. Thus for each value of μ, the gauge boson mass term is just like having a mass matrix for a $\dim(G)$ real-scalar field $\boldsymbol{\xi}$

$$\frac{1}{2}\boldsymbol{\xi}^\dagger(\mathsf{M}^2)\boldsymbol{\xi} = \frac{1}{2}\xi_a M^2_{ab}\xi_b \tag{7.2.12}$$

However, we have quadratic mixing of scalar $\boldsymbol{\eta}$ and gauge boson $W^{\mu a}$ fields in line (**??**) so preventing immediate identification of the particles. Notice that $\mathsf{W}^\mu = \mathsf{T}^a W^{\mu a}$ and so (remembering our choice (**??**)) these mixing terms are zero for $a = A \in \{1, 2, \ldots, \dim(H)\}$

$$\mathsf{T}'^A\boldsymbol{\phi}_0 = 0 \tag{7.2.13}$$

and only non-zero for $a = Z \in \{\dim(H) + 1, \ldots, \dim(G)\}$

$$\mathsf{T}''^Z\boldsymbol{\phi}_0 \neq 0 \tag{7.2.14}$$

Thus we will find we only have problems with the broken generators, the ones which in the global theory were linked to Goldstone bosons. We had a similar problem with the abelian example in the previous section 7.1 so let us try the same approach which worked in that case.

7.2.1 Unitary gauge

Following the abelian example, we split the scalar field fluctuations about the vacuum into those Goldstone like fluctuations $\theta^Z(x)$ along broken symmetry directions, and the rest. Unbroken generators have no effect (7.2.3). The flat directions of the scalar potential are given by the broken generators, precisely the reason why there is a Goldstone mode for each broken generator. Since it is these would-be Goldstone modes which are causing a problem it is a natural thing to do. We do this by noting that *only* the broken generators alter the vacuum (7.2.4) so out of all symmetry transformations, only they can be linked to some of the possible independent fluctuations in the scalar field. So we consider a subset of group elements of the form

$$\mathsf{U}'' = \exp\left\{i\varepsilon^Z\mathsf{T}''^Z\right\} \tag{7.2.15}$$

and since $\mathsf{U}''\boldsymbol{\phi}_0 \neq \boldsymbol{\phi}_0$ we can split the scalar field into two types of fluctuation

$$\boldsymbol{\phi}(x) = \exp\left\{i\theta^a(x)\mathsf{T}''^a\right\}(\boldsymbol{\phi}_0 + \boldsymbol{\sigma}(x)) \tag{7.2.16}$$

where the number of real fields $\theta^Z(x)$ is

$$b = g - h = \dim(G) - \dim(H) = \text{ number of broken generators.} \tag{7.2.17}$$

Since $\boldsymbol{\phi}(x)$ has d independent functions originally, and we have b independent $\theta^Z(x)$ functions, this means the d-dimensional $\boldsymbol{\sigma}(x)$ must be constrained to have just $d - b$ independent components if we are to avoid over counting. To remove those parts in $\boldsymbol{\sigma}$ lying along the broken symmetry directions, we demand that the vector $\boldsymbol{\sigma}$ is orthogonal (in the $i, j = 1, 2, \ldots, d$ space of the scalar field representation) to the direction of any of the fluctuations generated by any broken generator. Each broken generator defines an independent direction in the representation space of the Higgs field $\boldsymbol{\phi}$ as $\mathsf{U}''\boldsymbol{\phi}_0 - \boldsymbol{\phi}_0 \approx i\epsilon^Z\mathsf{T}^Z\boldsymbol{\phi}_0 \neq 0$ and as the ϵ^Z are independent, each $\mathsf{T}^Z\boldsymbol{\phi}_0$ is non-zero and defines another independent Goldstone fluctuation direction. So we must demand that $\boldsymbol{\sigma}$ obeys

$$\boldsymbol{\sigma}(x)^\dagger.\left(\mathsf{T}''^Z\boldsymbol{\phi}_0\right) = 0 \tag{7.2.18}$$

The point about using symmetry transformations to describe perturbations in the scalar field about the vacuum, is that we can make a gauge transformation with

$$\varepsilon^a(x) = \left\{ \begin{array}{cc} 0 & a = A \in \{1, 2, \ldots, \dim(H)\}, \mathsf{T}^a = \mathsf{T}'^a \\ -\theta^a(x) & a = Z \in \{\dim(H)+1, \ldots, \dim(G)\}, \mathsf{T}^a = \mathsf{T}''^a \end{array} \right. \tag{7.2.19}$$

Under this transformation

$$\boldsymbol{\phi}(x) \longmapsto \boldsymbol{\phi}'(x) = e^{i\epsilon^a \mathsf{T}^a} \boldsymbol{\phi}(x) = \boldsymbol{\phi}_0(x) = \boldsymbol{\phi}_0 + \boldsymbol{\sigma}(x) \tag{7.2.20}$$

and the gauge field transforms in the usual way, the $W^{\mu a}$ being replace by $W'^{\mu a}$. From gauge invariance of the Lagrangian we know that

$$|D_\mu \boldsymbol{\phi}|^2 = \left|D'_\mu \boldsymbol{\phi}'\right|^2 = \left|(\partial_\mu - ig\mathsf{W}')(\boldsymbol{\phi}_0 + \sigma)\right| \tag{7.2.21}$$

so that the original Lagrangian can be rewritten as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \boldsymbol{\sigma})^2 + ig\left[\boldsymbol{\phi}_0^{\mathsf{T}} \mathsf{T}^a (\partial_\mu \boldsymbol{\sigma}) - \left(\partial_\mu \boldsymbol{\sigma}^{\mathsf{T}}\right) \mathsf{T}^a \boldsymbol{\phi}_0\right] (W')^{\mu a} + \frac{1}{2} W'^a (\mathsf{M}^2)^{ab} W'^b + V(|\boldsymbol{\phi}_0 + \boldsymbol{\sigma}(x)|) \\ & -\frac{1}{4} F'^a_{\mu\nu} F'^{a\mu\nu} + O(\boldsymbol{\sigma}.\partial_\mu \boldsymbol{\sigma} W'^\mu, \boldsymbol{\sigma}^2 W'^\mu W'_\mu) \end{aligned} \tag{7.2.22}$$

The terms in the square brackets are zero by (7.2.18) as we are mixing small perturbations in the direction of the broken generators, $\mathsf{T}'^Z \boldsymbol{\phi}_0$, with those perturbations in a direction perpendicular to this, which was the definition of the $\boldsymbol{\sigma}$ condition (7.2.18). Thus *there are no mixing terms* and we can now interpret the particle content and the masses.

Gauge Boson Masses

The mass matrix for the physical gauge bosons $W'^{a\mu}(x)$ is

$$(\mathsf{M}^2)^{ab} \quad = \quad g^2 \boldsymbol{\phi}_0^{\mathsf{T}} \mathsf{T}^a \mathsf{T}^b \boldsymbol{\phi}_0. \tag{7.2.23}$$

Because we know the first $h = \dim(h)$ rows and columns $(a, b = A, B = 1, \ldots, h)$ are zero because these are associated with unbroken generators so we have

$$= \left(\begin{array}{cc} 0_{(h\times h)} & 0_{(h\times b)} \\ 0_{(b\times h)} & \mathsf{M}^2_{(b\times b)} \end{array} \right) \tag{7.2.24}$$

where $0_{(b\times h)}$ is a b row h column matrix of zeros and so forth. The $b \times b$ matrix $\mathsf{M}^2_{(b\times b)}$ is unconstrained by symmetry, the other zero blocks are forced to be zero because of the $h = \dim(H)$ unbroken local symmetry of the little group H. This implies that

- for each unbroken generator there is one massless gauge boson.
- for each broken generator there is one massive gauge boson.

The unbroken symmetry, H, is behaving exactly as unbroken gauge symmetries did in chapter 6 as each of its generators is linked to a massless gauge boson. To find the masses associated with the broken part of the group, we must diagonalise the b-dimensional matrix $\mathsf{M}^2_{(b\times b)}$, just as for any real fields. Gauge fields of equal mass, both massive and massless modes, must lie in representations of the unbroken symmetry H as before. The massless modes will be in the adjoint representation of H as we saw in section 6.3 for general unbroken local symmetry.

So with the broken generators in a theory of local symmetry, we have achieved our goal of describing short range forces (massive gauge bosons) within a gauge invariant theory.

Scalar masses

The scalar masses are determined exactly as in the global case as they are controlled solely by the scalar potential which we expand about the vacuum solution. The difference is that in unitary gauge we only expand in $\boldsymbol{\sigma}(x)$ fluctuations, the remaining b degrees of freedom, the $\theta^Z(x)$'s, have been removed by the unitary gauge transformation. Thus the scalar masses are the non-zero eigenvalues of

$$(\mathsf{M}^2)_{ij} = \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} \tag{7.2.25}$$

This $d\times d$ matrix has $b = g-h$ zero eigenvalues, as can be proved exactly as Goldstone's theorem was in the global case in section 5.2. However they are the masses of would-be Goldstone bosons, the $\theta^Z(x)$'s removed in the unitary gauge. The remaining scalars have non-zero masses and are called **Higgs particles**. The only constraint is that they *must* show the unbroken symmetry and so they must lie in representations of H.

Particle content

Before SSB we had

- d real scalars $\phi_i(x)$ with d degrees of freedom;
- $g := \dim(G)$ gauge fields $W^{\mu a}(x)$, which, being massless, have 2 degrees of freedom each, giving a total of $2g$ degrees of freedom.
- We thus have a total of $d+2g$ degrees of freedom.

After SSB

- $b = g-h (= \dim(G) - \dim(H))$ would-be Goldstone bosons, $\theta^a(x)$, which, being removable via a unitary gauge, are unphysical and hence provide 0 degrees of freedom to the total.
- $d-b$ physical scalar Higgs particles, $\sigma_i(x) \in \mathbb{R}$. These yield $d-b$ degrees of freedom to the total.
- The unbroken symmetry H ensures there are h massless gauge bosons, one for each generator T'^A, providing two degrees of freedom each, hence $2h$ degrees of freedom to the total.
- $g-h=b$ massive gauge bosons T''^a, one for every broken generator, providing three degrees of freedom each, hence $3(g-h)$ to the total.
- Our final sum is then
$$0 + (d-b) + 2h + 3(g-h) = d + 2g \tag{7.2.26}$$

Thus while the numbers of each type of particle has changed from the unbroken to the broken theory, the number of degrees of freedom has not. This is a general principle, that provided we are making exact mathematical redefinitions of fields,

$$\boxed{\text{The number of physical degrees of freedom } \textit{never} \text{ changes.}}$$

The nature of the particles can change and it is one of the most important features of field theory that particles can be transmuted into different forms. Here we say that these would-be Goldstone modes have been "eaten" by massive gauge bosons. The extra degree of freedom of a massive comes from the would-be-Goldstone bosons. Thus there are no physical massless scalars, as in the global case, but rather it is the appearance of massive gauge bosons which flags the existence of SSB when local symmetry is present.

7.3 Example of SSB in local $SO(3)$

Consider a theory with three scalar fields and a local O(3) symmetry[7]

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\{\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\} + (\mathsf{D}_\mu\boldsymbol{\phi})^\dagger(\mathsf{D}^\mu\boldsymbol{\phi}) - V(\boldsymbol{\phi}^\dagger\boldsymbol{\phi}) \tag{7.3.1}$$

where

$$\mathsf{D}_\mu(x) := \partial_\mu\mathbb{1} - ig\mathsf{W}_\mu(x), \quad \mathsf{W}^\mu(x) := W^{\mu a}(x)\mathsf{T}^a, \quad \mathsf{F}^{\mu\nu} := F^{\mu\nu a}(x)\mathsf{T}^a \tag{7.3.2}$$

When working with a real triplet of scalar fields in a local $SO(3)$ theory, i.e. $d = 3$ and $\dim(g) = 3$ in (7.2.23), we need to use pure imaginary generators, such as are given in (B.2.8)[8]. This is the special representation, the adjoint representation where $d = \dim(G)$ and the generators can be constructed from the structure constants. So we choose the basis where $T^a_{bc} = -\frac{i}{2}\epsilon_{abc}$ ($a, b, c = 1, 2, 3$, ϵ^{abc} is the completely anti-symmetric tensor with $\epsilon^{123} = +1$). Thus

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.3.3}$$

This basis also satisfies the equations

$$[T^a, T^b] = \frac{i}{2}\epsilon^{abc}T^c. \tag{7.3.4}$$

This basis is actually that of infinitesimal rotations (see below) but note none are diagonal so this field is not a simple charge eigenstate i.e. not all components can have definite charges. This is a basis where no generator is diagonal and this means the scalar fields are not the charge eigenstates. The rank of SO(3) is 1, i.e. the Cartan sub-algebra has dimension one, so we can diagonalise at most just one generator at any one time. If we were to do this to T^1 you would find you could only do this by working with complex eigenvectors e.g. $(0, 1, -i), (0, 1, +i), (1, 0, 0)$ i.e. the charge eigenstates are the combinations ϕ_1 and $\phi_2 \pm i\phi_3$.

(i) Suppose $V(x) = \frac{1}{2}m^2x + \lambda x^2$. Then *one* vacuum solution is[9]

$$\boldsymbol{\phi}_0 = v\boldsymbol{e}_1, \quad \boldsymbol{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v = \sqrt{\left(\frac{-m^2}{2\lambda}\right)} \tag{7.3.5}$$

(ii) Now find the most general *unbroken* generator by considering a general unbroken algebra element[10] A'

$$\mathsf{A}' = c^1\mathsf{T}^1 + c^2\mathsf{T}^2 + c^3\mathsf{T}^3, \quad c^a \in \mathbb{R} \tag{7.3.6}$$

and act it on the vacuum

$$\mathsf{A}'\boldsymbol{\phi}_0 = 0 \quad \Rightarrow \quad \begin{pmatrix} 0 \\ c_3 \\ c_2 \end{pmatrix} = 0 \tag{7.3.7}$$

Thus $\mathsf{A}' = c_1\mathsf{T}^1$ so that we see that the only unbroken generator is T^1.

It then follows that the Little group must be the only one-dimensional group $H = SO(2) \cong U(1)$.

[7] There is a global O(3) but for the local symmetry only the continuous parts of the symmetry are relevant.

[8] This is a special representation where the dimension of the representation equals the dimension of the group, here 3. It is called the adjoint representation and has some special properties. All groups have an adjoint representation and technically the gauge boson fields lie in the adjoint representation of the symmetry group.

[9] Multiply this $\boldsymbol{\phi}_0$ solution by any symmetry group element to find all other vacuum solutions, but note that only group elements made from broken generators will give distinct solutions.

[10] An arbitrary vector in the vector space that a Lie Algebra is.

(iii) Now we find the broken generators by completing the basis. Here we can see easily that a suitable orthogonal set of generators to choose for the broken ones are T^2 and T^3 so we have

$$\text{Unbroken } \mathsf{T}'^1 = \mathsf{T}^1; \quad \text{Broken } \mathsf{T}''^2 = \mathsf{T}^2, \mathsf{T}''^3 = \mathsf{T}^3; \tag{7.3.8}$$

(iv) Now rewrite the scalar field in the Unitary gauge manner. Here we have a geometric picture to help. Rotations about the 1 axis[11] leave the vacuum invariant. Thus it is the 2 and 3 axis rotations, generated by the $\mathsf{T}''^2, \mathsf{T}''^3$, which describe independent perturbations of the scalar field and which take the vacuum $\boldsymbol{\phi}_0$ round a sphere of constant $|\boldsymbol{\phi}_0|$, all of which are equally good vacuum solutions. The only other independent variation in $\boldsymbol{\phi}$ left after these two must be where we change the length of the $\boldsymbol{\phi}$. Thus we can write that

$$\boldsymbol{\phi} = \mathsf{U}''(x)\,(\boldsymbol{\phi}_0 + \boldsymbol{\sigma}) = \mathsf{U}''(x) \begin{pmatrix} v + \sigma(x) \\ 0 \\ 0 \end{pmatrix} \approx \begin{pmatrix} v + \sigma(x) \\ -\epsilon^3(x) \\ \epsilon^2(x) \end{pmatrix} \tag{7.3.9}$$

$$\mathsf{U}''(x) = \exp\{i\epsilon^2(x)\mathsf{T}''^2 + i\epsilon^3(x)\mathsf{T}''^3\} \qquad \boldsymbol{\sigma} = \sigma(x)\boldsymbol{e}_1\,, \tag{7.3.10}$$

where $\epsilon^2(x), \epsilon^3(x), \sigma(x) \in \mathbb{R}$ and $\boldsymbol{e}_1$ is a unit vector in the 1 direction. It is simple to check that different small perturbations are orthogonal, that is $\boldsymbol{\sigma}(x), (i\epsilon^2(x)\mathsf{T}''^2\boldsymbol{\phi}_0), (i\epsilon^3(x)\mathsf{T}''^3\boldsymbol{\phi}_0)$ are all perturbations which are orthogonal and are a complete but alternative way of describing the three $\phi_i(x)$ fluctuation fields. In particular we have

$$\left(\mathsf{T}''^Z\boldsymbol{\phi}_0\right)^\dagger \boldsymbol{\sigma} = 0, \quad Z = 2,3\,. \tag{7.3.11}$$

(v) One makes a gauge transformation with $\mathsf{U} = (\mathsf{U}'')^{-1}$.

The scalar field becomes $\boldsymbol{\phi} \to \boldsymbol{\phi}' = (\mathsf{U}'')^{-1}\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \boldsymbol{\sigma}$. This is the part of the scalar field perturbations not described by symmetry transformations so can not be the Goldstone fluctuations. Here its a radial fluctuation not an angular one. Clearly we have only one real scalar field mode left $\sigma(x)$ and this will have mass as in the global case since the scalar potential is still the only term effecting the scalar masses and it requires no changes in going from local to global case. Such modes are the Higgs modes and from the global results, or repeating the calculations, we find the Higgs mass is again $(-2m^2)$.

The Goldstone modes, the scalar modes associated with the perturbations written in terms of symmetry transformations, the $\mathsf{U}''(x)$ part, have been completely removed. Their degrees of freedom must have been shuffled into other fields. The only ones left are the gauge fields and indeed we should find two new degrees of freedom in the gauge fields now.

The gauge bosons will acquire a mass matrix M^2 coming from the $|\mathsf{D}^\mu\boldsymbol{\phi}|^2$ term, and we showed in the lectures that this has the general form

$$(\mathsf{M}^2)^{ab} = g^2\boldsymbol{\phi}_0^T\mathsf{T}^a\mathsf{T}^b\boldsymbol{\phi}_0\,. \tag{7.3.12}$$

Note that there are i,j indices on the vectors associated with the representation of the scalar field. These are completely summed over and have nothing to do with the space in which the M^2 matrix lives. The M^2 matrix is a matrix in the a,b indices of the generators and so living in the algebra[12]. It

[11] We are after all rotating the three real components of $\boldsymbol{\phi}$ into one another so it is just like a 3 dimensional rotation of space coordinates.

[12] Technically in the adjoint representation of the Lie Algebra.

is NOT a matrix in the i, j indices of the scalar field representation, though in this example both sets of indices run through 1,2 3[13]. In this case we can find that

$$(\mathsf{M}^2)^{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4}g^2v^2 & 0 \\ 0 & 0 & \frac{1}{4}g^2v^2 \end{pmatrix} . \tag{7.3.13}$$

Thus as advertised, there is one massless gauge boson $W^{\mu 1}(x)$ to go with the one unbroken generator T'^1, and two massive gauge bosons $W^{\mu 2}(x), W^{\mu 3}(x)$ to go with the two broken generators $\mathsf{T}''^2, \mathsf{T}''^3$, both of which have the same mass $gv/2$. Note that massless gauge bosons have two degrees of freedom, massive ones have three so there are two extra gauge boson degrees of freedom in the broken case, corresponding to the loss of two scalar Goldstone modes, so the total number of physical modes in the theory remains the same.

(vi) Calculating the mass matrix from the formula quoted above is often a tedious business. It is simpler to multiply matrices by a vector to get a vector, rather than starting with the matrix products. We can eliminate the former a, b indices by working with $T^a W^{\mu a}$ combinations directly. To do all this we calculate the whole gauge boson mass term and only indirectly the mass matrix. Its then quicker to consider the expression as the dot product of two vectors[14], $\boldsymbol{z}$

$$W^{\mu a}(\mathsf{M}^2)^{ab}W^b_\mu = \boldsymbol{z}^\dagger . \boldsymbol{z} = |\boldsymbol{z}|^2 \tag{7.3.14}$$

$$\boldsymbol{z}^\mu := \left(\mathsf{T}^1 W^{\mu 1} + \mathsf{T}^2 W^{\mu 2} + \mathsf{T}^3 W^{\mu 3}\right) \boldsymbol{\phi}_0 . \tag{7.3.15}$$

The vector $\boldsymbol{z}$ is quick to calculate if we chose $\boldsymbol{\phi}_0$ so as to simplify the problem, as we did here. We now see that T^1 annihilates the vacuum so the gauge boson field associated with it, W^1 never appears in the mass term, i.e. unbroken generators are linked with massless gauge bosons. To complete the calculation, we need only look at the first column of the T^2 and T^3 generators, as $\boldsymbol{\phi}_0$ is so simple, and we see that

$$\boldsymbol{z}^\mu = gv \left[\begin{pmatrix} 0 & \ldots \\ 0 & \ldots \\ -iW^{\mu 2} & \ldots \end{pmatrix} + \begin{pmatrix} 0 & \ldots \\ +iW^{\mu 3} & \ldots \\ 0 & \ldots \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ +iW^{\mu 3} \\ -iW^{\mu 2} \end{pmatrix} . \tag{7.3.16}$$

The modulus of this vector then gives the whole gauge boson mass term easily as

$$\frac{1}{2} g^2 v^2 \left((W^{\mu 2})^2 + (W^{\mu 3})^2 \right) \tag{7.3.17}$$

With this term in the Lagrangian and, in the unitary gauge, no quadratic mixing terms, we see that both $W^{\mu 2}$ and $W^{\mu 3}$ have mass gv and from its absence, we conclude $W^{\mu 1}$ is massless.

(vii) The charges of the physical modes in the broken case are interesting. Think of a fixed Lorentz index μ then look at the pair of real fields $W^{\mu 2}$ and $W^{\mu 3}$. They have the same mass (gv), a reflection of the unbroken $SO(2)$ symmetry. If we had seen two real scalar fields, say ϕ_2 and ϕ_3, with the same mass we would suspect an $SO(2)$ symmetry and we know that it is better to work with the complex field $\Phi = (1/\sqrt{2})(\phi_2 + i\phi_3)$ and its conjugate as these have $SO(2) \approx U(1)$ definite charge. The complex fields $\Phi, \Phi^\dagger$ would be the charge eigenstates rather than the real fields ϕ_2, ϕ_3. In exactly the same way, one can check and see that the physical gauge fields carry charge but the $W^{\mu a}$ forms are not the

[13]Here the scalar field is in the adjoint representation of SO(3).

[14]The Lorentz vector index plays no role here so we suppress it. For instance you can do this for each value of μ present in the implicit sum.

charge eigenstates. By analogy with the real scalar field problem we see that in the broken phase the charge eigenstates are the combinations

$$W^{\mu\pm} = \frac{1}{\sqrt{2}}\left(W^{\mu 2} \pm iW^{\mu 3}\right), \quad W^{\mu +} = \left(W^{\mu -}\right)^{\dagger} . \tag{7.3.18}$$

Thus these gauge fields are massive and have $U(1)$ charges ± 1. This could be checked by finding the contribution to the Noether currents from the $W^{\mu a}$ fields, though this is not easy as non-abelian gauge bosons give a complicated contribution to the Noether current.

The two other physical fields in the unbroken phase, $\sigma, W^{\mu 1}$ have distinct masses. Thus, like any other single real scalar field with a unique mass in an SO(2) theory, these must each transform in the trivial representation (i.e. they are invariant under the unbroken $SO(2)$ symmetry) and so have zero charge.

(viii) Finally, had we studied the unbroken model, we remember that SO(3) is rank one and therefore it can have one diagonal generator, and one charge. However the generators acting on the real scalar triplet have no diagonal generators though this form is convenient for the SSB calculations. To see the charges, one must diagonalise one generator, say T^1 in which case we see that $\Phi = (1/\sqrt{2})(\phi_2 + i\phi_3)$ and its conjugate are the charged scalar fields, and ϕ_1 is uncharged. In the unbroken phase all are physical and have equal mass a reflection of the $SO(3)$ symmetry. The gauge bosons behave in exactly the same way under SO(3) symmetry. The combinations $W^{\mu 1}, W^{\mu +}, W^{\mu -}$ are the gauge boson charge eigenstates with charges 0, +1, -1, they are all physical modes and they all have equal mass, zero.

Overall, whether we look at the broken or unbroken symmetry case, we see how symmetry controls relationships between masses and the possible charges, though this may require us to make field redefinitions in order to reveal the physics clearly.

Particle	Field	Mass	Observable Charge	Degrees of Freedom
Higgs	σ	$\sqrt{-2m^2}$	0	1
"Photon"	$W^{1\mu}$	0	0	2
"W's"	$W^{\pm\mu}$	$(gv/2)$	± 1	2×3
Scalar	ϕ_1	m	0	1
Triplet	$(\phi_2 \pm i\phi_3)/\sqrt{2}$	m	± 1	2×1
Non abelian	$W^{1\mu}$	0	0	2
Gauge bosons	$(W^{2\mu} \pm iW^{\mu 3})/\sqrt{2}$	0	± 1	2×2

Aside on SO(3) matrices

These generators are chosen to be infinitesimal rotations in three-dimensions. By considering the series expansion of an exponential and looking at terms with even and odd powers of ϵ separately, one can prove that in this basis T^a generates rotations about the a-th axis, i.e.

$$e^{i2\epsilon^1 T^1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ \cos(\epsilon^1)x_2 + \sin(\epsilon^1)x_3 \\ -\sin(\epsilon^1)x_2 + \cos(\epsilon^1)x_3 \end{pmatrix} \tag{7.3.19}$$

7.4 Questions

)7.1. Unitary Gauge: Abelian case

Consider a single complex scalar field and single gauge field model with no kinetic terms for a gauge field

$$\mathcal{L}[\Phi, \Phi^\dagger, B^\mu] = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi^\dagger \Phi) \tag{7.4.1}$$

(i) Show that this is the $e \to \infty$ limit of Scalar QED, provided we think in terms of the non-zero rescaled gauge field $B^\mu = eA^\mu$.

(ii) Hence show that the A^μ field can be eliminated by using its equation of motions to give

$$eA^\mu = -\frac{i}{2} \frac{\Phi^\dagger(\partial^\mu \Phi) - (\partial^\mu \Phi^\dagger)\Phi}{|\Phi|^2} \tag{7.4.2}$$

(iii) Thus show that if $\Phi = \eta e^{i\theta}$, $\eta \in \mathbb{R}$ then

$$eA^\mu = \partial^\mu \theta, \quad D^\mu \Phi = e^{i\theta}(\partial^\mu \eta) \tag{7.4.3}$$

(iv) Show that such a gauge field has no physical content, i.e. it is gauge equivalent to zero, contributes nothing to $F^{\mu\nu}$, and so represents zero electric and magnetic fields.

(v) Deduce that

$$\mathcal{L}[\Phi, \Phi^\dagger, B^\mu] = \mathcal{L}_{\text{new}}[\eta] = (\partial^\mu \eta)(\partial_\mu \eta) - V(\eta^2) \tag{7.4.4}$$

(vi) What happens if $\Phi = 0$? Hence what is the particle content of this model, i.e. what happens to the degrees of freedom in *rewriting* $\mathcal{L}$ as $\mathcal{L}_{\text{new}}$?

If we make a transformation of this form, i.e. transform the gauge field by the derivative of the phase of the scalar field, in normal models where an F^2 term is present, nothing physical changes in the gauge sector - it is a gauge transformation. Making such a gauge transformation, one which sets the scalar field to be real at ‘all’ points (except where Φ =0) is called the **Unitary Gauge**.

Chapter 8

Fermions

So far we have focused on scalar fields, both as fields representing ordinary particles with spin zero, and as Higgs fields, the essential element of any SSB scheme. We were also forced to introduce gauge fields when we considered local symmetry. However, most of the matter we know is made up of fundamental fermions, the quarks in neutrons and protons of a nucleus, and the electrons bound to nucleii. In this section we will give a brief review of the basic classical theory of fermions which centres on the Dirac equation. We will put the Dirac equation and its fields into the context of earlier chapters, that of local and global symmetry of Lagrangians.

In the second part of the chapter we look at different types of fermion, especially chiral fermions. It turns out that the weak force interacts with chiral fermions rather than simple Dirac fermions.

8.1 Dirac Fermions

Fields which satisfy the Dirac equation with *no* other constraints, describe particles called **Dirac Fermions**, in the limit where such particles have no interactions. Using $\psi^\alpha(x)$ for fermionic fields, the Dirac equation is

$$\left(i\gamma^\mu_{\alpha\beta}\partial_\mu - m\mathbb{1}_{\alpha\beta}\right)\psi^\beta(x) = 0 \tag{8.1.1}$$

where $\mu = 0, 1, 2, 3$ is a Lorentz vector index , spin indices $\alpha, \beta = 0, 1, 2, 3.$.

In four-momentum we have

$$\begin{aligned} \psi^\alpha(x) &= \int \frac{d^4p}{(2\pi)^2} e^{-ip_0t + i\boldsymbol{p}.\boldsymbol{x}}\psi(p) && (8.1.2)\\ 0 &= (\not{p} - m)\psi(p), \qquad \not{p} := \gamma_\mu p^\mu. && (8.1.3)\end{aligned}$$

For a physical particle, the fermion is on mass shell so that

$$p_0 = s\omega_p, \quad s = \pm 1, \quad \omega_p = \sqrt{\boldsymbol{p}^2 + m^2} \tag{8.1.4}$$

where $s = +1$ is a particle solution, $s = -1$ an anti-particle solution, both of physical energy ω_p.

Gamma matrices

[1]

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \left\{\gamma^5, \gamma^\nu\right\} = 0 \quad \gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{8.1.5}$$

where the metric is $g^{00} = +1$, $g^{11} = g^{22} = g^{33} = -1$ with other entries zero.

[1] I will *not* use my usual matrix notation of A etc. for the gamma matrices $\gamma^\mu_{\alpha\beta}$ with their spin indices.

The spin operator in spatial direction i is S_i $(i = 1, 2, 3)$ and for a particle with three-momentum p_i is

$$S_i = \frac{1}{4}\epsilon_{ijk}[\gamma_j, \gamma_k] = \frac{1}{2}\gamma_5\gamma_0\gamma_i, \quad \Leftrightarrow \quad [S_i, S_j] = \frac{i}{2}\epsilon_{ijk}S_k \tag{8.1.6}$$

and the last identity is the spin-algebra. This may be familiar from non-relativistic work where the Pauli matrices form another representation of the same algebra.

Dirac Basis

Many different matrices satisfy the algebra of the gamma matrices. All are equally valid, but depending on the physical problem some representations may be more convenient. It is also useful to have one explicit example such as the **Dirac** basis, which is often seen, and it is given by

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \boldsymbol{\tau}^i \\ -\boldsymbol{\tau}^i & 0 \end{pmatrix}, \quad \Rightarrow \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{8.1.7}$$

The $\boldsymbol{\tau}^i$ $(i = 1, 2, 3)$ are the three Pauli matrices of (B.2.4). In the Dirac basis, γ^5 has the following properties

$$\gamma_5^\dagger = \gamma_5, \quad \gamma_5{\cdot}\gamma_5 = 1. \tag{8.1.8}$$

Note that the spin operator in the Dirac basis is

$$S_i = \frac{1}{2}\begin{pmatrix} \boldsymbol{\tau}^i & 0 \\ 0 & \boldsymbol{\tau}^i \end{pmatrix} \tag{8.1.9}$$

Weyl Basis

Another basis which is particularly useful in the standard model is the **Weyl** basis where

$$\gamma^0 = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \boldsymbol{\tau}^i \\ -\boldsymbol{\tau}^i & 0 \end{pmatrix}, \quad \Rightarrow \quad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{8.1.10}$$

where the $\boldsymbol{\tau}^i$ $(i = 1, 2, 3)$ are the three Pauli matrices of (B.2.4). The spin operator in the Weyl basis is the same as in the Dirac basis (8.1.9).

Solutions of the Dirac Equation

There are two particle solutions, $u_r(\boldsymbol{p})$ $(r = 1, 2)$, and two anti-particle solutions, $v_r(\boldsymbol{p})$ $(r = 1, 2)$, to the Dirac equation. Thus the most general solution to the Dirac equation is made up of a linear combination of these four solutions

$$\psi(p) \;=\; 2\pi\delta(p_0 - \omega_p)\left(\sum_{r=1,2} f_r(\boldsymbol{p})u_r^\alpha(\boldsymbol{p})\right) + 2\pi\delta(p_0 + \omega_p)\left(\sum_{r=1,2} f_{-r}(\boldsymbol{p})u_r^\alpha(\boldsymbol{p})\right) \tag{8.1.11}$$

where $\omega_p = +\sqrt{\boldsymbol{p}^2 + m^2} > 0$. In the Dirac representation for the gamma matrices the four standard solutions u_r and v_r are

$$u_r(\boldsymbol{p}) = (\omega_p + m)^{\frac{1}{2}}\begin{pmatrix} \chi_r \\ \frac{1}{(\omega_p+m)}p^i\boldsymbol{\tau}^i\chi_r \end{pmatrix} \quad v_r(\boldsymbol{p}) = (\omega_p + m)^{\frac{1}{2}}\begin{pmatrix} \frac{1}{(\omega_p+m)}p^i\boldsymbol{\tau}^i\chi_r \\ \chi_r \end{pmatrix} \tag{8.1.12}$$

where spin up and spin down solutions (w.r.t. the third axis, note S_3 is diagonal in the Dirac basis) correspond to the two $r = 1, 2$ solutions

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.1.13}$$

Note that the two-by-two Pauli matrices act on the two dimensional spinors χ.

Degrees of freedom

The four independent real functions, f_{-2}, f_{-1}, f_1 and f_2 in the general solution (8.1.11) to the Dirac equation , indicate that there four independent or distinct ways of carrying quantum numbers (e.g. energy, momentum) through the system.. Thus *Dirac Fermions have 4 degrees of freedom.* These correspond to all permutations of spin up/spin down and particle/anti-particle.

One could try to compare this with the complex scalar field (two degrees of freedom) vs. real scalar field (one degree of freedom) case, and ask if there are fermionic fields with fewer degrees of freedom. The answer is yes.

A **Majorana** spinor has two degrees of freedom, each fermion is its own anti-particle but spin can be up or down. Thus the complex vs. real scalar fields have the same relation as the Dirac to Majorana fermion. These are very important when supersymmetry is present and in some models of massive neutrinos, but they will remain outside the scope of these notes.

Since the Dirac fermion has up/down spin as well as particle/anti-particle options, one could look for another way of halving the degrees of freedom. A **Chiral fermion** also has two degrees of freedom, but in this case the particle has its spin in one direction only, and the anti-particle has its spin ion the other direction. This is vital for the fundamental description of weak nuclear forces and we will study this in section 8.3.

8.2 Lagrangians for spin 1/2 particles

8.2.1 Single free spin 1/2 field

The Dirac equation is the equation of motion obtained from a Lagrangian quadratic in $\psi(x)$, namely

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi, \qquad \bar{\psi} = \psi^\dagger\gamma_0, \quad \partial\!\!\!/ = \partial_\mu\gamma^\mu \tag{8.2.1}$$

Thus this describes the propagation of a free i.e. non-interacting, fermion.

This Lagrangian for the Dirac fermionic field always has one global symmetry[2]

$$\textbf{Vector current:} \quad \psi \to \psi' = e^{i\theta}\psi \quad \Rightarrow \quad \mathcal{L}[\psi] = \mathcal{L}[\psi'], \quad J^\mu \propto \bar{\psi}\gamma^\mu\psi \tag{8.2.2}$$

This conserved charge is proportional to the fermion number (fermions minus anti-fermions)[3]. The Noether current has a single Lorentz index μ indicating that it transforms as a Lorentz vector under space-time symmetries, hence the name of the symmetry.

However, there is another important symmetry for the spacial case of a massless Dirac fermion. Supposed we make a **Chiral** or **Axial** transformation,

$$\psi_\alpha \to \psi'{}_\alpha = \left[e^{i\theta\gamma^5}\right]_{\alpha\beta}\psi_\beta \tag{8.2.3}$$

Then we find that in massless limit *only* this is a symmetry of the Lagrangian (8.2.1) the **Axial current**[4],

$$m = 0 \quad \Rightarrow \quad \mathcal{L}[\psi] = \mathcal{L}[\psi'], \quad J^\mu_A \propto \bar{\psi}\gamma^\mu\gamma^5\psi \tag{8.2.4}$$

The Noether current has a single Lorentz index μ but it also has the 5 of the γ^5. A careful check of Lorentz properties shows this current transforms as a Lorentz axial-vector. That is it behaves exactly as any other Lorentz vector except under reflections.

[2] **EFS 8.2.5**: Check this symmetry and derive this Noether current.
[3] **EFS 8.2.6**: Use a form for a free Dirac quantum field to confirm this.
[4] **EFS 8.2.7**: Check that the axial transformation of (**??**) is indeed a symmetry for massless free Dirac fermions. Also show showing in particular that the mass term is not invariant, and derive the Noether current.

8.2.2 Multiple free spin 1/2 fields

Just as we were able to generalise from a single complex scalar field, Φ, with U(1) symmetry, to a vector of scalar fields $\Phi_i(x)$ in (i.e. transforming under) a d-dimensional representation of some group, we can do the same with fermionic fields. Thus we consider $\psi_i^\alpha(x)$, with spin index $\alpha = 0,1,2,3$, internal symmetry index $i = 1,2,\ldots,d$ for a field in a d-dimensional representation. Note that we do not mix space-time transformations (acting on the α index) and internal symmetry transformations (acting on the i index), they are independent of each other (space-time and internal symmetry transformations factorise). We saw the same feature with non-abelian gauge fields $W^{\mu a}$ where the internal symmetry only acted on the a indices of the internal symmetry. We can easily construct a $U(d)$ globally symmetric generalisation of the U(1) symmetric free Dirac fermion Lagrangian, (8.2.1),

$$\mathcal{L} \;=\; \bar{\psi}_i\,(i\partial\!\!\!/ - m)\,\psi_i, \quad \psi_i^\alpha \to \psi'^\alpha_i = \left[e^{i\epsilon^a \mathsf{T}^a}\right]_{ij}\delta^\alpha_\beta\psi^\beta \quad\Rightarrow\quad \mathcal{L}[\psi] = \mathcal{L}[\psi'], \tag{8.2.5}$$

$$J^{\mu a} \propto \bar{\psi}_i^\alpha \gamma^\mu_{\alpha\beta} T^a_{ij}\psi_j^\beta \tag{8.2.6}$$

where we have left in some explicit unit matrices in spin δ^α_β to emphasise the points above, i.e. the symmetry transformation being discussed does not touch the spin indices. Such unit matrices will be left out in all remaining discussions.

8.2.3 Local symmetry and fermions

By now we might guess that we just have to make the usual replacement $\partial^\mu \to \mathsf{D}^\mu$ where $\mathsf{D}^\mu = \partial^\mu - ig\mathsf{W}^\mu$ is defined exactly as with scalar fields in (6.2.25).

$$\mathcal{L}_{\text{global}} \to \mathcal{L}_{\text{local}} \;=\; \bar{\psi}_i\left(i\mathsf{D}^\mu_{ij}\gamma_\mu - m\mathbb{1}_{ij}\right)\psi_j - \frac{1}{2}\mathrm{Tr}\left\{(\mathsf{F}_{\mu\nu})^2\right\} \tag{8.2.7}$$

$$= \bar{\psi}_i\,(i\partial\!\!\!/ - m)\,\psi_i + J_L^{\mu a}W^a_\mu - \frac{1}{2}\mathrm{Tr}\left\{(\mathsf{F}_{\mu\nu})^2\right\} \tag{8.2.8}$$

where $J_L^{\mu a} = gJ^{\mu a}$, the Noether current. Note that unlike the scalar case, the fermionic contribution to the Noether current is *independent* of the gauge fields W^a_μ[5]

8.2.4 Other fermionic Interactions

The local theory kinetic terms contain gauge field - fermion - anti-fermion interactions. There are two other basic examples:

• $g_Y\phi\bar{\psi}\psi$ is renormalisable in four space-time dimensions, and it is invariant under the U(1) symmetry (8.2.2) and Lorentz symmetry. It represents scalar-fermion-anti-fermion interactions with interaction strength g_Y (not a gauge coupling strength). There are more sophisticated variants with complex fields, fields in higher dimensional representations (see chapter 10 on the EWmodel) but all such terms are called **Yukawa** terms or interactions. Yukawa used such interactions in 1935 to model the strong nuclear force as the exchange of a pion (a scalar) between protons or neutrons (fermions).

In the modern context they are important as they generate mass corrections for fermions when there is SSB present as

$$g_Y\phi\bar{\psi}\psi = (g_Y v)\bar{\psi}\psi + \text{ cubic interaction} \tag{8.2.9}$$

[5] **EFS 8.2.8**: Consider the Lagrangian for the free Dirac fermion with a local symmetry of (8.2.7). (a) Check that the is symmetry. (b) Derive the Noether current. (c) Derive the equations of motion without any kinetic term for the gauge fields but for general non-abelian symmetry, showing from the equation of motionfor the gauge field that it is an auxiliary field and easily eliminated. (d) Consider a U(1) local theory but with the gauge field kinetic term present, and derive the eom - two of Maxwell's equations.

if $\phi(x)$ is set equal to a constant vev v plus fields for the fluctuations.
• $g(\bar{\psi}\psi)^2$ is a four fermion interaction and can describe directly scattering of fermions off fermions without any intermediate particle. Again, many sophisticated variations are possible with different fermionic fields and fields lying in higher dimensional representations. It is even common to see gamma matrices or group generators appearing to make more sophisticated terms. With such a term, one could imagine modelling the strong force fermion-proton interactions mentioned above without a scalar field.

However in a modern context such a term is *not* renormalisable in four-dimensions and would not be included as a part of a fundamental theory. It does have a very important role to play in effective theories which are powerful tools. This type of term is the basis of the Nambu-Jona/Lasinio models [14], which are important as simple and early examples of *dynamical symmetry* breaking, an important alternative to the spontaneous symmetry breaking considered in most of these notes.

8.3 Chiral Fermions

We have already mentioned chiral fermions as fermions of two degrees of freedom, and chiral symmetry in the context of massless fermions and a γ^5 transformation. We will come back to these ideas later but we will start from a formal set of definitions.

Chiral Projection Operators and Eigenstates

The left and right chiral projection operators are defined to be

$$P_R := \frac{1}{2}(1+\gamma_5), \quad P_L := \frac{1}{2}(1-\gamma_5). \tag{8.3.1}$$

Note that in the Weyl basis (8.1.10) these are particularly simple

$$P_R = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_L = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{8.3.2}$$

The projection operators (in any basis) satisfy the usual properties of projection operators, namely[6]

$$P_R.P_R = P_R, \quad P_L.P_L = P_L, \quad P_R.P_L = P_L.P_R = 0, \quad 1 = P_R + P_L. \tag{8.3.3}$$

The last identity ensures we can split *any* arbitrary Dirac spinor into a left- and right-handed part,

$$\psi = \psi_R + \psi_L, \quad \psi_R := P_R\psi, \quad \psi_L := P_L\psi, \tag{8.3.4}$$

Thus ψ_L and ψ_R are both eigenstates of both the projection operators, with eigenvalues 0 and 1, each having the opposite eigenvalue for a given projection operator

$$P_R\psi_L = P_R\psi_L = 0, \quad P_R\psi_R = \psi_R, \quad P_L\psi_L = \psi_L. \tag{8.3.5}$$

Note that the Weyl basis is the basis for gamma matrices where the eigenvectors for chirality and for the z-component of spin are proportional to the simplest unit vectors, $(1,0,0,0)$ (spin up, RH fermion), $(0,1,0,0)$ (spin down, RH fermion), $(0,0,1,0)$ (spin up, LH fermion), and $(0,0,0,1)$ (spin down, LH fermion). This is why its so useful when dealing with chiral fermions.

One has to be *very* careful with the conjugate left- and right-handed fields field, $\bar{\psi}_R$ and $\bar{\psi}_L$, which we will define by

$$\bar{\psi}_R = (\psi_R)^\dagger\gamma_0, \quad \bar{\psi}_L = (\psi_L)^\dagger\gamma_0. \tag{8.3.6}$$

[6] **EFS 8.3.9**: Prove that the left- and right-projection operators (8.3.1) satisfy the properties (8.3.3) of projection operators.

Other definitions of $\bar{\psi}_R$ and $\bar{\psi}_L$ are in use[7]. If in doubt as to their meaning, refer back to equation (8.3.6). From our definitions, for any ψ the following hold:

$$\bar{\psi}_R P_L = \bar{\psi}_R, \quad \bar{\psi}_R P_R = 0, \quad \bar{\psi}_L P_R = \bar{\psi}_L, \quad \bar{\psi}_L P_L = 0. \tag{8.3.7}$$

8.3.1 Helicity, Chirality and Mass

Helicity is the component of the spin lying along the direction of travel of a particle, i.e. along the direction of the unit vector $\boldsymbol{p}/|\boldsymbol{p}|$. The helicity operator, Λ, is therefore given by

$$\Lambda = \frac{2p_i S_i}{|\boldsymbol{p}|}, \quad S_i = \frac{1}{2}\gamma_5\gamma_0\gamma_i \tag{8.3.8}$$

where $\boldsymbol{p}$ is the three momentum with components p_i and S_i is the spin operator defined in the second part of the equation above. The index i here runs over the three space directions.

It turns out that for *massless* particles, the states with definite handedness, i.e. eigenstates of P_L and P_R, are also eigenstates of helicity. For simplicity we will consider the non-interacting case and let ψ be a solution to the massless Dirac equation, $m = 0$ in (8.1.1). Working in four-momentum p_μ coordinates, with $m = 0$ in (8.1.3), we have

$$p_0\psi(p) = p_i\gamma^0\gamma^i\psi(p) \quad \Rightarrow \quad \psi(p) = \frac{p_i}{p_0}2\gamma^5 S_i\psi(p) \tag{8.3.9}$$

using the definition of the spin matrix S_i in (8.1.6). Now the energy is equal to the size of the momentum for a physical massless particle, solutions to Dirac's equation are "on-shell". Thus $\omega_p = p = |\vec{p}|$ in (8.1.4) and so

$$p_0 = s|\boldsymbol{p}|, \quad s = \pm 1 \tag{8.3.10}$$

where $s = +1$, is a particle solution, $s = -1$ an anti-particle solution. Remember that the physical energy is $|p_0|$ but its convenient to work with negative p_0 values, e.g. in the general free field expression (8.1.11). We then note that we have the helicity operator in (8.3.9)

$$\Lambda := \frac{2\boldsymbol{p}.\boldsymbol{S}}{|\boldsymbol{p}|} \tag{8.3.11}$$

This operator has eigenvalues ± 1 as the $\boldsymbol{p}/|\boldsymbol{p}|$ just projects out the component of spin along the direction of the velocity, while the 2 corrects for fact that the eigenvalues of any spin component are $\pm 1/2$. Thus

$$\psi(p) = s\gamma^5\Lambda\psi(p) \tag{8.3.12}$$

Using

$$P_R\gamma_5 = P_R, \quad P_L\gamma_5 = -P_L \tag{8.3.13}$$

and some more manipulation we find the following eigenstate equations[8].

$$\begin{aligned} \Lambda\psi_L(p) &= -s\psi_L(p) \\ \Lambda\psi_R(p) &= +s\psi_L(p) \end{aligned} \tag{8.3.14}$$

Thus the chiral eigenstates (8.3.5) are also helicity eigenstates but with a non-trivial relationship between the positive 'energy' particle solutions and the negative 'energy' particle solutions. The free field is usually written in terms of two particle free field solutions, $u_\chi \exp\{-\omega t - i\boldsymbol{p}.\boldsymbol{x}\}$ $(\chi = 1, 2)$, and two anti-particle

[7]The obvious alternative is $\bar{\psi}'_R = (\psi)^\dagger\gamma_0 P_R = \bar{\psi}_L$.

[8] **EFS 8.3.10**: plrg5Prove that the chiral fermions ψ_L and ψ_R are eigenstates of helicity as given in (8.3.15)

solutions, $v_\chi \exp\{-\omega t - i\boldsymbol{p}.\boldsymbol{x}\}$ ($\chi = 1, 2$) with some normalisation . The two solutions can be chosen using the $P_{L/R}$ operators to be chiral eigenstates, $u_{L/R}(p), v_{L/R}(p)$, in which case we also know that they are also eigenstates of helicity, namely[9]

$$\begin{aligned} \Lambda u_L(p) &= -u_L(p) \\ \Lambda v_L(p) &= +v_L(p) \\ \Lambda u_R(p) &= +u_L(p) \\ \Lambda v_R(p) &= -v_L(p) \end{aligned} \tag{8.3.15}$$

A purely left-handed solution with *positive* p_0, denoted u_L, is an eigenstate of helicity with eigenvalue -1, while a pure right-handed solution with *negative* p_0, v_l, has helicity eigenvalue $+1$. The right-handed solutions have the opposite helicity.

This explains why we use the word "chiral" for the right- and left-handed fermions. Consider a moving particle and measure its spin along the direction of travel $\boldsymbol{p}/|\boldsymbol{p}|$. If this component of spin points in the same direction as the velocity, we could picture the particle as a spinning ball, and a point on its circumference would mark out a left-handed corkscrew as the particle traveled along[10] If the spin of the particle had been in the opposite direction, we would represent this by having the ball spinning in the opposite sense so the point on the equator of the ball would sketch out a corkscrew of the opposite sense. As chiral means handedness, or an object which is not equivalent to its mirror image, the term is appropriate in this context.

Note that the particle must be *massless* for this link between chirality and helicity to be identical for different observers and therefore a fundamental property of a particle. Suppose we, observe an eigenstate of helicity in one frame. For a massive particle, we could consider a second observer moving faster than the particle. To this second observer the spin would be the same but the direction of travel of the particle has switched. This observer concludes that the particle as the opposite helicity from that deduced by the original observer. Thus massive fermions can not be eigenstates of helicity.

8.3.2 Chiral fermions, Lagrangians and Masses

Can chiral fermions, ψ_L, exist in their own right as a fermionic field with two degrees of freedom? It is certainly possible to find solutions of the massless Dirac equation which are pure left-handed or righthanded states, the standard normalised u_L and v_L solutions for instance, and their existence does not depend on the other. Thus the answer is yes. We can see this from the equation of motion for a chiral state, say ψ_L,

$$0 = (i\partial\!\!\!/ - m)\,\psi_L \tag{8.3.16}$$

which implies $i\partial\!\!\!/\psi_L = m\psi_L$ and

$$0 = (i\partial\!\!\!/ - m)\,P_L\psi_L = P_R(i\partial\!\!\!/\psi_L) - P_L(m\psi_L) \tag{8.3.17}$$

$$\Rightarrow \quad \text{(a)} \quad m = 0 \qquad \text{or (b)} \quad P_R\psi_L = P_L\psi_L \quad \Rightarrow \quad 0 = P_L\psi_L \tag{8.3.18}$$

Thus a chiral fermion can satisfy Dirac's equation but only if it is massless. Again we see that only massless fermions can be chiral.

This can be seen in another way. The Lagrangian for a general Dirac field can be written as follows

$$\begin{aligned} \mathcal{L} &= \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi && (8.3.19) \\ &= \bar{\psi}_R\,(i\partial\!\!\!/)\,\psi_R + \bar{\psi}_L\,(i\partial\!\!\!/)\,\psi_L - m\left(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R\right) && (8.3.20) \end{aligned}$$

[9] **EFS 8.3.11**: Calculate the four explicit forms of these eigenstates. Remember they are solutions to massless Dirac equation, and if you wish to exploit the solutions in (**??**) then pick spatial coordinates such that $\boldsymbol{p} = (0, 0, p > 0)$.

[10] Its a matter of convention whether we think of it as a left or right handed corkscrew. Its convention as spin is fundamentally quantum and special relativistic phenomena and thinking of it as any sort of spinning classical object is at best an analogy, one which turns out to be of quite limited use in general.

Note that the **mass term mixes chiral eigenstates**, but the kinetic term does not. A mass term requires both left- and right-handed parts to be present. A pure left-handed particle must be massless. Thus a valid free Lagrangian for a left-handed chiral fermion is

$$\mathcal{L} \;=\; \bar{\psi}_L i\partial\!\!\!/\psi_L = \frac{1}{2}\bar{\psi}i\partial_\mu\gamma^\mu\left(1-\gamma^5\right)\psi \tag{8.3.21}$$

The first form is compact, the second is a clearer reminded that we do not just have a Dirac fermion. To see what interactions we can have, it is best to put chirality into our general language of symmetries and transformation

8.3.3 Axial currents and chiral symmetry

The existence of chiral fermions is related to the Axial or chiral symmetry of (8.2.4) as we can show[11]

$$\psi_L \to \psi'_L = U_A\psi_L = e^{+i\theta}\psi_L, \quad \psi_R \to \psi'_R = U_A\psi_R = e^{-i\theta}\psi_R, \quad U_A := \exp\{i\theta\gamma^5\} \tag{8.3.22}$$

This transformation is distinct from the usual phase symmetry of the Dirac equation, $\psi \to \exp\{i\theta\}\psi$, and in this langauge this is called a **vector** U(1) symmetry. The axial symmetry rotates left- and right-handed parts by the opposite phase factors, while a vector transformation does not

$$\psi_L \to \psi'_L = U_V\psi_L = e^{+i\theta}\psi_L, \quad \psi_R \to \psi'_R = U_V\psi_R = e^{+i\theta}\psi_R, \quad U_V := \exp\{i\theta\mathbb{1}\} \tag{8.3.23}$$

Thus for a Lagrangian to be chirally invariant we must not mix left- and right-handed fields, in free or interacting terms. The usual mass term of the Dirac fermion fails this test and we have already noted that chiral fermion must be massless. Put another way, if we want a Lagrangian describing a world of purely left-handed fermionic particles, we must not have any individual terms which require right-handed particles i.e. no left-right mixing terms. Such a Lagrangian will have an axial symmetry.

Parity and Chirality

Finally, chirality is linked with another discrete symmetry, that of parity — an improper discrete space-time symmetry where one considers the reflected system

$$P: \quad \boldsymbol{x} \to -\boldsymbol{x}, \quad \boldsymbol{p} \to -\boldsymbol{p}, \quad S_i \to S_i \tag{8.3.24}$$

and spin is unchanged under parity. Thus we see that the helicity operator switches sign under parity, $P\Lambda = -\Lambda$, and thus parity switches left-handed massless fermion states into right-handed massless fermion states. For a theory to be invariant under parity, two experiments identical except they are mirror images of each other, must behave the same. If a theory treats left- and right- handed particles differently — a chiral theory — then it must also break parity, as in two parity related experiments, the left- and right-handed particles are swapped around. Thus experiments which reveal that the EWtheory is chiral (it treats left- and right-handed parts of fermions very differently) are often discussed in terms of parity violation.

[11] **EFS 8.3.12**: Show that the Axial or chiral symmetry of (8.2.4) is indeed a symmetry of the Lagrangian of a massless free Dirac fermion. Do this first using infinitesimals, and then for arbitrary θ [Hint prove that $U_5 = \cos(\theta)\mathbb{1} + i\sin(\theta)\gamma^5$].

Chapter 9

Strong interactions and QCD

The first part of the standard model of particle physics we shall look at is QCD, the modern theory of the strong interactions. The Lagrangian of QCD is rather simple, yet it seems to describe most of the particles discovered in the twentieth century. Thus QCD is a great example of a successful unification process. Its history charts the rise in particle physics of the concepts of global and local symmetry and gives an example of the successful use of explicit symmetry breaking. Only the ideas of local symmetry breaking, the Higgs-Kibble mechanism, have no role to play for the strong interactions.

9.1 Early theories and experiments

In 1935 Yukawa suggested that nuclear interactions could be described by the exchange of a particle or mass around 200GeV. This particle was later called the **pion** and leads to a picture of a nucleon as cloud of virtual pions of size around 1fm. The pion was found experimentally over the following decade by observing scattering of cosmic rays.

By the early fifties, studies of nuclear interactions showed that there was a symmetry under interchange of protons and neutrons, provided other exchanges between particles of almost similar mass, e.g. the pions, were made at the same time. This can not be expressed in terms of a simple conservation law such as electric charge conservation or baryon number conservation, i.e. it was not an abelian symmetry. However it can be expressed in terms of a non-abelian SU(2) symmetry. It can not be an exact symmetry of nature as EMdistinguishes the proton and neutron, and the different types of pion. Today this symmetry is explained in terms of a global symmetry between up and down quarks, i.e. this is an SU(2) **flavour symmetry**. Each flavour, or type, of quark has a distinct combination of electric charge and mass. The up and down quarks have almost the same mass, and same strong force charges, only EMand weak force effects distinguish them. A simple example of a model in this formalism might take the form for fermionic particles

$$\mathcal{L} = \bar{\boldsymbol{\psi}}_i \left(i\partial\!\!\!/ - m\right) \boldsymbol{\psi}_i - g(\bar{\boldsymbol{\psi}}_i \boldsymbol{\psi}_i)^2 \tag{9.1.1}$$

Here the $\boldsymbol{\psi}$ transforms in an $SU(2)$ representation[1]. For instance this could be two-dimensional for the nucleons $\boldsymbol{\psi} = (p, n)$ or three-dimensional for the fermionic Sigma baryons $\boldsymbol{\psi} = (\Sigma^+, \Sigma^0, \Sigma^-)$. Today we could use this model for the fundamental up and down quarks where $\boldsymbol{\psi} = (u, d)$ and u and d are the up and down quark fields. Note we often use a notation where the usual label for a particle, e.g. p for a proton, is used as a shorthand for the associated field, so $p \equiv p^\alpha(x)$ is a Dirac fermion field.

As time went on, increasing numbers of particle were being discovered. In particular from 1947 particles such as the Kaons and the Λ^0 were found. They had comparatively long life times. This suggested they

[1] Of course if $i = 1, 2, \ldots, d$ then the full symmetry of this Lagrangian is a $U(d) \cong U(1) \times SU(d)$ global symmetry. However, if $d \geq 2$ then an $SU(2)$ subgroup always exists so in reality this would be one piece of a larger Lagrangian and we imagine that terms in that Lagrangian limit the overall symmetry to this $SU(2)$ subgroup.

were the lightest hadrons which had a non-zero charge of new quantum number, **strangeness** S. It would be conserved under strong interactions, so that decays of the Kaons and the Λ^0 had to be via slower weak interactions. Strangeness obeyed the simple conservation law which comes from an abelian U(1) symmetry. This we now know to be counting the number of constituent strange quarks in the hadrons. In 1953 it was noted that strangeness plus baryon number B, a quantity dubbed the **hypercharge** $Y := B + S$ (also related to a U(1) symmetry since both B and S are), linked the isospin charges of the SU(2) up/down flavour symmetry to the EMcharges, Q, through the Gell-Mann-Nishijima relation

$$Q = T^3 + \frac{Y}{2} \tag{9.1.2}$$

Increasing numbers of new particles were being found, and the particles formed small sets where they had almost identical masses, the same strangeness and baryon numbers, had isospin charges appropriate for an SU(2) representation and with electric charges satisfying (9.1.2). It led both Gell-Mann and Ne'eman in 1961 to suggest that in fact the SU(2) isospin symmetry and the U(1) hypercharge symmetry are a subgroup of a weaker symmetry, a global SU(3) flavour symmetry. This approach to particle physics was known as the **Eightfold-Way**. An SU(3) symmetric theory can not be as good an approximation as the SU(2) symmetry as it groups particles together in the same SU(3) representations even though they have different mass. The point was that if you regarded the model as having a dominant SU(3) symmetric part and then a small correction term breaking the symmetry to the much better $SU(2) \times U_Y(1)$ symmetry of isospin and hypercharge, then this enabled one to make predictions about the mass differences. In particular, Gell-Mann (1961) and Okubo (1962) showed how this could be done, culminating in the prediction of the Ω^- baryon before it was found in 1964. A simple version for fermionic fields transforming in the three-dimensional SU(3) representation might look like

$$\begin{aligned}
\mathcal{L} &= \bar{\boldsymbol{\psi}}_i \left(i\not{\partial} - m_u\right) \boldsymbol{\psi}_i - g(\bar{\boldsymbol{\psi}}_i \boldsymbol{\psi}_i)^2 - (\Delta m)\bar{\psi}_3 \bar{\psi}_3 \quad (i = 1, 2, 3) && (9.1.3)\\
&= \bar{\boldsymbol{\psi}}_j \left(i\not{\partial} - m_u\right) \boldsymbol{\psi}_j - \bar{\psi}_3 \left(i\not{\partial} - (m_u + \Delta m)\right) \bar{\psi}_3 \\
&\quad -g(\bar{\boldsymbol{\psi}}_j \boldsymbol{\psi}_j)^2 - g(\bar{\boldsymbol{\psi}}_3 \boldsymbol{\psi}_3)^2 - 2g(\bar{\boldsymbol{\psi}}_j \boldsymbol{\psi}_j)(\bar{\boldsymbol{\psi}}_3 \boldsymbol{\psi}_3) \quad (j = 1, 2) && (9.1.4)
\end{aligned}$$

From our modern perspective we would assign the up, down and strange quarks as $\boldsymbol{\psi} = (u, d, s)$. The strange-up/down mass difference $\Delta m = m_s - m_u$ breaks the global SU(3) symmetry of the first two terms of (9.1.3) to $SU(2) \times U(1)$, the up and down retaining the SU(2) symmetry of (9.1.1) while the $\psi_3 = s$ has the usual U(1) phase invariance of a Dirac fermion. The second form (9.1.4) shows this most clearly. On the other hand, if Δm gives only small corrections, we could calculate results with just the first two terms of (9.1.3), and add in small corrections due to the mass difference term later. In this way we exploit the approximate SU(3) symmetry. However, when this approach was first put forward, all the fields were hadrons and none of those fitted into the simplest three dimensional representations, a point we shall pick up again in a moment.

The subsequent discovery of new conservation laws, what we would now call new quark flavours, is not terribly important for our discussion. The discovery in 1974 of the J/ψ particle, the lowest mass $c\bar{c}$ bound state, established the existence of the charmed quark. This confirmed earlier theoretical suggestions of Glashow, Iliopoulos and Maiani in 1970 that such a quark should exist to explain why certain strangeness changing weak interactions were suppressed. The bottom quark was found in a similar way by finding the lightest $b\bar{b}$ meson, the Υ, in 1977. The top, on the other hand, was only found in 1995 at Fermilab. However, even the lightest of these quarks, the charm, is so much heavier than the strange, up and down quarks, that extending the SU(3) flavour symmetries to SU(4) with charm is of no practical use, the symmetry is too badly broken.

A more important direction to follow from the Eightfold-Way is the establishment of the existence of quarks themselves, something I have taken for granted so far. In fact in the early 60's, it was not obvious that all these hadrons and their flavour symmetries could be explained in terms of some smaller constituent parts. However, it was noticeable that none of the hadrons in the SU(3) model fitted into the simplest representations, the defining or fundamental three-dimensional representations of SU(3). In group theoretical terms it is possible to build all other representations out of combinations of the three dimensional representations. Thus it is therefore natural, theoretically, to suggest that such building blocks might correspond to a more fundamental particle, and the idea of quarks was born in 1964 (Gell-Mann and Zweig). However, it was not clear if such entities really existed. On the other hand, ever since Yukawa's theory emerged, it was known that nucleons had a finite size of about 1fm and so it is natural to look for substructure. The confirmation that nucleons have point like constituents, dubbed **partons** by Feynman, can be done in the same way that the structure of the nucleus was revealed, that is by studying high-energy inelastic scattering of electrons on protons and neutrons. Such studies also show that the partons have charges $+2/3$ and $-1/3$ and so it is natural to associate them with the quarks suggested from the SU(3) flavour symmetry ideas.

9.2 Gauge Theory for strong interactions

The picture of the strong interactions based on different flavours of quarks still leaves us significantly short of QCD, today's accepted model for the strong interactions. We have no force carrying gluons, no colour charges linked with the gluons and no gauge theory. The historical discussion so far has focused on the types or flavours of the quark and these have been related (at least for up, down and strange quarks) to the conserved quantities associated with *global* symmetries involving the interchange of the flavours of quark, as in (9.1.1) and (9.1.3). Even with the identification of the point like constituents of nucleons, the partons, as the building blocks of the Eightfold-Way, the quarks, there were problems with the quark model.

The most obvious problem is that we see bound states of one quark and one anti-quark (the mesons) or bound states of three quarks (the baryons), but no other combinations are seen, e.g. quark-quark bound states. There is no reason in the theories discussed so far why other bound states should not exist, after all the nucleons bind in a wide variety of combinations to give the different nucleii, electron pairs can bind as Cooper pairs in superconductors.

However most illuminating is the existence of certain spin 3/2 baryons. A spin 3/2 bound state of three up quarks such as the Δ^{++} (mass 1232MeV), is consistent only with symmetric wave functions. For instance, the Δ^{++} state with total component of spin $S_z = +3/2$ could be composed from three up quarks as follows

$$\Psi_{\Delta^{++}}(x_1, x_2, x_3; S_z = +3/2) = u_\uparrow(x_1)u_\uparrow(x_2)u_\uparrow(x_3) \tag{9.2.1}$$

where $u_\uparrow$ is a $S_z = +1/2$ spin up up-quark wavefunction. The trouble with this wavefunction is that it is symmetric and thus we violate the fundamental spin-statistics theorem, namely that the fermionic Δ^{++} should have an anti-symmetric wavefunction.

The way forward is to give the quarks another label, a hidden degree of freedom. Then we can construct the same Δ^{++} state as before but now use this extra label to provide the required anti-symmetry , e.g.

$$\Psi_{\Delta^{++}}(x_1, x_2, x_3; S_z = +3/2) = \epsilon^{ijk} u_{\uparrow,i}(x_1)u_{\uparrow,j}(x_2)u_{\uparrow,k}(x_3) \tag{9.2.2}$$

where the ϵ^{ijk} is the totally anti-symmetric symbol ($\epsilon^{ijk} = +\epsilon^{jki} = -\epsilon^{jik}$) and the labels run $i, j, k = 1, 2, 3$. The three values required for these extra labels on the quarks are commonly referred to as the **colours** of the quarks, and the values of the labels can be called red, blue and green. The reason for the notation is more than just whimsical, as we will note below in section **??**. The fact that we duplicate each quark three times in the three different 'colours', suggests that we are working with a U(3) symmetry, though one completely different from the SU(3) flavour symmetry of the eight-fold way. This is a natural model for a triplet of Dirac fermions sharing the same mass and other properties. Let us now look at how this is implemented

9.2.1 One flavour

Let us first focus on a single flavour of quark, q. There is no evidence for parity breaking in strong interactions so we need to work with Dirac fermion fields $q \equiv q^\alpha(x)$. However the evidence discussed above suggests that each flavour of quark appears in three types or colours. Thus our single flavour of quarks should appear as a triplet of quarks q_i, all of the same flavour but with a new label $i = 1, 2, 3$, called **colour**. The three different types of coloured quark are often referred to as red green and blue quarks[2]. The one flavour QCD Lagrangian would then be

$$\mathcal{L}_{\mathrm{qcd},1} = \bar{q}_i \left(i\mathsf{D}^\mu_{ij}\gamma_\mu - m\mathbb{1}_{ij} \right) q_j - \frac{1}{2}\mathrm{Tr}\left\{\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\right\} \tag{9.2.3}$$

$$\mathsf{D}^\mu = \partial^\mu - ig_s\mathsf{G}^\mu \qquad \mathsf{G}^\mu = G^{\mu a}(x)\mathsf{T}^a, \quad \mathsf{T}^a \in SU(3), \quad a = 1, \dots, 8 \tag{9.2.4}$$

$$\mathsf{F}^{\mu\nu} = -\frac{i}{g_s}\left[\mathsf{D}^\mu, \mathsf{D}^\nu\right] = \partial^\mu\mathsf{G}^\nu - \partial^\nu\mathsf{G}^\mu - ig_s\left[\mathsf{G}^\mu, \mathsf{G}^\nu\right] \tag{9.2.5}$$

The first thing to note is that the classical theory has a global symmetry (most easily seen by setting the gauge fields to zero $W^{\mu a} = 0$) where it is invariant under full three-by-three unitary matrix transformations. Such matrices can always be factorized into a phase factor equal to the determinant, and then a special unitary matrix (of determinant one). So this Lagrangian has a global symmetry of $U(3) \cong U(1) \times SU(3)$. In the fully gauged theory (9.2.3) we have included the eight gauge fields necessary for the eight-dimensional local SU(3) group, but no gauge field for the U(1) global symmetry. The eight gauge fields are those of the eight **gluons** which mediate the strong interactions.

Since only the SU(3) part of the symmetry is gauged, one will often see QCD referred to as an SU(3) gauge theory, or even a theory of SU(3) symmetry. Both labels neglect the important U(1) global symmetry. This U(1) has a simple charge which is equal to the number of quarks minus the number of anti-quarks. It means that the mesons will always have a zero charge under this U(1) symmetry. On the other hand, the baryons with their three quarks will have charge +3, the anti-baryons charge -3 if we count +1 per quark. Since we are always free to set the scale of U(1) charges, we see that if we assign a charge of +1/3 (-1/3) for each quark (anti-quark) under this U(1) symmetry, then what we have is baryon number. So this global U(1) is the symmetry which ensures we reproduce one of the best conservation laws we have.

It is still not very obvious from this Lagrangian if there are bound states, and if they appear in the correct form, namely mesons and baryons only but to address this issue we will have to address a vital aspect of QCD — confinement and asymptotic freedom. This we postpone till section 9.3 and for now we will look at how to add the remaining flavours.

9.2.2 Additional flavours

The up and the down quarks have approximately the same mass as mentioned above. We just need to add an additional label to the quarks to reflect the up and down flavours

$$\mathcal{L}_{\mathrm{qcd}} = \sum_{f=u,d} \bar{q}_{fi} \left(i\mathsf{D}^\mu_{ij}\gamma_\mu - m\mathbb{1}_{ij} \right) q_{fj} \tag{9.2.6}$$

The labels $i, j = 1, 2, 3$ are the colour labels as in (9.2.3). The label f runs through two flavours of quark, up and down, and the quark field is again a Dirac fermion field, $\boldsymbol{q}$, which is a colour triplet and a flavour doublet. The strong interactions are not effected by differences in the flavours. The differences between proton and neutron, or the differences between the three pions, are due to the differences in the electromagnetic and weak properties of quarks (see section (**??**)). This tells us that the gluons are flavour

[2] It doesn't matter which numerical value we associate to which colour, as we can shuffle the order of the coloured quarks in the vector $\boldsymbol{q}$ by using a global symmetry transformation.

neutral and the gauge symmetry is completely separate from the flavour symmetry. So the gluons carry no flavour labels. The symmetry is now the local $SU_c(3)$ together with the pure global symmetry of the flavours, the $U_I(2) \cong U_I(1) \times SU_I(2)$ isospin symmetry, reflecting the approximate equality in the up and down quark masses. The overall $U(1)$ global symmetry of (9.2.3) which counted quark number, was just a simple phase transformation of the quark field. Such a transformation is part of the isospin symmetry, afterall, a phase transformation of either the up or the down quark field alone is still a symmetry. However, rather than working in terms of the number of up and the number of down quarks as the conserved number (under strong interactions at least), one works traditionally with the conserved numbers of the $U_I(1) \times SU_I(2)$. The diagonal generator of the $SU_I(2)$ symmetry, T^3, is half the third Pauli matrix so that the conserved quantity, the third component of isospin, is counting half the difference between the number of up and down quarks

$$J_I^{\mu 3} = \bar{q}_{fj}\gamma^\mu T^3_{jk} q_{fk} = \frac{1}{2}\bar{u}_j\gamma^\mu u_k - \frac{1}{2}\bar{d}_j\gamma^\mu d_k \tag{9.2.7}$$

The overall $U_I(1)$ is a transformation of both up and down field by the same phase

$$q_{fj} \to q'_{fj} = e^{i\theta} q_{fj} \tag{9.2.8}$$

so this counts the total number of up and down quarks. However, it is traditional to do this in units of 1/2 (remember we are free to scale our definitions of abelian charges) i.e. total isospin is half the number of up plus the number of down quarks

$$J_I^{\mu} = \bar{q}_{fj}\gamma^\mu T^I_{jk} q_{fk} = \frac{1}{2}\bar{u}_j\gamma^\mu u_k + \frac{1}{2}\bar{d}_j\gamma^\mu d_k \tag{9.2.9}$$

Of course this is just half the baryon number whose conservation should be encoded in the Lagrangian. The reason for using these two numbers rather than their sum and difference, i.e. the number of up and the number of down quarks, which is an equally good is largely historical. Isospin was introduced before the quark model.

Following the historical track it is worth noting that with the lightest three quarks, up, down and strange, we have

$$\mathcal{L}_{\mathrm{qcd}} = \sum_{f=u,d,s} \bar{q}_{fi}\left(i\mathsf{D}^\mu_{ij}\gamma_\mu - m_f \mathbb{1}_{ij}\right) q_{fj} \tag{9.2.10}$$

where $m_u = m_d < m_s$ with the first equality being a very good approximation. The symmetry is $SU_c(3) \times U_I(1) \times SU_I(2) \times U_s(1)$, the first factor being local, the remaining just global and the last is conservation of strange quarks, the strangeness quantum number. Had $m_s = m_u = m_d$ the global symmetry would have been a $U_B(1) \times SU(3)$ symmetry and this is the explicit breaking of this symmetry by the $m_s > m_u = m_d$ which is discussed in the eightfold way. As it is we see that the total number of up and down quarks, ...

To add the remaining quarks it is just a matter of extending the sum over flavours and allowing the quark masses to vary so we have

$$\mathcal{L}_{\mathrm{qcd}} = \sum_{f=u,d,c,s,b,t} \bar{q}_{fi}\left(i\mathsf{D}^\mu_{ij}\gamma_\mu - m_f \mathbb{1}_{ij}\right) q_{fj} \tag{9.2.11}$$

$$\mathsf{D}^\mu = \partial^\mu - ig\mathsf{W}^{\mu a}(x)\mathsf{T}^a, \quad \mathsf{T}^a \in SU(3) \tag{9.2.12}$$

where the label f now runs through each of the six flavours of quark, up, down, charm, strange, top and bottom. Each flavour has a colour triplet of Dirac fermion fields. The colour and flavour symmetries again factorise. The local $SU_c(3)$ of colour is linked to the quark triplets in i,j colour labels. If each quark mass is kept distinct (as it may be) then there is an independent U(1) phase transformation associated with each quark flavour, so we also have $U_u(1) \times U_d(1) \times U_s(1) \times U_c(1) \times U_b(1) \times U_t(1) = [U_f(1)]^6$. That is each quark flavour is exactly conserved under strong interactions. This in turn implies their sum is also conserved and this is baryon number conservation in QCD. The conserved quantities of the U(1) factors associated with the heaviest four flavours symmetries are the ones quoted for hadrons, e.g. strangeness is linked to $U_s(1)$ and is just the number of strange minus anti-strange quarks present.

If we take the up and down quarks to be the same, a very good approximation, then the $U_u(1) \times U_d(1)$ part of the global symmetry becomes a subgroup of a larger global symmetry, a global $U(2) \cong SU_I(2) \times U_Y(1)$ called isospin symmetries. The up and down quark numbers are still conserved in this symmetry, as they must be under the strong interactions. It is just that the conservation of up and of down quarks is usually specified via isospin and baryon number. Isospin number is $+1/2$ for up quarks and $-1/2$ for down quarks, from the standard diagonal element of the two-dimensional SU(2) algebra. Thus isospin is equivalent to half the difference between up and down quark numbers. This, with the numbers of the heavier quarks, the quantum numbers strangeness, charm etc., together with baryon number, is sufficient to fix the number of up and down quarks individually. One could of course specify the number of each quark flavour rather than this mixture of flavour number, isospin and baryon number but the latter notation predates the quark model given here.

9.3 Confinement and asymptotic freedom in QCD

A standard calculation in QFTusing perturbation theory and the renormalisation group leads us to the following formula for the running coupling constant of QCD $\alpha_s = g_s^2/4\pi$ at energy scale μ

$$\alpha_s(\mu) = \frac{4\pi}{(11 - \frac{2}{3}n_f)\ln(\mu^2/\Lambda_{\mathrm{qcd}}^2)} \tag{9.3.1}$$

where n_f is the number of flavours. The energy scale Λ_{qcd} is something which is a must appear in the quantum theory but which can be defined in a variety of ways. Here we are defining it to be the scale at which the coupling constant becomes infinite[3]. In reality this infinity is a symptom of the failure of perturbation theory for QCD at low energy scales $\mu \lesssim \Lambda_{\mathrm{qcd}}$, or equivalently long distances $\gtrsim$ 1fm and the formula is invalid for such energies.

At high energy scale, short distances, the analysis leading to formula (9.3.1) is valid (we assume the coupling was weak) and this suggests that the strength of QCD is weak in this regime, so that the quarks and gluons become free in the limit of high energies. This is the property of **asymptotic freedom** and it enables us to make good predictions using perturbation theory for high energy QCD processes[4]. For these types of process it is possible to get good experimental checks of perturbative theoretical calculations which gives us much faith in QCD as the theory of the strong interactions.

At high energy scale, short distances, the analysis leading to formula (9.3.1) is invalid (the coupling is blowing up yet the derivation assumes it is weak). However the breakdown does indicate that QCD has interesting long distance behaviour. At such low energies, long distances, it is better to think in terms of the following analogy.

Picture a bar magnet with the lines of force, the magnetic flux, linking its two poles. The flux lines use the whole of space though they are strongest nearest the magnet. Suppose we could put the bar magnet[5] in a superconductor. The Meissner effect tells us that magnetic field can not exist in a superconductor yet the two poles must have flux lines starting from one and ending at the other. The most energetically favourable way to do this is to create a narrow tube between the two poles where the superconductor is in the higher

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[3]All *non-abelian* gauge theories all have running coupling constants of a similar form. For all other theories the couplings blow up at large energies and perturbation theory is a good low energy approximation.

[4]In practice, this analysis is valid for the infinite *Euclidean* energy region but it it is still of great physical importance for high energy QCD processes.

[5]We could also think of the poles of the magnet as a magnetic monopole and anti-monopole pair.

energy (so usually metastable) normal state (the local U(1) symmetry is restored in this region) and there the magnetic flux can exist. Obviously the smaller this region of normal state is, the lower the energy of the configuration, so a tube of normal state linking the two poles is optimal. The amount of energy needed to create this tube of normal conductor with the magnetic flux lines is going to be roughly proportional to the length of the tube and the flux lines now look more like a string linking the two poles. The quarks also act as colour charges and we can think as them being linked by lines colour flux. Of course there are many additional features for our non-abelian SU(3) colour theory of QCD when compared to the picture of magnetic forces in U(1) electromagnetism. Nevertheless, it does appear that at larger distances we can picture the colour flux lines between quarks as being a string between the two and the potential energy stored is simply proportional to the separation of the quarks

$$V(r) \approx \sigma r \tag{9.3.2}$$

The parameter σ is called the **string tension**. Thus as you try to separate two quarks you need more and more energy. Eventually, it becomes energetically favourable to create a quark/anti-quark pair each linked to the original quarks via two shorter strings of colour flux. This gives us a picture where we can never see a free quark, just as we can never get a single pole of a bar magnet or see a magnetic monopole. If we try to separate a quark from a meson, we just get new pairs, new mesons instead. This lack of free quarks in QCD is called **confinement**. Of course we have not proved this with our analogy, merely illustrated what several calculations have suggested. In fact, the short distance confinement property has never been rigourously proven to be a property of QCD (and all other non-abelian theories), unlike whereas asymptotic freedom at long distances is proven. Nevertheless, all the theoretical evidence suggests QCD is confining and few doubt the theory has this property. Thus we feel confident that QCD also describes the experimental fact that we have never seen a free quark.

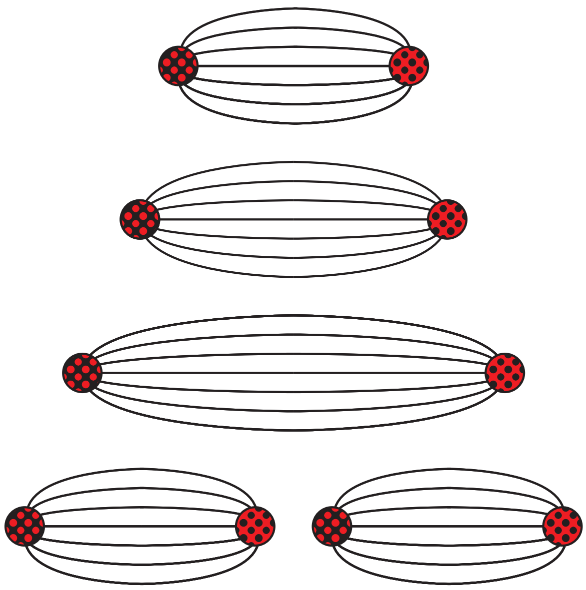


Figure 9.1: As a meson can be pictured with colour flux being in confined to a narrow tube between the coloured quark and the anti-quark. As they are pulled apart, the energy rises, essentially linearly until it is energetically favourable to create a new quark/anti-quark pair to reduce the total length of the colour flux tubes..

The analogy above was appropriate for a quark/anti-quark pair, that is a meson. Its less appropriate for baryons as one does not expect two or three magnetic monopoles to form a bound state (their flux lines each must end on a magnetic anti-monopole). This is where the different group behind the symmetries becomes important. For the bar magnet, the net magnetic charge or the magnet is zero, and all flux lines which start from one pole can end on the other. The mathematical way of saying we have a bag of quarks with no flux lines emerging from the bag is to say that the collection together has zero colour charge. Colour flux links colour charges just as magnetic flux links magnetic poles. So if we take the wavefunction of such a bag of confined quarks, and make an SU(3) colour transformation, each quark transforms no trivially (each has a colour charge) but the combined wavefunction must remain unchanged so that no flux need emerge. The bag must therefore have no net colour charge and is a colour singlet. In the same way a bar magnet has no net magnetic charge as it has two opposite poles. *Confinement means that the only allowed combinations of quarks are colourless ones.*

Confinement and allowed bound states

Confinement has told us that we can see no free quark, as each quark has a colour. It has however told us much more that quarks can only appear in certain combinations which also have no overall colour. In practice we have seen only mesons and baryons so let us look at how the SU(3) local symmetry of QCD explains this observation.

Clearly a quark and its anti-quark have opposite charges (all anti-particles must) so they can form colourless combinations. Thus mesons like the π^0, made from a quantum superposition of up/anti-up quarks and down/anti-down quarks can exist. Such mesons would always have zero flavour related charges like isospin, strangeness, charm etc. We can think of the anti-quarks carrying one of three 'anti-colours' which can cancel the colour of the quarks.

Taking this further one could then take quark/anti-quark pairs of different flavours but opposite colour charges to form bags with no colour flux emerging. These would be mesons such as the π^+, an up/anti-down combination. These would be the mesons with net isospin, strangeness, charm etc.

However, it is harder to see from the electromagnetic analogue why Baryons should exist. Afterall, three magnetic poles would need three opposite poles. However, this is where the details of the different symmetry groups become important. In SU(3) it is possible to form a colourless combination (colour chargeless) from three quarks but one needs to understand the group theory to see why. It turns out that if the each of the three quarks has a different charge then this combination can result in no overall colour charge. This is why the SU(3) symmetry uses the colour labels as a useful mnemonic since three primary colours combine to give white or no colour i.e. a colour chargeless state.[6] In fact we have already have already given the combination of three coloured quark wavefunctions needed to give an overall colourless combination. This was given in (9.2.2) noted in connection with the Δ^{++} particle used to motivate the introduction of colour. One could check that this combination is invariant under the SU(3) three dimensional matrix transformations of (**??**), the quark colour transformations, but it is far more efficient to use standard group theoretical techniques which are beyond the scope of this book (see [2] for instance).

Group theory would also tell us that far more complicated combinations of quarks are also colourless, (four quarks and one anti-quark, i.e. a meson+baryon combination, is one obvious possibility) but in practice these have not been seen. This lack of observation may simple be a matter of energetics as while group theory explains what is possible, it still does not confirm that such particles can form a stable or long-lived bound state. That must come from a more detailed study of low energy, long distance QCD. Since that is already beyond the reach of our best theoretical tool, perturbation theory, and multi-body bound states

[6] To be more precise, the quarks live in the fundamental 3 representation of SU(3), that is they were a simple triplet in (**??**). If one multiplies together three functions in such a representation, the combination can behave like a 10 dimensional or 8 dimensional SU(3) representation, i.e. ones which transform non-trivially and so have colour charge, or like a 1 dimensional representation, i.e. a singlet something which has no charge. The notation used for this is that $3 \times 3 = 1 + 8 + 8 + 10$. Likewise, anti-quarks live in the conjugate three dimensional representation, denoted $\bar{3}$ as it is distinct from the 3 representation of the quarks. It then turns out that for SU(3) we have $3 \times \bar{3} = 1 + 8$.

are already a complicated problem, there is no strong theoretical evidence for such larger bound states. However, theory is unable to exclude them either and some analyses point to more exotic quark bound states, such as quark matter [**?**].

9.4 Questions

Q9.1. QCD string tension

Hadrons have a size of around $1\text{fm} \sim 5\text{GeV}^{-1}$. Use this to estimate the QCD string tension σ of (9.3.2).

Chapter 10

Weak interactions and EW theory

The development of QCD brought great simplicity to a complicated zoo of particles in terms of bound states of more elementary constituents, the quarks, but it has always been the physics of one fundamental force, the strong nuclear force. The EW theory does not unify a vast collection of particles, but rather it provides us a successful example whereby two forces, previously thought of as distinct, are brought together under one theory. At the same time, local symmetry breaking and the Higgs-Kibble mechanism play a crucial role in EW theory, the one trick we didn't need for QCD.

10.1 Early theories and experiments

The classic example of the weak nuclear force is seen in β-decay of various radioactive nuclei, in which a neutron is converted to a proton[1]

$$\mathrm{n} \longrightarrow \mathrm{p}^+ + \mathrm{e}^- + \bar{\nu}_\mathrm{e} \tag{10.1.1}$$

The existence of the neutrino was postulated by Pauli in 1930. However its existence was confirmed by direct observation only in 1956 since it interacts so weakly that most cosmic neutrinos pass through the whole earth without interacting. So weak nuclear physics and the β-decay in particular have been stimulating advances in particle physics for a long time. This process also suggests a possible link between weak and EM forces as EM charges are clearly involved in the decay. However other characteristics of the weak force such as its range — of order the size of a nucleus $r_{\mathsf{EW}} \sim 10^{-15}$m — are quite different from EM — a very long range force. These apparently conflicting clues were only finally reconciled using QFT during the 60's and 70's.

Early theory – Fermi theory

One of the earliest models is due to Fermi . It describes some of the low energy physics quite well. In a modern relativistic language, we work with four Dirac fermion fields of the type (8.1.1), ψ_e, ψ_ν, ψ_n and ψ_p describing the electron, electron neutrino, neutron and proton respectively. Their dynamics are described by the Lagrangian

$$\mathcal{L} = \sum_{j=e,\mu,n,p} \bar{\psi}_j (i\partial\!\!\!/ - m_j)\psi_j - \frac{G_F}{\sqrt{2}}(J^\mu)^\dagger J_\mu \tag{10.1.2}$$

The J^2 term contains the relevant interaction terms written in terms of the current[2]

$$J^\mu = \bar{\psi}_\mathrm{e}\gamma^\mu\psi_\nu + \bar{\psi}_\mathrm{p}\gamma^\mu\psi_\mathrm{n} \tag{10.1.3}$$

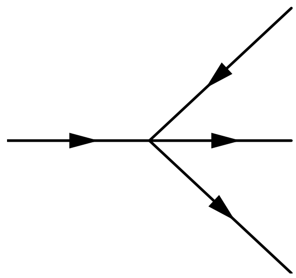
[1] The reverse process are very rare in nature, yet positron emission is important in some forms of medical imaging.

[2] The **current algebra** approach to QFT, often used in the 1960's, focused on currents and their commutation relations rather than Lagrangians and fields. Discussions of early theories are often phrased in this language. Also note that ignoring the mass terms, the current J^μ is a Noether current of this Lagrangian (10.1.2).

and its hermitian conjugate. Expanding the Lagrangian, we find interaction terms such as $\left(\bar{\psi}_{\mathrm{p}}\gamma^{\mu}\psi_{\mathrm{n}}\right)\left(\bar{\psi}_{\nu}\gamma^{\mu}\psi_{\mathrm{e}}\right)$ which give a contribution to the β-decay process at the tree level, i.e.

$$\mathcal{M} = \langle \mathrm{p}, \mathrm{e}, \bar{\nu}; t = +\infty | \mathrm{n}; t = -\infty \rangle \propto G_F \tag{10.1.4}$$

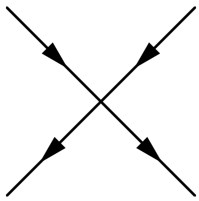
In terms of Feynman diagrams, if we were treating the neutron, proton, electron and neutrino as fields the lowest order contribution to the matrix element $\mathcal{M}$ would come from



(10.1.5)

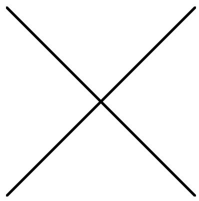
Thus the β-decay rate is just $\sim |\mathcal{M}|^2 \propto G_F^2$.

We can take this a step further. The interaction in this model is a four-fermi interaction.



(10.1.6)

In principle it is similar to the four scalar interaction of simple scalar $\lambda\phi^4$ theory where the lowest order contribution to the matrix element $\mathcal{M}$ for $\phi \to \phi^3$ scattering would come from



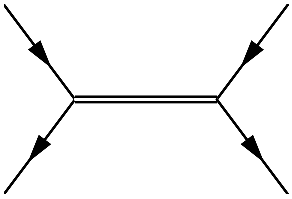
(10.1.7)

However, the coupling constant is dimensionless in $\lambda\phi^4$ theory, while G_F has dimensions of inverse mass squared reflecting the different dimensions of the scalar and fermion fields. More importantly, this tells us that unlike $\lambda\phi^4$ theory, the four-fermi interaction is *not* renormalisable in four dimensions. So from our modern perspective, Fermi theory of (10.1.2) can *only* be a low energy approximation. That is, the lack of renormalisability implies that Fermi theory goes wrong at high energies and another mechanism is required to cut off at the ultraviolet infinities. Again, our modern perspective suggests using a gauge theory

$$\mathcal{L} = \bar{\boldsymbol{\psi}} i \not{D} \boldsymbol{\psi} + \ldots = \bar{\boldsymbol{\psi}} i \not{\partial} \boldsymbol{\psi} + g J_N^{\mu a} W_\mu^a + \ldots \tag{10.1.8}$$

where $\boldsymbol{\psi}$ contains p, n, e and ν fields. D contains couplings g, gauge fields $W^{\mu a}$ and generators of some local symmetry. For fermions, we can always write out the gauge field/fermion interactions in terms of the conserved Noether currents as sketched in the second form above and noted in (8.2.8).

The lowest contribution to the β-decay process would then come from two fermion-fermion-gauge particle interactions linked by a gauge field propagator[3]



(10.1.9)

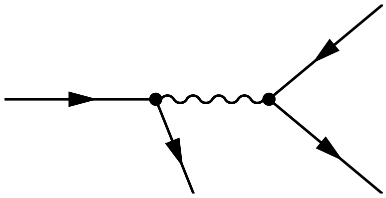
so that the matrix element for β-decay would be

$$\mathsf{M} \propto g\frac{1}{k^2 - M^2}g \tag{10.1.10}$$

where the gauge boson is assumed to massive. It can't be massless otherwise we would have long range weak nuclear forces just as we have long range EM forces carried by the massless photon. Thus we will have to break the local symmetry as described above. In QFT such theories are renormalisable and the proof of this, due to t'Hooft and Veltman in 1971[4], was the final step to a complete EW theory.

If we are studying energies much lower than the gauge boson mass, then $k^\mu \ll M$, so

$$\mathsf{M} \propto \frac{g^2}{M^2} \equiv G_F \tag{10.1.11}$$



(10.1.12)

Low energy studies can not distinguish between a simple four-fermi model and a theory of broken gauge symmetry, only interactions at energy scales of order the gauge field mass M can decide which is correct.

We can estimate the required mass scale if we guess that EM and weak forces are related as discussed at the start, and estimating that $g \sim e$. Then using the known value of Fermi's constant, $G_F = 1.2 \times 10^{-5}\text{GeV}^{-2}$, we see that $M \sim 100\text{GeV}$. Note that this corresponds to a range of $r \sim 1/M \approx 10^{-3}\text{fm}$.

We have couched this discussion in terms of modern theories with broken local symmetry. Only when renomalizability of broken symmetry gauge fields was proved, did this option emerge as the overwhelming theoretical favourite, subsequently vindicated by experiments. The whole discussion can be phrased in terms of a generic vector field W^μ in a theory valid for energies below some scale. The new field $W^\mu(x)$ need not be a gauge field, it just mediates the weak force. The J^2 interaction term of (10.1.2) is again replaced by a $G_F W^\mu J_m u$ type term and simple kinetic and mass terms for W^μ added. In fact Fermi theory can be written in terms of many other types of fermionic current e.g. axial currents $J^{\mu 5} = \bar{\psi}_e\gamma^\mu\gamma^5\psi_\nu + \bar{\psi}_p\gamma^\mu\gamma^5\psi_n$. The Lagrangian of (10.1.2) in terms of such currents would still give β-decay like interactions. Again all these current-current interaction terms could be replaced by appropriate $G_F W^{\mu 5} J^5_\mu$ new field/current interaction terms with simple kinetic and mass terms for the new fields. In all cases, the analysis would lead to similar conclusion about the mass of the force carrying particles, whatever they might be. The choice of current or field type does effect predictions and in the 1950's further experimental evidence started to discriminate between these options. This led to **V-A theory**. We will not study these in detail, but the new physical insights which lay behind them are crucial to EW theory and we now turn to consider what these were.

[3]Recall $\bar{\psi}\not{D}\psi$ contains $\bar{\psi}gW\psi$, i.e. anything above a cubic power appears as a vertex.
[4]The Nobel prize for physics was given for this in 1999.

10.2 Parity, helicity, and chirality in EW interactions

Parity P is a discrete space-time symmetry $\mathsf{P} : \boldsymbol{x} \longmapsto -\boldsymbol{x}$. Many theories show this mirror symmetry including EM. Thus if we have a solution, $\{A^\mu(t, \boldsymbol{x}), \psi(t, \boldsymbol{x})\}$ to Maxwell's equations, or to its QFT extension QED, then because parity is a symmetry of QED, $\{A^\mu(t, -\boldsymbol{x}), \psi(t, -\boldsymbol{x})\}$ is also a solution. We say that *QED is parity invariant.*

We saw in section 8.3 that Dirac fermions were parity invariant but chiral fermions were not. We noted that this could be visualised by making the link between helicity Λ, chirality P_L, P_R and parity P

$$\Lambda = 2\boldsymbol{S}_0 \frac{\boldsymbol{p}}{|\boldsymbol{p}|} \qquad\qquad P_{R(L)} = \frac{1}{2}\left(1 + (-)\gamma_5\right) \tag{10.2.1}$$

Using this type of language we can now discuss the experiment of Wu et. al (1957). For simplicity, imagine that the electron is massless (a full analysis in QMterms is given in Perkins [18] §7.4). The experimental setup is given in **Figure 10.1**.

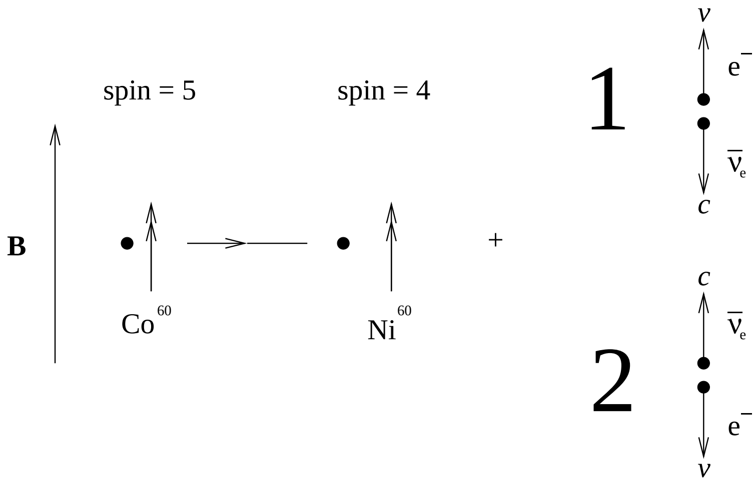


Figure 10.1: The electron is expected to have all possible directions for $\boldsymbol{v}$, but with spin fixed by $\boldsymbol{B}$.

The magnetic field $\boldsymbol{B}$ causes the spin of the Cobalt atom to become aligned with it. The Cobalt atom undergoes β-decay and the resulting Nickel nucleus must have it spin similarly aligned so the spin-half leptons produced must both have their spins aligned with the magnetic field. Ignoring the slight recoil of the heavy Nickel nucleus means that in the lab frame the two leptons have equal and opposite momentum. Pretending the electron is massless and assuming the neutrino is too, means the spins must be aligned parallel or anti-parallel with the momentum, i.e. the leptons are in helicity eigenstates. Since the spins must be parallel with the magnetic field, this means there are only two possible sets of velocities for the lepton pair - one lepton runs parallel to the field and has helicity $+1$ while the other moves anti-parallel and has helicity -1. The two solutions are related by parity, $\mathsf{P} : 1 \longleftrightarrow 2$, as this symmetry leaves spins unchanged $S \longrightarrow S$ but reverses velocities $\boldsymbol{p} \longrightarrow -\boldsymbol{p}$. Thus if the weak force was invariant under parity we will see the electron come out parallel to the field as often as it comes out in the opposite direction.

What was seen was case 2 *only*[5]! The weak force does not only break parity by favouring one direction for the electron over the other, but it breaks it as much as possible by producing only one of the two parity related solutions. We say that parity is **maximally violated** by the weak nuclear force.

[5]Of course, we have simplified both the theoretical model in several ways and glossed over the experimental difficulties and limitations, but the clear asymmetry in the position of the emerging electron seen in the real experiments led to the same conclusion.

In terms of helicity, Goldhaber in 1958 confirmed that the helicity +1 partner of the electron was an anti-neutrino. Thus we see that EW interactions only interact with left-handed fermions here just e_L with ν_L and not the e_R we know to exist from —EM interactions which respect parity and so do not distinguish between left- and right-handed electrons.

In the language of the Fermi theory (10.1.2) the lepton contribution to the relevant currents appearing in the interactions are of the form

$$J^\mu = \left((P_L\psi_\nu)^\dagger\gamma_0\right).\gamma^\mu\psi_\mathrm{e} = \bar{\psi}_\nu\gamma^\mu P_L\psi_\mathrm{e} = \frac{1}{2}\bar{\psi}_\nu\gamma^\mu\psi_\mathrm{e} - \bar{\psi}_\nu\gamma^\mu\gamma_5\psi_\mathrm{e} \tag{10.2.2}$$

The last version shows that the current is a Lorentz vector minus a Lorentz axial-vector, and this led to what was called **V-A theory**. Rather than discuss this, we will move straight to the full gauge theory.

10.3 Gauge theory for weak and EM interactions

To help reduce the clutter of subscripts and superscripts, we shall now use a particle's letter to denotes the corresponding field: thus the electron is now described by

$$\mathrm{e} \equiv \mathrm{e}^\alpha(x) = \psi^\alpha_{(\mathrm{e})}(x) \tag{10.3.1}$$

If we want to try to use a gauge theory to describe weak and EM forces, we have to specify the gauge group and the representations into which the physical particles will go, equivalent to fixing the charges of the particles. Of course today the wealth of experimental evidence means that most of the EW theory is fixed. However, if one was limited to basic evidence reviewed here, one would find a much richer range of possibilities. At the very least, many models could not be ruled out with out much more detailed calculations than we will consider. Of course this mimics the historical situation, the information available in the 60's was limited and patchy. Therefore we will first see how the few clues presented might be used to decide on a plausible gauge group and representations. Of course ultimately we will pick the answer we now know to be correct, but the lesson is that the EW model was not obvious historically, a unique answer does not just fall out from theoretical considerations. Likewise, any attempts to use these ideas on a grander scale for higher energy theories, will not provide a unique route forward, but one with several likely options that can only be eliminated after considerable calculational effort. Should a theory survive basic theoretical consistency, it still has to pass future experimental tests, though these are often suggested by the proposed theories. It is one of them most remarkable features of EW theory is that it has survived increasingly stringent experimental tests yet the simplest version[6] written down in the 60's, renormalised in the 70's and presented in graduate level texts of the 80's has been sufficient for our needs. Only with the emergence of firm evidence for a more complicated neutrino physics at the turn of the millennium do we have to update the model. We will return to the status of the EW model at the end of the chapter.

10.3.1 What gauge symmetry should we use?

From the experiments demonstrating parity violations of weak processes, we see that the weak forces can be represented by terms in the Lagrangian such as

$$J^\mu W_\mu(x), \quad J^\mu \sim \bar{\boldsymbol{e}}_L\gamma^\mu\nu_L + \mathrm{h.c.} \tag{10.3.2}$$

in the Lagrangian. We have already seen, (8.2.8), that fermion gauge field interactions in non-abelian gauge theories are of the form

$$\bar{\psi}i\not{D}\psi = i\bar{\psi}\not{\partial}\psi + gJ^{\mu a}W^a_\mu(x) \tag{10.3.3}$$

[6] One Higgs doublet, no neutrino masses or mixing.

where the currents are the Noether currents of the form

$$\bar{\psi}\gamma^\mu \mathsf{T}^a \psi \tag{10.3.4}$$

Thus if we are going to encode the experimental information of (10.3.2) using a non-abelian gauge theory of (10.3.3) and (10.3.4) it suggests that we need *at least* a doublet of left-handed leptons mixed by generators T^a which are at least 2×2 in size. The simplest form would be to choose a doublet such as

$$\psi = \begin{pmatrix} \nu_L \\ \mathrm{e}_L \end{pmatrix} \tag{10.3.5}$$

together with a gauge group of $SU(2)$ or bigger.

The other simple handle on the gauge group is the number of gauge bosons required as there is one gauge boson per generator. We have already seen that if we are to mediate beta decay through a gauge field then it must have electric charge equal to that of the electron This must therefore have a distinct anti-particle with the same charge as the positron, so we see that we need at least *two* charged gauge bosons, $W^\mp$ for the weak forces.

The fact that the force carrying particles of the weak nuclear force have an electromagnetic charge is also one of the reasons why it makes sense to try and describe the two forces in a single theory. The difference in the range of the forces would suggest otherwise, but we have already seen how this can be accommodated in a gauge theory using SSB, an idea not proven for those who first tried to construct a unified EW gauge theory. Given that we are trying to unify the weak forces with the electromagnetic forces, which are well described by QED, we also include the photon. The gauge group must therefore have a dimension of at least three.

Some Possible Choices of Symmetry Group

i) $SU(2)$
EW and EM all in one $SU(2)$ theory. This is the **Glashow $SO(3)$ model**, since $SU(2)$ and $SO(3)$ are locally isomorphic — they have the same generators.
At the time this model was postulated, there was no indication of the Z^0 particle so this does not invalidate this case. However, the big problem was that EM conserves parity, while weak forces do not, in which case how can their generators be part of the same gauge group?

ii) $SU_W(2) \times U_{\mathsf{EM}}(1)$
That is, add a $U(1)$ EM symmetry to a pure weak force symmetry $SU(2)$. $SU(2)$ has three generators, and $U(1)$ has one, so a further gauge boson is required by this model: the Z^0.
The problem here is how do the $W^\pm$ gauge bosons get an EM charge when EM symmetry commutes with weak $SU(2)$ symmetry (i.e. they are completely independent).

iii) Larger symmetries
higher dimensional groups than the previous two and the $SU(2) \times U_Y(1)$, which follows, were tried, but these require the introduction of more gauge bosons for each extra dimension in the group, and thus were aesthetically unappealing.

iv) $SU(2) \times U_Y(1)$
where $U_Y(1)$ is *not* EM symmetry. The $U_{\mathsf{EM}}(1)$ which is a good (unbroken) symmetry in todays world must therefore 'lie between' the $SU(2)$ and $U_Y(1)$ groups. The single generator $U_{\mathsf{EM}}(1)$ generator will be a linear combination of generators of the other groups. This turns out to be the correct choice.

10.3.2 Choosing representations

Having chosen a gauge group to study, we still have to choose into which representations the physical particles will go. Even with a given gauge group, choosing different representations for the particles has a big effect on the physics. We will work the other way round, starting from the experimental clues, to suggest one plausible solution. Once again, many other theoretical possibilities suggest themselves at the level of evidence presented here, but we will have he luxury of using hind sight to choose the correct solution.

The representation needed is encoded in the covariant derivatives and revealed though the interactions of the fermions or scalars with the gauge boson fields. The particles will not all go into one irreducible representation, and we will need a separate covariant derivative for each. Also note that since we have chosen the product group $SU(2) \times U_Y(1)$, we need *two* gauge couplings, g and g'. We will define the generic covariant derivative to be of the form

$$\mathsf{D}^\mu = \partial^\mu \mathbb{1} - igW^{a\mu}(x)\mathsf{T}^a - ig'B^\mu(x)\left(\frac{1}{2}\mathsf{Y}\right) \quad a = 1,2,3 \tag{10.3.6}$$

The $SU(2)$ generators are the three T^a and have the three gauge boson fields $W^{\mu a}$ associated with them. While there are representations of all dimensions, we will only need the one-dimensional trivial representation where $\mathsf{T}^a = 0$, or the fundamental two-dimensional representation using the Pauli matrices $\mathsf{T}^a = \frac{1}{2}\tau^a$.

The $U_Y(1)$ symmetry has the gauge field $B^\mu(x)$ and the generator $\mathsf{Y} = q_Y\mathbb{1}$. As with all U(1) symmetries, the choice of representation of $U(1)$ is all in the choice of the normalisation constant: $q_Y \in \mathbb{Q}$, i.e. we have to choose the overall phase symmetry of our fields:

$$\left(e^{iq_Y\frac{Y}{2}\theta}\right)\boldsymbol{\psi} = e^{i\frac{1}{2}q_Y\theta}\boldsymbol{\psi} \tag{10.3.7}$$

Note that we have included a factor of $\frac{1}{2}$ in the explicit definition of the $U_Y(1)$ part of our covariant derivative. This is included for historical reasons but could easily have been included in the definition of the g' coupling or the q_y charges, again exploiting the freedom to rescale $U(1)$ generators. This factor of $1/2$ is a matter of convention and not all texts use the same conventions for in the $U_Y(1)$ sector.

We must therefore choose doublets and singlets for $SU(2)$ representations of particles *and* we must choose the q_Y of each irreducible representation, which is equivalent to choosing the $U_Y(1)$ representation.

10.3.3 Leptonic Sector

We have already noted how the experimental evidence suggests that the left and right-handed leptons behave completely differently under weak forces. Let us examine each part separately.

Left-handed leptons

(a) $SU(2)$ symmetry
We have already noted above in (10.3.5) that since β decay sees currents/interactions of the form

$$W^-\bar{\nu}_L\gamma^\mu \mathrm{e} \tag{10.3.8}$$

that we need to put left-handed electrons, e_L, and neutrinos, ν_L, in the same doublet. This ensures they will be mixed together by the same gauge boson and can be produced together. Therefore, try

$$\boldsymbol{l}_L{}^\alpha = \begin{pmatrix} \nu_L^\alpha(x) \\ \mathrm{e}_L^\alpha(x) \end{pmatrix} \tag{10.3.9}$$

where we shall from this point drop the α spin indices for convenience. Defining the covariant derivative for this left handed doublet to be D_L with $\mathsf{T}^a = \frac{1}{2}\boldsymbol{\tau}^a$ we see that the interactions are of the form

$$\bar{\boldsymbol{l}}_L i \not{D} \boldsymbol{l}_L = +\frac{g}{2} W^{\mu a} \bar{\boldsymbol{l}}_L \gamma^\mu \tau^a \boldsymbol{l}_L + \ldots \tag{10.3.10}$$

$$= +\frac{g}{2}\left(W^{\mu 1} - iW^{\mu 2}\right)\bar{\nu}_L \gamma^\mu e + \frac{g}{2}\left(W^{\mu 1} + iW^{\mu 2}\right)\bar{\nu}_L \gamma^\mu e + \ldots \tag{10.3.11}$$

which is the form required provided we define

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}\left(W_{1\mu} \pm iW_{2\mu}\right) \tag{10.3.12}$$

as the gauge fields carrying the electromagnetic charge. This is exactly the same sort of redefinition required when wanting to identify the charge eigenstates in a scalar theory with an SO(2) global symmetry, where one has to switch to the complex scalar field representation usually used for $U(1) \cong SO(2)$ symmetries. The trick works here for gauge bosons (which are just real fields at one level) in exactly the same way. In terms of the generators associated with each of the gauge bosons, we are switching to another familiar basis of SU(2), namely that of the raising/lowering operators

$$\tau^{\pm} = \frac{1}{\sqrt{2}}\left(\tau_1 \pm i\tau_2\right) \tag{10.3.13}$$

(b) $U_Y(1)$ symmetry
To fix this part is equivalent to fixing the q_Y charge for the left handed-doublet, and we will denote this as q_{YL}. Thus it contributes

$$J^\mu_Y = \ldots + \frac{1}{2}\bar{\boldsymbol{l}}_L \gamma^\mu \mathsf{Y} \boldsymbol{l}_L + \ldots \qquad \mathsf{Y}_{(l_L)} = q_{YL}\mathbb{1} \tag{10.3.14}$$

It is not an observable charge because of the SSB required in the model and the $U_Y(1)$ will not turn out to be the unbroken electromagnetic symmetry. With no direct handle on this charge, we will fix this once we have identified its relationship to the electric charges, which we do know.

Right-handed leptons

So far, we only have e_R: no ν_R have been see in experiments to date (and, unless we think ν has mass, we will not see any).

(a) $SU(2)$
Put in a singlet (i.e. a trivial representation). This is one-dimensional, hence

$$\mathsf{T}^a = 0 \tag{10.3.15}$$

Thus, there is no $SU(2)$ charge, thus no $W^{\mu a}$ interactions, and so no $gW^{\mu a}$ term in D_R.

(b) $U_Y(1)$ This is tackled in exactly the same manner as in the left-handed case but we use q_{YR}, so

$$J^\mu_Y = \ldots + \frac{1}{2}\bar{\mathrm{e}}_R \gamma^\mu \mathsf{Y}_{eR} \mathrm{e}_R + \ldots \qquad \mathsf{Y} = q_{\mathrm{e}_R} 1 \tag{10.3.16}$$

EM lepton current

We have now fixed the lepton/gauge boson terms to be of the form

$$\bar{\boldsymbol{l}}_L i \not{D}_L \boldsymbol{l}_L + \bar{e}_R i \not{D}_R \mathrm{e}_R \tag{10.3.17}$$

with the generators chosen as above.

However this theory is meant to represent EM so it must contain QEDits usual photon/elecron term $A_\mu J^\mu_{\mathsf{EM}}$, where

$$\bar{\psi}_\mathrm{e} i \not{D}_{\mathsf{EM}} \psi_\mathrm{e} = \bar{\psi}_\mathrm{e} i \not{\partial} \psi_\mathrm{e} + \mathsf{e} J^\mu_{\mathsf{EM}} \tag{10.3.18}$$

Note the zero charge on the neutrinos means they do not interact with the photon A^μ. The constant of proportionality will be e, the electron charge, with the Noether current being given by

$$J^\mu_{\mathsf{EM}} = -\bar{\boldsymbol{\psi}}_\mathrm{e} \gamma \boldsymbol{\psi}_\mathrm{e} = -\bar{e}_R \gamma^\mu \mathrm{e}_R - \bar{e}_L \gamma^\mu \mathrm{e}_L - 0 \bar{\nu}_L \gamma^\mu \nu_L \tag{10.3.19}$$

A quick and dirty way of finding this is to say that this must be a combination of conserved currents, the weak isospin $J^{\mu a}$ and the weak hypercharge J^μ_Y. Comparing (10.3.17) with (10.3.18) we see that we are not interested in the terms which mix the electron and neutrino fields which come only from T^1 and T^2 terms. So we can focus on just the T^3 and Y Noether currents. For the left-handed doublet we see that if we take $c^3 J^{\mu a=3} + c^4 J^\mu_Y$ then we get

$$c^3 J^{\mu a=3}_L + c^4 J^\mu_{YL} = \frac{1}{2}(c^3 + q_{YL} c_4) \bar{e}_L (\gamma^\mu \mathrm{e}_L) + \frac{1}{2}(-c^3 + q_{YL} c_4) \bar{\nu}_L \gamma^\mu \nu_L \tag{10.3.20}$$

Thus we see we must choose $c^3 = c^4 q_{YL}$. Likewise for the righthanded fields for the same combination $c^3 J^{\mu a=3} + c^4 J^\mu_Y$ but with the appropriate matrices for the right-handed case we see that

$$c^3 J^{\mu a=3}_R + c^4 J^\mu_{YR} = \frac{1}{2}(q_{YR} c_4) \bar{e}_L (\gamma^\mu \mathrm{e}_L) \tag{10.3.21}$$

Thus if we choose a normalisation[7] $c_3 = c_4 = 1$ then we must have that the EM generator is

$$\mathsf{Q} := T^3 + \frac{1}{2}\mathsf{Y} \tag{10.3.22}$$

and the weak hypercharges of left-handed leptons is $q_{YL} = -1$ and for the right-handed electron it is $q_{YL} = -2$. So this suggests that the generator of the unbroken electromagnetic symmetry is Q but it is related to the generators of a $U(2) \cong SU(2) \times U(1)$ symmetry in exactly the same way this was done for flavour symmetry in the hadrons [8] summarised by the **Gell-Mann/Nishijima** formula (9.1.2) which is of the same form. Its for this historical reason we have used a convention with the weak-hypercharge coming with a factor of 1/2. Likewise the SU(2) symmetry is called weak-isospin, and the $U_Y(1)$ is called weak-hypercharge after the corresponding names for the flavour symmetries, isospin and hypercharge.

If we make a table of the charges we have[9]

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[7] The freedom to rescale the charges of U(1) symmetry means there is a lot of freedom here. We will confirm that the identification made here does indeed work with a more detailed calculation later. Other conventional choices are possible but this will not change the physics.

[8] This is a global $SU(3)$ symmetry between up, down and strange quarks, explicitly broken by the strange quark having a slightly larger mass than the other two. See section **??**.

[9] Note also that, as ν_L and e_L are in the same doublet, they carry the same charge q_Y.

	weak isospin T^3	weak hypercharge Y (q_Y)	EM charge Q
ν_L	$+\frac{1}{2}$	-1	0
e_L	$-\frac{1}{2}$	-1	-1
e_R	0	-2	-1

10.4 Fermion masses

Unbroken symmetry ensures particles in same irreducible representations, for example the doublet (e_L, ν_L), have the same mass. Likewise, this mass will be unrelated to masses of particles in other representations, such as the singlet (e_R), without some additional symmetry or unlikely accident. Here we do have additional symmetry at the classical level, the axial or chiral symmetry demands that all these chiral fermions are massless. If we tried to add a mass like term we would find it was zero e.g.

$$m\bar{\boldsymbol{l}}_L\boldsymbol{l}_L = m\left(\bar{e}_L\mathrm{e}_L + \bar{\nu}_L\nu_L\right) = m\left(\bar{e}_L P_R P_L \mathrm{e}_L + \ldots\right) = 0 \tag{10.4.1}$$

However, we know that in QEDwe do have a mass term for a Dirac fermion field representing the electron, so we *must* have a term

$$m_e\bar{\psi}_e\psi_e = m_e(\bar{e}_R\mathrm{e}_L + \bar{e}_L\mathrm{e}_R) \tag{10.4.2}$$

How do we find such a term in EW, where the left- and right-handed electrons are in different multiplets and how do we avoid giving the neutrino a mass?

The solution is to use the same SSB process which is needed to give the gauge bosons different masses. Thus let us add a scalar field, $\boldsymbol{\Phi}$ the **Higgs field**, to our theory. In four dimensions we can have a scalar-fermion-fermion interactions term, a **Yukawa interaction** (see section 8.2.4). With SSB such terms can generate a mass as indicated in (8.2.9). Here we need to mix the left-handed electron and the right-handed electron to get a term like (10.4.2), so consider

$$\mathcal{L}_{\text{Yukawa}} = g_e\boldsymbol{\Phi}\bar{\boldsymbol{l}}_L\mathrm{e}_R + (\text{h.c.}) = g_e(\boldsymbol{\Phi})_i(\bar{\boldsymbol{l}}_L)_i^\alpha(\mathrm{e}_R)_\alpha + (\text{h.c.}) \tag{10.4.3}$$

where

$$\Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}\Phi^{(+)}\\ \Phi^0\end{pmatrix} \tag{10.4.4}$$

We only want to have the electron component of the left-handed doublet involved in the mass term, the term coming from the vev of the Higgs field. So we clearly want to choose the Higgs to be of the form

$$\Phi(x) = \Phi_0 + \text{fluctuations} \qquad\qquad \Phi_0 = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \tag{10.4.5}$$

i.e. in the complex two-dimensional representation of SU(2) with a real vev $\boldsymbol{\Phi}_0$ (real so the hermitian conjugate term is correct too). With this we have

$$\mathcal{L}_{\text{Yukawa}} = -\frac{g_e v}{\sqrt{2}}\left(\bar{e}_L\mathrm{e}_R + \text{h.c.}\right) + \text{cubic fluctuation terms} \qquad \Rightarrow \qquad m_e = \frac{g_e v}{\sqrt{2}} \tag{10.4.6}$$

Note that such a term is Lorentz invariant — the spin indices on the spinors are contracted out — and $SU(2)$ invariant — the doublets of the Higgs and the left-handed leptons are contracted to give an invariant under $SU(2)$ transformations, i.e.

$$\mathcal{L}_{\text{Yukawa}} = g_e(\boldsymbol{\Phi})_i(\bar{\boldsymbol{l}}_L)_i^{\alpha}(\mathrm{e}_R)_{\alpha} + (\text{h.c.}) \tag{10.4.7}$$

To ensure it is invariant under the $U_Y(1)$, we look at pure $U_Y(1)$ transformations and see that the first term becomes

$$\left(\exp\left\{+i\frac{1}{2}q_{Y,L}\epsilon^4(x)\right\}\boldsymbol{l}_L\right)^{\dagger}\gamma_0 \exp\left\{i\frac{1}{2}q_{Y,R}\epsilon^4(x)\right\}\mathrm{e}_R \exp\left\{i\frac{1}{2}q_{Y\Phi}\epsilon^4(x)\right\}\Phi \tag{10.4.8}$$

We can rewrite this as

$$\exp\left\{i\frac{1}{2}\epsilon^4(x)\left(-q_{Y,L}+q_{Y,R}+q_{Y,\Phi}\right)\right\}\bar{\boldsymbol{l}}_L\mathrm{e}_R\Phi \tag{10.4.9}$$

For this term to be invariant under $U_Y(1)$ we require

$$-q_{Y,L}+q_{Y,R}+q_{Y,\Phi} = 0 \tag{10.4.10}$$

So we can now complete the table of charges for the Higgs field We can therefore deduce the electric charge of the two Higgs components, +1 for the top entry and zero for the bottom.

	weak isospin T^3	weak hypercharge $\mathsf{Y} = q_Y\mathbb{1}$	EM charge Q
Φ_1	$+\frac{1}{2}$	$+1$	$+1$
Φ_2	$-\frac{1}{2}$	$+1$	0

In particular, we will see that the radial fluctuation is the only one to remain after SSB 'eats' the other scalar degrees of freedom. This means the only Higgs particle observable today has no electric charge.

The table of charges for the Higgs can be summarised by saying that we have found the appropriate covariant derivative for the Higgs field namely

$$\mathsf{D}^{\mu}_{(\Phi)} = \partial^{\mu} - ig\left(\frac{1}{2}\tau^a\right)W^{\mu a}(x) - \frac{1}{2}ig'\mathbb{1}B^{\mu}(x) \qquad a = 1,2,3 \tag{10.4.11}$$

where we note the $\frac{1}{2}$ factor for the g' coupling term[10] . The $q_{Y,\Phi}$ lies inside the $-\frac{1}{2}ig'\mathbb{1}$ term.

10.5 The Gauge Bosons and Higgs Mechanism

We know we only observe *one* conserved $U(1)$ charge in EW interactions: the EM charge. We thus know that, under symmetry breaking

$$SU(2)\times U_Y(1) \longrightarrow U_{\mathsf{EM}}(1) \tag{10.5.1}$$

where four generators give rise to one unbroken generator and three broken generators. We therefore need three real scalar modes to be would-be Goldstone bosons. The smallest $SU(2)$ scalar of three real modes is the complex doublet. In fact a Higgs in this fundamental representation can only break $SU(2)\times U_Y(1)$ to $U(1)$. Thus the Higgs field suggested by the fermion masses is consistent with the requirements for the gauge bosons

[10]Note that in terms of the covariant derivative we are free to set the scale of the weak-hypercharges at the expense of changing the scale of the g' coupling. This is part of the freedom in conventions noted above when deciding the Q matrix in (10.3.22).

From the analysis of the fermion masses we know that the vev is

$$\langle\Phi\rangle = \Phi_0 = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \tag{10.5.2}$$

Further it is easy to check that this is a vacuum solution for a Higgs potential of the form $m^2|\boldsymbol{\Phi}|^2 + \lambda|\boldsymbol{\Phi}|^4$ when $m^2 < 0$.

To find the unbroken generator, its convenient to extend our notation and treat the weak hypercharge generator as the fourth T^4 matrix,

$$\mathsf{T}^4 = \frac{1}{2}\mathsf{Y} = \frac{1}{2}\mathbb{1} \tag{10.5.3}$$

Given the weak-hypercharge of the Higgs, this means $\mathsf{T}^4_{(\Phi)} = \frac{1}{2}q_{Y\Phi}\mathbb{1} = \frac{1}{2}\mathbb{1}$. This gives us a generator of the standard normalisation too, at least for the Higgs field[11]. Thus we find that

$$c^{\bar{a}}\mathsf{T}^{\bar{a}}_{(\Phi)}\Phi_0 = 0 \qquad \Rightarrow \qquad c_1 = c_2 = 0 \qquad c_3 = c_4 = 1, \qquad \bar{a} = 1,2,3,4 \tag{10.5.4}$$

and so the unbroken generator is

$$\mathsf{Q} = \mathsf{T}_3 + \frac{1}{2}\mathsf{Y} \tag{10.5.5}$$

i.e. this is the EM generator, exactly as we noted earlier in (10.3.22). Its covariant derivative follows as noted above (10.4.11).

We could also have chosen a different vacuum from that used above, though it would be related to our chosen solution by a global unitary transformation. Working with such an alternative is equivalent to say that we must be using a different labelling of the two components of the left-handed lepton doublet, i.e. whatever linear combination the vev projects, $\boldsymbol{\Phi}^\dagger.\boldsymbol{l}_L$ is going to be the physical left-handed electron. Such a global unitary transformation makes no difference to the physics, only the labels in our model[12].

With the covariant derivative for the Higgs $\mathsf{D}^\mu_{(\Phi)}$ (10.4.11), we can immediately extract the Gauge boson masses in the usual way, provided we take care of the complication that we now have two coupling constants. However, it's straight forward to see that the generalisation of the gauge boson mass matrix for a simple group (7.2.11) is

$$\begin{aligned}(\mathsf{D}^\mu\Phi)^\dagger(\mathsf{D}_\mu\Phi) &= |\mathsf{D}_\mu\Phi_0|^2 + (\text{cubic, quartic}) & (10.5.6)\\ &= \frac{1}{2}W^{\mu\bar{a}}(x)(\mathsf{M}^2)^{\bar{a}\bar{b}}W^{\bar{b}}_\mu(x) & (10.5.7)\\ (\mathsf{M}^2)^{\bar{a}\bar{b}} &= 2\Phi_0^\dagger\mathsf{t}^{\bar{a}}\mathsf{t}^{\bar{b}}\Phi_0 & (10.5.8)\end{aligned}$$

where $W^{\mu 4} = B^\mu$, $\bar{a},\bar{b} = 1,2,3,4$ and

$$\mathsf{t}^{\bar{a}} = \begin{cases}\frac{1}{2}g\boldsymbol{\tau}^a & \bar{a} = a = 1,2,3\\ \frac{1}{2}g'\mathbb{1} & \bar{a} = 4\end{cases} \tag{10.5.9}$$

It's a useful trick for non-simple groups to absorb the different gauge couplings into four new generators $\mathsf{t}^{\bar{a}}$ at the expense of making the orthogonality relations more complicated.

[11] The U(1) part can have different normalisations for different particles, and this may not fit the standard normalisation used for the non-abelian sectors of the symmetry.

[12] Indeed one can make a symmetry transformation on the fields to remove this global transformation, and to return to our standard definitions.

In principle, one can calculate the entries in this four-by-four matrix M^2 but with many zeros in the expressions for the generators and the vev, it is easier to calculate the result directly by first evaluating the vector $D_\mu \Phi_0$ and then taking its modulus. We find that

$$D_\mu \Phi_0 = \frac{v}{\sqrt{2}} \left(-i\frac{1}{2} g \tau^a W^a_\mu - i\frac{1}{2} g' \mathbb{1} B_\mu \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10.5.10}$$

$$= \frac{-iv}{2\sqrt{2}} \begin{pmatrix} gW^3_\mu + g'B_\mu & gW^1_\mu - igW^2_\mu \\ gW^1_\mu + igW^2_\mu & -gW^3_\mu + g'B_\mu \end{pmatrix} \tag{10.5.11}$$

$$= \frac{-iv}{2\sqrt{2}} \begin{pmatrix} g(W^1_\mu + iW^2_\mu) \\ -gW^3_\mu + g'B_\mu \end{pmatrix} \tag{10.5.12}$$

so that

$$|D_\mu \Phi_0|^2 = \frac{v^2}{2} \left(\frac{1}{4} \left(gW^2_\mu \right)^2 + \frac{1}{4} \left(gW^1_\mu \right)^2 + \frac{1}{4} \left(-gW^3_\mu + g'B_\mu \right)^2 \right) \tag{10.5.13}$$

These are the only quadratic terms containing the gauge boson fields, so analysing these terms will tell us the gauge boson masses.

For the W^1_μ and W^2_μ fields this is easy as they are not mixed with any other field at the quadratic level. Thus we deduce that $m_W := gv/2$ is the mass for two massive gauge bosons, the W^1 and W^2 gauge bosons. The first two terms in (10.5.13) then become

$$\frac{1}{2} m_W^2 \left(W^1_\mu \right)^2 + \frac{1}{2} m_W^2 \left(W^2_\mu \right)^2, \quad m_W := m_W := \frac{gv}{2} \tag{10.5.14}$$

The physical W masses are 80.42 ± 0.04GeV. However, $W^{1\mu}$ and $W^{2\mu}$ are real fields with equal masses. We have already seen that if we had two real scalar fields, say ϕ_1 and ϕ_2, with equal mass, then we would suspect an unbroken $U(1)$ symmetry. Of course we have an unbroken EM $U(1)$ symmetry here and a check of the full Lagrangian will confirm that this remaining unbroken symmetry, generated by the Q of (10.5.5), does indeed mix W^1_μ and W^2_μ. Of course, just like the simple scalar field case, the charge eigenstates are the complex fields built out of these two real fields, c.f. the complex scalar case $\Phi = 1/(\sqrt{2})(\phi_1 + i\phi_2)$. Thus the W gauge boson fields of definite mass m_W and carrying electric charges ± 1 are

$$W^\pm = \frac{1}{\sqrt{2}} \left(W^\mu_1 \pm i W^\mu_2 \right) \tag{10.5.15}$$

with mass term $(m_W)^2 W^{+\mu} W^-_\mu$.

Following on from this discussion we can identify the charges of the gauge bosons without doing a long non-abelian charge calculation and we can summarise them as follows:

	weak isospin T^3	weak hypercharge $\mathsf{Y} = q_Y \mathbb{1}$	EM charge Q
W^+	+1	0	+1
W^3	0	0	0
W^-	-1	0	-1
B^0	0	0	0

The quadratic term with W^3_μ and B_μ in (10.5.13) contains mixing terms. However, this calculation presents us automatically with the suitable redefinition of the fields needed to eliminate the mixing term.

Suppose we just define a new gauge boson field to be the combination appearing in the mass term of (10.5.13), i.e. if we chose the Z_0 gauge boson

$$Z_\mu(x) \propto -gW^3_\mu(x) + g' B_\mu(x) \tag{10.5.16}$$

This field will have a mass set by the coefficients in (10.5.13).

As we have two independent fields, the we can define a second field to be a linearly independent combination of W^3_μ and B_μ. This does not appear in the mass terms, it will be massless and is therefore our candidate for the photon $A^\mu(x)$.

To be more precise, given that W^3_μ and B_μ have the standard normalisation, we want to define new linear combinations which also have the correct normalisation and which are orthogonal[13]. We discussed how this was done for real scalar fields in section 4.2.1 but the principle is the same here. Given our rough definition (10.5.16) we see that the desired orthogonal combinations are given by

$$\begin{pmatrix} A^\mu(x) \\ Z^\mu(x) \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B^\mu(x) \\ W^{3\mu}(x) \end{pmatrix} \tag{10.5.17}$$

where

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \qquad \Rightarrow \qquad \tan\theta_W = \frac{g'}{g} \tag{10.5.18}$$

The parameter θ_W is called the **Weinberg angle** and is around 28°. The A_μ and Z^0_μ gauge bosons are each their own anti-particles and so have zero electric charge. Again a more careful check of the transformation generated by Q will confirm this.

This gives a mass term for the Z gauge boson of $m_Z = \frac{1}{2}v\sqrt{g^2+g'^2} = m_W/\cos(\theta_W)$ and a physical mass of 91.188 ± 0.002GeV.

10.6 EM current revisited

The contribution to the physical electric current coming from the leptons can be read off directly from the term linear in the photon field and bilinear in the lepton fields (this is not true for scalars). That is in the Lagrangian we only need look at $\bar\psi \not{D} \psi = J^\mu_{\mathsf{EM}} A_\mu + \ldots$ terms. We know the current should be of the form

$$J^\mu_{\mathsf{EM}} = -\mathsf{e}\left(\bar e(x)\gamma^\mu e(x)\right) = -\mathsf{e}\left(\bar e_L(x)\gamma^\mu \mathrm{e}_L(x)\right) - \mathsf{e}\left(\bar e_R(x)\gamma^\mu \mathrm{e}_R(x)\right) \tag{10.6.1}$$

where $-\mathsf{e}$ is the charge on the electron (so $\mathsf{e} = +1.6 \times 10^-19C$) and the fields are $e(x)$ etc. Expanding the lepton covariant derivatives then we find that

$$\begin{aligned} &\bar e_R i\not{D}_R \mathrm{e}_R + \bar{\boldsymbol{l}}_L i\not{D}_L \boldsymbol{l}_L \\ &= -g'\bar e_R\gamma^\mu \mathrm{e}_R B_\mu - \frac{1}{2}\bar{\boldsymbol{l}}_L\gamma^\mu \begin{pmatrix} g'B - gW^3 & -g\left(W_1 - iW_2\right) \\ -g\left(W_1 + iW_2\right) & \left(g'B + gW^3\right) \end{pmatrix} \bar{\boldsymbol{l}}_L \end{aligned} \tag{10.6.2}$$

We are only interested in the A^μ terms so if we substitute from (10.5.17) we get

$$\bar e_R i\not{D}_R \mathrm{e}_R + \bar{\boldsymbol{l}}_L i\not{D}_L \boldsymbol{l}_L \tag{10.6.3}$$

$$= -g'\bar e_R\gamma^\mu \mathrm{e}_R\cos(\theta_W)A - \frac{1}{2}\bar{\boldsymbol{l}}_L\gamma^\mu \begin{pmatrix} g'\cos(\theta_W)A - g\sin(\theta_W)A & 0 \\ 0 & g'\cos(\theta_W)A + g\sin(\theta_W)A \end{pmatrix} \bar{\boldsymbol{l}}_L \tag{10.6.4}$$

$$= -g'\cos\theta_W\left(\bar e_R\gamma^\mu \mathrm{e}_R\right)A^\mu - \frac{1}{2}\left(g\sin\theta_W + g'\cos(\theta_W)\right)\left(\bar e_L\gamma^\mu \mathrm{e}_L\right)A_\mu + +\left(g\sin\theta_W - g'\cos(\theta_W)\right)\left(\bar\nu_L\gamma^\mu\nu_L\right)A_\mu \tag{10.6.5}$$

[13]This comes from maintaining commutation relations in QFT.

This implies that the charge on the positron e is

$$\mathsf{e} = g' \cos\theta_W = g \sin\theta_W \tag{10.6.6}$$

as this then gives us

$$J^\mu_{\mathsf{EM}} = -\bar{\psi}_{\mathrm{e}}\gamma\psi_{\mathrm{e}} = -\bar{e}_R\gamma^\mu \mathrm{e}_R - \bar{e}_L\gamma^\mu \mathrm{e}_L - 0\bar{\nu}_L\gamma^\mu\nu_L \tag{10.6.7}$$

We could in fact derive the weak-hypercharges of the leptons and find the correct linear combination of gauge fields that gives the photon, just from the currents. From (10.3.17) we get

$$\begin{aligned}
&\bar{\boldsymbol{l}}_L i\not{D}_L \boldsymbol{l}_L + \bar{e}_R i\not{D}_R \mathrm{e}_R \\
&= \frac{1}{2}\left(gW^3_\mu + g'q_{YL}B_\mu\right)(\bar{\nu}_L\gamma^\mu\nu_L) + \frac{1}{2}\left(-gW^3_\mu + g'q_{YL}B_\mu\right)(\bar{e}_L\gamma^\mu \mathrm{e}_L) \\
&\quad + \frac{1}{2}g'q_{YR}B_\mu(\bar{e}_R\gamma^\mu \mathrm{e}_R) + \ldots
\end{aligned} \tag{10.6.8}$$

In the same way the photon field must be a linear combination of W^3_μ and B_μ fields. The most general linear combination of these two real fields which preserves their normalisation is

$$\begin{pmatrix} B^\mu(x) \\ W^{3\mu}(x) \end{pmatrix} = \begin{pmatrix} \cos(\theta_W) & -\sin(\theta_W) \\ \sin(\theta_W) & \cos(\theta_W) \end{pmatrix} \begin{pmatrix} A^\mu(x) \\ Z^\mu(x) \end{pmatrix} \tag{10.6.9}$$

where A^μ is the photon field, Z^μ is a second independent gauge field and θ_W is a parameter to be fixed. Substituting this into (10.6.8) and studying the coefficient of A_μ we see that

$$\begin{aligned}
&\bar{\boldsymbol{l}}_L i\not{D}_L \boldsymbol{l}_L + \bar{e}_R i\not{D}_R \mathrm{e}_R \\
&= \frac{1}{2}\left(g\sin(\theta_W) + g'q_{YL}\cos(\theta_W)\right)A_\mu(\bar{\nu}_L\gamma^\mu\nu_L) + \frac{1}{2}\left(-g\sin(\theta_W) + g'q_{YL}\cos(\theta_W)\right)A_\mu(\bar{e}_L\gamma^\mu \mathrm{e}_L) \\
&\quad + \frac{1}{2}g'q_{YR}\cos(\theta_W)A_\mu(\bar{e}_R\gamma^\mu \mathrm{e}_R) + \ldots
\end{aligned} \tag{10.6.10}$$

Demanding that the neutrino does *not* couple to the photon, i.e. it has no electromagnetic charge, gives us $g\sin(\theta_W) = -g'q_{YL}\cos(\theta_W)$ and so

$$\begin{aligned}
\bar{\boldsymbol{l}}_L i\not{D}_L \boldsymbol{l}_L + \bar{e}_R i\not{D}_R \mathrm{e}_R &= -g'q_{YL}\cos(\theta_W)A_\mu(\bar{e}_L\gamma^\mu \mathrm{e}_L) - \frac{1}{2}g'q_{YR}\cos(\theta_W)A_\mu(\bar{e}_R\gamma^\mu \mathrm{e}_R) \\
&\quad + \ldots
\end{aligned} \tag{10.6.11}$$

Comparing with (10.3.18) we see that we require $2q_{YL} = q_{YR}$ and $e = q_{YL}g'\cos(\theta_W)$. We always have a freedom to scale $U(1)$ charges and here changes in the overall size of the $U_Y(1)$ charges q_Y can be compensated for by scalings of the g' coupling. However let us fix a scale by setting $q_{YL} = -1$ which is a common convention. Then we can summarise our findings so far

$$q_{YL} = -1, \quad q_{YR} = -2 \tag{10.6.12}$$

$$\tan(\theta_W) = \frac{g'}{g} \tag{10.6.13}$$

10.7 Higgs masses

The three massive gauge bosons will absorb three of our four scalar modes in the Higgs doublet, $\mathbf{\Phi}$. These will be related to modes which can be described in terms of local symmetry transformations of the scalar field, i.e. the three would-be Goldstone bosons ‘eaten’ by the three massive gauge bosons. Since these are unitary transformations, they do not alter the modulus of the Higgs field. Thus the one scalar mode left must be related to fluctuations in the size of the scalar field and we deduce that in the unitary gauge the Higgs field is reduced to

$$\mathbf{\Phi} = \frac{v + \sigma(x)}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10.7.1}$$

where $\sigma(x) \in \mathbb{R}$ is our one degree of freedom. As a single real scalar mode, it must have no electric charge (the charge of the unbroken symmetry), and in particular it must be electromagnetically uncharged. This is the **Higgs particle** and the last major particle of the EW puzzle to be discovered. The mass is simple to calculate as it comes purely from the scalar potential. If we take the only possible form in four-dimensions making the usual assumptions about renormalisability etc., we see that

$$V\left(\mathbf{\Phi}, \mathbf{\Phi}^\dagger\right) = m^2|\mathbf{\Phi}|^2 + \lambda|\mathbf{\Phi}|^4 \tag{10.7.2}$$

$$\Rightarrow \quad m^2_{\text{Higgs}} = -2m^2 = 2\lambda v^2 \tag{10.7.3}$$

Note again that even though we have used the simplest possible model for the Higgs, a single complex doublet, the experimental predictions are limited as the mass is expressed in terms of another free parameter) λ. Only if we had also information on pure Higgs scattering rates either directly or indirectly through its virtual contributions to other processes, could we compare the λ calculated there against the λ needed to get the correct Higgs mass. This relationship between scattering rates and masses in the Higgs sector of a unified model is the type of prediction being made by such a unified model of EM and weak interactions.

10.8 The Quarks and Weak Physics

The quarks present several additional problems over the leptons. The most obvious one is that non of the quarks is massless or even approximately so, unlike the neutrino. In a similar way, all the quarks are involved in strong and EM interactions and there is no indication of parity violation there. This suggests that we must include left and right handed parts for all of the quarks but we must ensure that both parts interact in the same way with the photon. Still there is a very natural way to do this and we will start by focusing on the first generation of quarks, the up and down quarks, just as we did with the leptons.

When we discussed beta decay and the parity violation experiments such as that of Wu et al, we focused on the lepton output and did not discuss the fact that a neutron is decaying into a proton. This part of the process is a weak decay of a down quark into an up quark with emission of a W^-. Thus it suggests that just we couple a left-handed e^-, μ doublet to the W bosons, that we should do the same for the up and down, i.e. we want a left handed doublet with the up and down quarks. Let us denote this as $\boldsymbol{q}_L$. This leaves us with the right handed parts, and if we follow the pattern set by the leptons we might try right handed singlets for them. Thus this suggests a quark/gauge-boson Lagrangian of the form

$$\mathcal{L}_q = \bar{\boldsymbol{q}}_L i \mathsf{D}^\mu_{qL} \gamma_\mu \boldsymbol{q}_L + \bar{u}_R i D^\mu_{uR} \gamma_\mu u_R + \bar{d}_R i D^\mu_{dR} \gamma_\mu d_R \tag{10.8.1}$$

where

$$\boldsymbol{q}_L = \left(\begin{array}{c} u_L(x) \\ d_L(x) \end{array} \right), \tag{10.8.2}$$

$$u_R = u_R(x), \qquad d_R = d_R(x), \tag{10.8.3}$$

There can be no explicit mass term for these chiral fields so again we will have to use Yukawa terms to create these which we will study shortly. First we need to fix the weak isospin and weak hypercharge charges, or equivalently the covariant derivatives, for the quarks. This is straightforward once we have fixed the representations, as we already know the electromagnetic charges and the relationship between the weak charges and EM charges (10.5.5), $Q = T^3 + Y/2$. Reading off the weak isospin values from $SU(2)$ third generator, combining with the known EM charges we get the following table

	weak isospin T^3	weak hypercharge Y (q_Y)	EM charge Q
u_L	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{2}{3}$
d_L	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$
u_R	0	$+\frac{4}{3}$	$+\frac{2}{3}$
d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$

Now we need to look at the mass terms. If we were to mimic the lepton sector we would try quark Yukawa terms of the form

$$\mathcal{L}_{q\Phi,\mathrm{trial}} = +g_u[\bar{\boldsymbol{q}}_L \boldsymbol{\Phi} u_R + \boldsymbol{\Phi}^\dagger \bar{u}_R \boldsymbol{q}_L], +g_d[\bar{\boldsymbol{q}}_L \boldsymbol{\Phi} d_R + \boldsymbol{\Phi}^\dagger \bar{d}_R \boldsymbol{q}_L], \tag{10.8.4}$$

With two right-handed singlets we can have two Yukawa constants. This would give

$$m_u = \frac{g_u v}{\sqrt{2}}, \quad m_d = \frac{g_d v}{\sqrt{2}}, \tag{10.8.5}$$

and so each new Yukawa term added corresponds to the a new quark flavour added. Again we see that this is not reducing the number of unknowns but merely switching one, a quark mass, for another, a Yukawa coupling.

10.9 Summary: One Generation

Lagrangians

The contribution to the electroweak Lagrangian coming from the gauge boson, lepton (first generation only) and scalar sectors can be written as

$$\mathcal{L} = \mathcal{L}_{W,B} + \mathcal{L}_{\boldsymbol{\Phi}} + \mathcal{L}_l + \mathcal{L}_q \tag{10.9.1}$$

where

$$\mathcal{L}_{W,B} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}, \tag{10.9.2}$$

$$\mathcal{L}_{\boldsymbol{\Phi}} = (\mathsf{D}_\mu \boldsymbol{\Phi})^\dagger (\mathsf{D}^\mu \boldsymbol{\Phi}) - V(\boldsymbol{\Phi}^\dagger.\boldsymbol{\Phi}) \tag{10.9.3}$$

$$\mathcal{L}_l = \bar{\boldsymbol{l}}_L i \mathsf{D}^\mu_L \gamma_\mu \boldsymbol{l}_L + \bar{e}_R i D^\mu_R \gamma_\mu e_R + g_e[\bar{\boldsymbol{l}}_L \boldsymbol{\Phi} e_R + \boldsymbol{\Phi}^\dagger \bar{e}_R \boldsymbol{l}_L], \tag{10.9.4}$$

$$\begin{aligned}\mathcal{L}_q = {} & \bar{\boldsymbol{q}}_L i \mathsf{D}^\mu_{qL} \gamma_\mu \boldsymbol{q}_L + \bar{u}_R i D^\mu_{uR} \gamma_\mu u_R + \bar{d}_R i D^\mu_{dR} \gamma_\mu d_R \\ & + g_d[\bar{\boldsymbol{q}}_L \boldsymbol{\Phi} d_R + \boldsymbol{\Phi}^\dagger \bar{d}_R \boldsymbol{q}_L] + g_u[\bar{\boldsymbol{q}}_L \boldsymbol{\Phi}^c u_R + \boldsymbol{\Phi}^{c\dagger} \bar{u}_R \boldsymbol{q}_L],\end{aligned} \tag{10.9.5}$$

Field Strengths

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{10.9.6}$$

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g f^{abc} W^b_\mu W^c_\nu \tag{10.9.7}$$

Fields

$$\begin{aligned}
\boldsymbol{\Phi} &= \begin{pmatrix} \Phi^{(+)}(x) \\ \Phi^{(0)}(x) \end{pmatrix} = \frac{\mathsf{U}''(x)}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h(x) \end{pmatrix}, \quad \Phi^{(+)}(x), \Phi^{(0)}(x) \in \mathbb{C}, \quad v, h(x) \in \mathbb{R} && (10.9.8)\\
\boldsymbol{\Phi}^c &:= i\tau^2 \boldsymbol{\Phi}^*(x) && (10.9.9)\\
\boldsymbol{l}_L &= \begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}, && (10.9.10)\\
e_R &= e_R(x), && (10.9.11)\\
\boldsymbol{q}_L &= \begin{pmatrix} u_L(x) \\ d_L(x) \end{pmatrix}, && (10.9.12)\\
u_R &= u_R(x), \quad d_R = d_R(x), && (10.9.13)
\end{aligned}$$

The field e_R and the components of L_l are the spinors representing the appropriate chiral eigenstates of the relevant particles which for the first generation are: $e(x)$ = electron, $\nu(x)$ = neutrino, $u(x)$ = up quark, $d(x)$ = down quark. Spinor indices are suppressed throughout. Likewise the quarks carry SU(3) colour indices but these are suppressed as the EW symmetry does not mix colours. $h(x)$ is a real scalar field for the **Higgs** particle, while the $U''(x)$ is a local symmetry transformation built purely out of broken symmetries, so contains three real parameters for the three would-be Goldstone modes. The $\boldsymbol{\Phi}^c$ field is the charge conjugate of $\boldsymbol{\Phi}$ and explicitly $\Phi_1^c = \Phi_2^*$ and $\Phi_2^c = -\Phi_1^*$. As with all SU(2) doublets, this field has weak isospin of $+\frac{1}{2}$ $(-\frac{1}{2})$ for the top (bottom) entry but has weak isospin -1. This ensures the Yukawa term for the up quark mass is SU(2) and $U_Y(1)$ invariant.

The fields $\boldsymbol{l}_L$, $\boldsymbol{q}_L$ and $\boldsymbol{\Phi}$ are vectors and lie in the two-dimensional SU(2) representation.

By diagonalising the gauge boson mass matrix you find that the A^μ (photon) and Z^μ fields are given as follows

$$\begin{pmatrix} A^\mu(x) \\ Z^\mu(x) \end{pmatrix} = \begin{pmatrix} \cos(\theta_W) & \sin(\theta_W) \\ -\sin(\theta_W) & \cos(\theta_W) \end{pmatrix} \begin{pmatrix} B^\mu(x) \\ W_3^\mu(x) \end{pmatrix} \qquad (10.9.14)$$

where the **Weinberg angle** is

$$\tan(\theta_W) = \frac{g'}{g} \qquad (10.9.15)$$

By studying the E/M current, the interaction between the photons and the leptons, we see that the charge on the electron, e, is given by

$$e = g\sin(\theta_W) = g'\cos(\theta_W) \qquad (10.9.16)$$

Covariant Derivatives

$$\begin{aligned}
\mathsf{D}_L^\mu &= \partial^\mu - ig\frac{1}{2}\boldsymbol{\tau}^a W^{a,\mu}(x) + \frac{i}{2}g'\mathbb{1}B(x), && (10.9.17)\\
D_R^\mu &= \partial^\mu + ig'B(x), && (10.9.18)\\
\mathsf{D}^\mu &= \partial^\mu - ig\frac{1}{2}\boldsymbol{\tau}^a W^{a,\mu}(x) - \frac{i}{2}g'\mathbb{1}B(x)., && (10.9.19)\\
\mathsf{D}_{qL}^\mu &= \partial^\mu - i\frac{g}{2}\boldsymbol{\tau}^a W^{a,\mu}(x) - i\frac{g'}{6}\mathbb{1}B(x), && (10.9.20)\\
D_{uR}^\mu &= \partial^\mu - i\frac{2g'}{3}\mathbb{1}B(x), && (10.9.21)\\
D_{dL}^\mu &= \partial^\mu + i\frac{g'}{3}\mathbb{1}B(x). && (10.9.22)
\end{aligned}$$

There are two parts to every covariant derivative: (i) the *weak isospin* SU(2) part with generators T^a and (ii) the *weak hypercharge* $U_Y(1)$ part.

Generators

The generators for the Higgs and left handed lepton fields are $\mathsf{T}^a = \frac{1}{2}\boldsymbol{\tau}^a$, a two-dimensional representation of the SU(2) generators where τ^a are the usual Pauli spin matrices (B.2.4). Note that in principle there are generators associated with the e_R and the gauge bosons. e_R is a singlet i.e. the relevant representation is the trivial one where the generators are equal to zero.

The fourth generator of the $SU(2) \times U_Y(1)$ group is the $U_Y(1)$ generator. In my conventions Y is equal to the unit matrix multiplied by the weak hypercharge. For fields in a two-dimensional SU(2) representation this means it is also a two by two matrix. This means Y does not usually have the standard normalisation that non-abelian generators have, but this is the usual freedom of choosing the charge for abelian generators.

With the gauge bosons I have chosen to write out everything in terms of the real fields rather than the Lie valued gauge bosons.

From the vacuum, we see that there is **ONE** unbroken generator, the $U_{\mathrm{EM}}(1)$ generator

$$\mathsf{Q} = \left(\mathsf{T}^3 + \frac{1}{2}\mathsf{Y}\right) \tag{10.9.23}$$

Note that the factor of one half associated with the weak hypercharge generator, Y, is purely a matter of convention, as it just rescales the weak hypercharges assigned to each particle. Not everyone uses this convention. It is chosen to be compatible with Gell-Mann-Nishijima formula expressing the relationship found between charges of hadrons reflecting the approximate flavour symmetries of the quarks.

This means that there are **three** broken generators

$$\mathsf{T}^1, \mathsf{T}^2, \left(\mathsf{T}^3 - \frac{1}{2}\mathsf{Y}\right) \tag{10.9.24}$$

Charges

First a complete list of all the distinct fields (anti-particles have the opposite charges) and then a list of other fields derived from the first set.

	weak isospin T^3	weak hypercharge $\mathsf{Y} = q_Y \mathbb{1}$	EM charge $\mathsf{Q} = \mathsf{T}^3 + \frac{1}{2}\mathsf{Y}$
ν_L	$+\frac{1}{2}$	-1	0
e_L	$-\frac{1}{2}$	-1	-1
e_R	0	-2	-1
u_L	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{2}{3}$
d_L	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$
u_R	0	$+\frac{4}{3}$	$+\frac{2}{3}$
d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$
Φ_1	$+\frac{1}{2}$	+1	+1
Φ_2	$-\frac{1}{2}$	+1	0
W^1	Undefined	0	+1
W^2	Undefined	0	-1
W^3	0	0	0
B^0	0	0	0
$\Phi_1^c = \Phi_2$	$+\frac{1}{2}$	-1	0
$\Phi_2^c = -\Phi_1$	$-\frac{1}{2}$	-1	-1
$W^+ = (W^1 + iW^2)/\sqrt{2}$	+1	0	+1
$W^- = (W^1 - iW^2)/\sqrt{2}$	-1	0	-1
$A = \cos(\theta_W)B + \sin(\theta_W)W^3$	0	0	0
$Z = -\sin(\theta_W)B + \cos(\theta_W)W^3$	0	0	0

For the fermions and scalars, these can be read straight from the covariant derivatives. Note that non-abelian gauge fields always lie in the adjoint representation so we deduce that the W's lie in the three-dimensional representation of $SU(2)$. The diagonal form (with T^3 diagonal) produces the SU(2) charge assignments and tells use the charge eigenstates are the $W^{\pm}$ combinations not W^1, W^2. The $U_Y(1)$ charges follow from the fact that the W's are invariant under those transformations. The B^μ field is likewise invariant under the $SU(2)$ transformations, and we know from the example of QED that gauge bosons of abelian symmetries carry no abelian charge even though they are not invariant under these transformations. These charge assignments give another indication why the W^3 and B^μ field can be combined in other linear combinations, as they all share the same charges.

Masses

With these conventions we see that

$$m_e = \frac{g_e v}{\sqrt{2}}, \quad m_W = \frac{gv}{2}, \quad m_Z = \frac{m_W}{\cos(\theta_w)}, \quad m_u = \frac{g_u v}{\sqrt{2}}, \quad m_d = \frac{g_d v}{\sqrt{2}}, \tag{10.9.25}$$

The single remaining physical Higgs particle has a mass which depends on the precise form of V. This is the same as encountered in the global case. Thus if we choose $V = m^2|\mathbf{\Phi}|^2 + \lambda|\mathbf{\Phi}|^4$ we will find that the physical Higgs has mass squared of $m_h^2 = -2m^2$.

10.10 Three generations

If we assume that there is no right-handed neutrino and no neutrino masses, then we can just duplicate the lepton sector three times, once per generation, each with its own Yukawa terms. There is still the one Higgs doublet $\mathbf{\Phi}$ and the same four gauge bosons $W^{\mu a}$, B^{μ}, and each generation mixes with the gauge bosons through the same gauge couplings g and g'. Thus

$$\mathcal{L} = \mathcal{L}_{W,B} + \mathcal{L}_{\mathbf{\Phi}} + \sum_{l=e,\mu,\tau} \mathcal{L}_l + \mathcal{L}_{q3} \tag{10.10.1}$$

where in $\mathcal{L}_l$ is (10.9.4) with the electron and its neutrino replaced by the charged lepton and neutrino pair appropriate for generation $l = e, \mu, \tau$. Note there is *no* mixing of the lepton generations.

The mass of the quarks, and so the requirement for a right-handed singlet for each flavour, allows a more general solution:

$$\begin{aligned}\mathcal{L}_{q3} \;&=\; \bar{\boldsymbol{Q}}_L i \mathsf{D}^{\mu}_{qL}\gamma_{\mu}\boldsymbol{Q}_L + \bar{\boldsymbol{D}}_R i D^{\mu}_{dR}\gamma_{\mu}\boldsymbol{D}_R + \bar{\boldsymbol{U}}_R i D^{\mu}_{uR}\gamma_{\mu}\boldsymbol{U}_R \\ &\quad + \bar{\boldsymbol{Q}}_L \mathbf{\Phi} \mathsf{N} \boldsymbol{D}_R + \mathbf{\Phi}^{\dagger}\bar{\boldsymbol{D}}_R \mathsf{N}^{\dagger}\boldsymbol{Q}_L + \bar{\boldsymbol{Q}}_L \mathbf{\Phi}^c \mathsf{M} \boldsymbol{U}_R + \mathbf{\Phi}^{c\dagger}\bar{\boldsymbol{U}}_R \mathsf{M}^{\dagger}\boldsymbol{Q}_L,\end{aligned} \tag{10.10.2}$$

where

$$\boldsymbol{Q}_L \;=\; \begin{pmatrix} \boldsymbol{U}_L(x) \\ \boldsymbol{D}_L(x) \end{pmatrix}, \quad \boldsymbol{U}_R = \boldsymbol{U}_R(x), \quad \boldsymbol{D}_R = \boldsymbol{D}_R(x), \tag{10.10.3}$$

but now the U and D are triplets in generation space, that is

$$\boldsymbol{U} = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \quad \boldsymbol{D} = \mathsf{V}\begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{10.10.4}$$

with the same generation structure for the left- and right-handed parts[14]. Here u, d, c, s, t, b are the *mass eigenstates* of the quarks. The V is a three-by-three unitary matrix known as the **Kobayashi-Maskawa matrix**. One can always make a unitary redefinition of fields and absorb any similar matrix in the $\boldsymbol{U}$ case but then one can't remove the V in the $\boldsymbol{D}$ case. Further field redefinitions can reduce V to a form involving three angles and one phase. If this last phase (and there is no symmetry reason for it to be zero) then there is **CP violation** in the EW model, as observed in experiments e.g. in the neutral K meson system.

The Yukawa couplings of the single generation are replaced by matrices in generation space of couplings, N and M which are related to masses

$$\frac{v}{\sqrt{2}}\mathsf{N} = \mathsf{V}\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad \frac{v}{\sqrt{2}}\mathsf{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \tag{10.10.5}$$

10.11 The status of EW model today

We noted in the derivations, that by its very nature as the field that breaks the symmetry yet leaves EM unbroken, the Higgs *must* be electromagnetically uncharged. This makes it difficult to detect. It also turns out that the sensitivity to scalar couplings in any measurable quantity (for instance through the effect of

[14]Thus the left handed doublet has four indices: SU(2) ($i = 1, 2$), generation ($f = 1, 2, 3$), and the suppressed colour ($c = 1, 2, 3 \equiv$ red, green, blue) and spin $\alpha = 0, 1, 2, 3$. The other matrices and vectors have to be given appropriate indices and are assumed to be unit matrices in the irrelevant indices.

virtual Higgs particles) is generally $\ln(\lambda)$. By way of contrast there is usually a polynomial dependence on gauge couplings. So precision measurements give relatively weak constraints on the Higgs whereas they enabled us to home in on the mass of say the top quark long before it was detected directly. Thus the data we have today of $E \lesssim 100\text{GeV}$ can accommodate very different types of Higgs sectors than the one we have used which is the simplest. There are models with the Higgs in other representations, more Higgs multiplets (at least two are required for supersymmetric models) and so forth.

For this reason, there could be a much more complicated pattern in the Higgs sector than the simplest one we have outlined. However it should be clear that this simplest Higgs sector is completely compatible with the low energy data $E \lesssim 100\text{GeV}$ at the time of writing (2005). Thus if a more complicated scalar sector is found, probably by the end of the decade in the new generation of accelerators, more Higgs, different representations, it may well be a pointer to new physics present at a deeper level.

Another area we have hardly touched on is CP violation. This is necessary to explain why there is more matter than anti-matter in the universe. It is present in small amounts in the electro-weak model though attempts to use the EW CP violation to explain the cosmic preference for matter have so far failed. Nevertheless it again is an area where our measurements are poor and so far this is the only place where such CP violation is known. So this represents another active area of experimentation.

The most interesting challenge to the standard model today (2005) comes in the neutrino sector. The model considered in this chapter assumes that the neutrinos are massless. Since the neutrinos interact so weakly (they have no colour charges and no electromagnetic charges) this sector has little effect on most high-energy particle physics experiments or on cosmology. However there is now definite evidence that neutrinos have a mass difference of around 1eV no more than one flavour can be massless. This means we can get a similar type of physics from the leptons as from the quarks, perhaps something which should not be so surprising. In particular the individual lepton flavours can not be conserved only the total lepton number appears to be. Adding neutrino masses in a sensible way seems to point to aspects of higher energy physics and ideas such as Majorana fermions become relevant. .

Chapter 11

Questions and Answers

11.1 From data to model

The questions in this section are of a general nature. The idea is that they try to give a crude simulation of the type of problem encountered when trying to build a model to fit real data. They require one to understand all the relationships between particle properties and theoretical Lagrangians that have been set out in all the previous chapters. For this reason, the first question must be completed before tackling the remaining ones as it asks you to summarise all the key information gathered in the earlier chapters.

11.1.

(i) Give a list of rules relating masses of particles to various aspects of symmetry. (If particles X have masses Y then Z is true/maybe true/is never true.) I wrote down ten such rules but you may find more or less.

(ii) Give one rule linking observed particle charges with symmetry.

(iii) Give an example Lagrangian showing each of your rules above.

11.2.

In the future, experimentalists see two new scalar particles and their distinct anti-particles. The particles all have the same mass. They also carry a new conserved quantum number, "Sillyness" and the two particles have equal but opposite sillyness. To describe the interactions only one additional parameter seems to be required.

(i) Roughly speaking, why does the particle/anti-particle structure suggest the use of two complex scalar fields?

(ii) Why does the single mass parameter rule out a possible U(1) symmetry?

(iii) Construct a Lagrangian $\mathcal{L}$ which describes this data noting how your answer does this.

There are *many* possible answers to this question, but invariably there will be one particularly simple one which requires the smallest number of extra parameters. This is the one I have in mind but I will always give full credit to any answer which satisfies all the data and limitations that I may specify.

11B. Repeat question 11 assuming that 3 particles and their three distinct anti-particles were observed, all with the same mass and with relative charges on the particles of +1,0,-1. What other charges do your particles have?

11.3.

(i) Suppose that experimentalists found some new physics and particles, working up to some maximum energy scale. Two scalar particles of different non-zero masses, each one is its own anti-particle, have been seen. They have also found evidence for a charge associated with a new long range force, though the two scalar particles mentioned have charge zero (we'll suppose that there are fermions around too carrying non-zero charges). Write down an appropriate renormalisable Lagrangian for the bosonic fields only which describes this particle content. You should briefly note the following:-

- the continuous symmetry group associated with your Lagrangian,
- how your answer explains the masses and charges given,
- how many new particles remain to be found by the experimentalists
- an exemplary interaction which ensures that all particles interact with those of different masses, and which does not break the required symmetries.

Avoid unnecessary additional particles, additional charges and continuous symmetries.

(ii) Suppose that there is evidence for the existence of a short range force. Give a new solution for all the evidence in this case following the same principles as above.

11.2 Answers to questions

Q??. In the following, any constants have been dropped as they do not alter the equation of motion.

(i)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 \tag{11.2.1}$$

$$\phi = \eta + \frac{v}{\mu^2} \tag{11.2.2}$$

(ii)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_j\partial^\mu\phi_j - \frac{1}{2}\mu^2\phi_j\phi_j, \quad (j = 1, 2) \tag{11.2.3}$$

$$\eta = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2 - \frac{v}{\mu^2}) \tag{11.2.4}$$

(iii)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_j\partial^\mu\phi_j - \frac{1}{2}\phi_j B_{jk}\phi_k, \quad (j, k = 1, 2, \ldots N) \tag{11.2.5}$$

$$B_{jk} = \frac{1}{2}(A_{jk} + A_{kj}), \quad \phi_j = \eta_j - \frac{1}{2}(B^{-1})_{jk}v_k \tag{11.2.6}$$

B must be diagonalised to find the masses of particles.

Q??.

Consider a Lagrangian $\mathcal{L}[\{F_i(x)\}]$ for complex fields $F_i(x) \in \mathbb{C}$. The equations of motion are

$$\partial_\mu \frac{\partial}{\partial\mathcal{L}}(\partial_\mu F_i) - \frac{\partial}{\partial\mathcal{L}}F_i + \partial_\mu \frac{\partial}{\partial\mathcal{L}}\left(\partial_\mu F_i^\dagger\right) - \frac{\partial}{\partial\mathcal{L}}F_i^\dagger = 0 \tag{11.2.1}$$

Now suppose

$$\begin{aligned} F_i(x) &\rightarrow F_i'(x) = U_{ij}F_j(x), \quad \boldsymbol{F}(x) \rightarrow \boldsymbol{F}'(x) = \mathsf{U}\boldsymbol{F}(x) & (11.2.2)\\ F_i^\dagger(x) &\rightarrow F_i'^\dagger(x) = F_j^\dagger(x)(\mathsf{U}^\dagger)_{ji}, \quad \boldsymbol{F}^\dagger(x) \rightarrow \boldsymbol{F}'^\dagger(x) = \boldsymbol{F}^\dagger(x)\mathsf{U}^\dagger & (11.2.3)\end{aligned}$$

is a continuous (global) symmetry, where U is finite-dimensional and unitary, i.e. $\mathsf{U}\mathsf{U}^\dagger = \mathbb{1}$, a property of compact Lie groups. Thus we can always write

$$\mathsf{U} = \exp\{i\varepsilon^a \mathsf{T}^a\} \qquad a = 1, \ldots, \dim(G) \tag{11.2.4}$$

where $\varepsilon^a \in \mathbb{R}$, $\partial_\mu \varepsilon^a = 0$ and T^a are generators of the Lie algebra.

For a general small variation of $\boldsymbol{F}$, i.e. for $\boldsymbol{F} \mapsto \boldsymbol{F}' = \boldsymbol{F} + \delta\boldsymbol{F}$, the variation of $\mathcal{L}$ is

$$\delta\mathcal{L} = \frac{\partial}{\partial\mathcal{L}}F_i\delta F_i + \frac{\partial}{\partial\mathcal{L}}(\partial_\mu F_i)\delta(\partial_\mu F_i) + \frac{\partial}{\partial\mathcal{L}}F_i^\dagger\delta F_i^\dagger + \frac{\partial}{\partial\mathcal{L}}(\partial_\mu F_i^\dagger)\delta(\partial_\mu F_i^\dagger) \tag{11.2.5}$$

The variation in the Lagrangian is due to the variation in the fields, their hermitian conjugates (as these are independent degrees of freedom) and their derivatives (a second order ODE requires the value of fields and their derivatives to specify a solution).

Now let us specialise to a small symmetry transformation where

$$\delta\boldsymbol{F} = \boldsymbol{F}' - \boldsymbol{F} \approx (\mathbb{1} + i\varepsilon^a\mathsf{T}^a)\,\boldsymbol{F} - \boldsymbol{F} + O\left(\varepsilon^2\right). \tag{11.2.6}$$

A quick check shows here that $\delta\left(\partial_\mu\boldsymbol{F}\right) = \partial_\mu\left(\delta\boldsymbol{F}\right)$ and Thus, to first order,

$$\delta\boldsymbol{F} = i\varepsilon^a\mathsf{T}^a\boldsymbol{F} \tag{11.2.7}$$

Likewise

$$\delta(\boldsymbol{F}^\dagger) = -i\varepsilon^a\boldsymbol{F}^\dagger\mathsf{T}^a \tag{11.2.8}$$

where we have used the Hermitian property of the generators $\mathsf{T}^a = (\mathsf{T}^a)^\dagger$. Under a symmetry transformation

$$\begin{aligned} 0 = \delta\mathcal{L} &= \frac{\partial}{\partial\mathcal{L}}F_i \cdot i\varepsilon^a T_{ij}^a F_j + \frac{\partial}{\partial\mathcal{L}}(\partial_\mu F_i)\cdot i\varepsilon^a T_{ij}^a\left(\partial_\mu F_j\right)\\ &\quad -i\varepsilon^a F_j^\dagger T_{ij}^a\frac{\partial}{\partial\mathcal{L}}F_i^\dagger - i\varepsilon^a\left(\partial_\mu F_j^\dagger\right)T_{ij}^a\frac{\partial}{\partial\mathcal{L}}\left(\partial_\mu F_i^\dagger\right) & (11.2.9)\end{aligned}$$

(where we have used the fact that $\partial_\mu\varepsilon^a = 0$ characterizes a global symmetry).

The equation of motion yields, for the partial derivative of $\mathcal{L}$ with respect to F_i

$$\begin{aligned} 0 = \delta\mathcal{L} &= \left(\partial_\mu\frac{\partial}{\partial\mathcal{L}}(\partial_\mu F_i)\right).i\varepsilon^a T_{ij}^a F_j + \frac{\partial}{\partial\mathcal{L}}(\partial_\mu F_i).i\varepsilon^a T_{ij}^a\left(\partial_\mu F_j\right)\\ &\quad -i\varepsilon^a F_j^\dagger T_{ij}^a\left(\partial_\mu\frac{\partial}{\partial\mathcal{L}}\left(\partial_\mu F_i^\dagger\right)\right) - i\varepsilon^a\left(\partial_\mu F_j^\dagger\right)T_{ij}^a\frac{\partial}{\partial\mathcal{L}}\left(\partial_\mu F_i^\dagger\right) & (11.2.10)\end{aligned}$$

We can combine the first two terms as a complete derivative, and likewise with the last two terms, and then we recognize the form

$$i\varepsilon^a\partial_\mu J^{\mu a} = 0 \tag{11.2.11}$$

Since the ε^a are arbitrary, each of the $\dim(G)$ components are conserved

$$\partial_\mu J^{\mu a} = 0 \tag{11.2.12}$$

We conclude that the conserved currents, Noether's currents, for complex fields are

$$\boxed{\partial_\mu J^{\mu a} = 0 \qquad J^{\mu a} = i\frac{\partial}{\partial L}(\partial_\mu F_i)T_{ij}^a F_j} \tag{11.2.13}$$

11.2.

(i) Need to consider four group axioms for the set of 2×2 orthogonal real matrices:-

Closure. The product of two orthogonal matrices A, B is also orthogonal,

$$(\mathsf{AB})^T.(\mathsf{AB}) = \mathsf{B}^T\mathsf{A}^T.\mathsf{AB} = \mathsf{B}^T \mathbb{1} \mathsf{B} = \mathbb{1} \tag{11.2.1}$$

and the product of real matrices is real.

Associativity. A property of matrix multiplication.

Identity. Always the unit matrix for matrix representations, $\mathbb{1}$. It is real and orthogonal and so in the group.

Inverse. The inverse group element is the inverse matrix. This is also orthogonal and real and so is in the group.

$$\mathsf{A}^T\mathsf{A} = \mathbb{1} \quad \rightarrow \quad \mathsf{A}^{-1} = \mathsf{A}^T \quad (\mathsf{A}^{-1})^T.\mathsf{A}^{-1} = \mathsf{A}.\mathsf{A}^T = \mathbb{1} \tag{11.2.2}$$

(ii) From the explicit form of the entries in the matrices U making up the representation, none of them are larger than 1, $|U_{ij}| \leq 1$. Thus the group must be compact. It is a group (proved above) and, if we look only at the subgroup with the Z_+ factor (the U_+ elements making up the $SO(2)$ subgroup), then all elements are "close" to one another, i.e. varying θ moves us smoothly through all two-by-two real orthogonal matrices. Thus is it is a Lie group. With only one continuous parameter, it is a one-dimensional group.

Looking at the terms with the Z_+ factor (the U_+ elements of the $SO(2)$ subgroup, a Lie group) we see that taking θ to zero gives the identity element. Looking close to this, $|\theta| \ll 1$, gives

$$\mathsf{U}_+ = \exp\{i\theta\mathsf{T}\} \quad \approx \quad \mathbb{1} + i\theta\mathsf{T} \tag{11.2.3}$$

$$\mathsf{U} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad \approx \quad \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} + O(\theta^2) \tag{11.2.4}$$

$$= \quad \mathbb{1} + i\theta \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{11.2.5}$$

Comparing (11.2.5) and (11.2.3) we see the generator is as given

$$\mathsf{T} = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \tag{11.2.6}$$

(iii) Let us consider the Lie group part of the group, namely the U_+ part. The eigenvalues are $\lambda_\pm = \exp\{\pm i\theta\}$ with eigenvectors $\boldsymbol{e}_\pm \propto (1, \pm i)$. *NOTE* that the eigenvectors are the same for all the symmetry matrices U, only the eigenvalues depend on the continuous parameter θ. Thus we can diagonalise all the symmetry matrices U at the same time with one transformation. Let us choose

$$\boldsymbol{e}_+ := \frac{1}{\sqrt{2}}(1, i), \quad \boldsymbol{e}_- := \frac{1}{\sqrt{2}}(i, 1) \tag{11.2.7}$$

We can therefore write the $SO(2)$ rotation matrices as

$$\mathsf{U} = \mathsf{B}\boldsymbol{\Lambda}\mathsf{B}^{-1}, \quad \boldsymbol{\Lambda} := \begin{pmatrix} e^{+i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{11.2.8}$$

where the B has the eigenvectors as columns (we scaled $\boldsymbol{e}_-$ in (11.2.7) to make B unitary for computational simplicity)

$$\mathsf{B} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \mathsf{B}^\dagger.\mathsf{B} = \mathbb{1} \tag{11.2.9}$$

What does it all mean?

The two real fields of an $O(2)$ invariant Lagrangian always appear in the invariant form $\phi_i \phi_i$ $(i = 1, 2)$. Under a symmetry transformation they behave, of course, as $\boldsymbol{\phi}^T.\boldsymbol{\phi} \to (\mathsf{U}\boldsymbol{\phi})^T\,\mathsf{U}\boldsymbol{\phi}$. However the diagonalised form of B suggests we work in terms of the eigenvectors of U. Replacing the transpose by a hermitian conjugation ($\boldsymbol{\phi}$ is real so this is allowed)

$$\begin{aligned}
\boldsymbol{\phi}^\dagger.\boldsymbol{\phi} = (\mathsf{U}\boldsymbol{\phi})^\dagger.(\mathsf{U}\boldsymbol{\phi}) &= \left(\mathsf{B}\boldsymbol{\Lambda}\mathsf{B}^\dagger\boldsymbol{\phi}\right)^\dagger.\left(\mathsf{B}\boldsymbol{\Lambda}\mathsf{B}^\dagger\boldsymbol{\phi}\right) && (11.2.10)\\
&= \left(\boldsymbol{\Lambda}[\mathsf{B}^\dagger\boldsymbol{\phi}]\right)^\dagger.\mathsf{B}^\dagger\mathsf{B}.\left(\boldsymbol{\Lambda}[\mathsf{B}^\dagger\boldsymbol{\phi}]\right) && (11.2.11)\\
&= = (\boldsymbol{\Lambda}.\boldsymbol{\Phi})^\dagger(\boldsymbol{\Lambda}.\boldsymbol{\Phi}) = \boldsymbol{\Phi}^\dagger\boldsymbol{\Phi} && (11.2.12)
\end{aligned}$$

where

$$\boldsymbol{\Phi} := \mathsf{B}^\dagger\boldsymbol{\phi} = \begin{pmatrix} \Phi^\dagger \\ -i\Phi \end{pmatrix}, \quad \Phi := \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad (11.2.13)$$

Note the entries of B, the eigenvectors of U the symmetry matrices, suggest we work with the new Φ_i components, new linear combinations of the real fields ϕ_i. Put another way we can express the two real fields in terms of these orthogonal eigenvalues as follows

$$\phi_1 := \Phi\boldsymbol{e}_+ - i\Phi^\dagger\boldsymbol{e}_-, \quad \phi_2 := -i\Phi\boldsymbol{e}_+ + \Phi^\dagger\boldsymbol{e}_- \qquad (11.2.14)$$

The B matrix is transforming the basis we are using in the vector space, that is the two-dimensional space in which the two fields live, the internal space.

The point is that the symmetry transformations for the $\boldsymbol{\Phi}$ fields are $\boldsymbol{\Lambda}$ and is **diagonal**. The Lagrangian would now look like

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - V(\frac{1}{2}\phi_i\phi_i) && (11.2.15)\\
&= \frac{1}{2}(\partial_\mu\boldsymbol{\Phi})^\dagger(\partial^\mu\boldsymbol{\Phi}) - V(\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi}) && (11.2.16)
\end{aligned}$$

with symmetry transformations given by $\boldsymbol{\Lambda}$

$$\boldsymbol{\Phi} \to \boldsymbol{\Phi}' = \boldsymbol{\Lambda}\boldsymbol{\Phi} = \begin{pmatrix} e^{i\theta}\Phi^\dagger \\ -ie^{-i\theta}\Phi \end{pmatrix} \qquad (11.2.17)$$

Thus we have transformed the symmetry matrices U into symmetry matrices $\boldsymbol{\Lambda} = \mathsf{B}^\dagger\mathsf{U}\mathsf{B}$, i.e.we have found a *similarity* transformation which makes the group matrices U block diagonal and so U is now a reducible representation. **HOWEVER** this was *ONLY* at the cost of working with Φ_i complex combinations of our real fields, ϕ_i. If we had stuck to the original real vector space, i.e. used matrices which acted on two-dimensional real vectors and rotated/reflected them into other real two-dimensional real vectors, we would not allowed to choose such complex combinations Φ as basis vectors. Thus we say the rotation matrices are not "reducible over the field of reals", i.e. when sticking to pure real vectors, but they are reducible if we allow ourselves to use complex vectors.

Once we decide to use this complex representation, then we may as well exploit the reducibility. Writing out the components of $\boldsymbol{\Phi}$ we see that the Lagrangian can be written out as

$$\mathcal{L} = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - V(2\Phi^\dagger\Phi) \qquad (11.2.18)$$

Note that the coefficients in the Lagrangian have exactly the factors of two which we have stated are appropriate for the standard form for such complex field representations. Here, these came from the factors of $1/\sqrt{2}$ in the definition of B, and were chosen to make the B unitary. In terms of the fields in QFT, such a linear unitary mapping from $\boldsymbol{\phi}$ to $\boldsymbol{\Phi}$ ensures that if the $\boldsymbol{\phi}$ fields were properly normalised and obey the equal time commutation relations (the foundations of QFT from one point of view) then so will the $\boldsymbol{\Phi}$ fields.

Q??.

(i)

$$\boldsymbol{\phi} \to \boldsymbol{\phi}' = \mathsf{U}\boldsymbol{\phi}, \qquad \Rightarrow \quad \boldsymbol{\phi}^T \to \boldsymbol{\phi}^T\mathsf{U}^T, \tag{11.2.1}$$
$$\Rightarrow \quad \boldsymbol{\phi}^T.\boldsymbol{\phi} \to \boldsymbol{\phi}^T\mathsf{U}^T.\mathsf{U}\boldsymbol{\phi} = \boldsymbol{\phi}^T.\boldsymbol{\phi} \tag{11.2.2}$$

since $\mathsf{U}^T.\mathsf{U} = \mathbb{1}$ is the definition of orthogonal matrices — they keep the length of real vectors constant. This is sufficient to see that the potential terms $V(\boldsymbol{\phi}^T.\boldsymbol{\phi}/2) = V(\boldsymbol{\phi}'^T.\boldsymbol{\phi}'/2)$ are invariant. For the kinetic terms we must show a little more

$$\partial_\mu(\boldsymbol{\phi}) \to \partial_\mu(\boldsymbol{\phi}') \quad = \quad \partial_\mu(\mathsf{U}\boldsymbol{\phi}) = (\partial_\mu\mathsf{U})\,\boldsymbol{\phi} + \mathsf{U}\,(\partial_\mu\boldsymbol{\phi}), \tag{11.2.3}$$
$$= \quad \mathsf{U}\,(\partial_\mu\boldsymbol{\phi}) \ \text{ if } \partial_\mu\mathsf{U} = 0 \tag{11.2.4}$$

Thus we see that for **global** symmetry transformations, i.e. $\partial_\mu\mathsf{U} = 0$ we have that

$$(\partial^\mu\boldsymbol{\phi}')^T(\partial_\mu\boldsymbol{\phi}') \quad = \quad (\partial^\mu\boldsymbol{\phi})^T\,\mathsf{U}^T.\mathsf{U}\,(\partial_\mu\boldsymbol{\phi}), \tag{11.2.5}$$

and now the whole Lagrangian has been shown to be invariant.

(ii) We showed in the lectures that the Noether currents for real fields are of the form[1]

$$J^{\mu a} = i\phi_i\overrightarrow{\partial}^\mu T^a_{ij}\phi_j - i\phi_i\overleftarrow{\partial}^\mu T^a_{ij}\phi_j \tag{11.2.6}$$

In an exam, the context of a question will tell you if you are required to derive this form. Substituting in the form of the $O(2)$ generator (11.2.6) we get the form for the current given in the question.

(iii) One could proceed as in the previous question, but once one knows the relationship between real and complex representations, we may as well proceed directly.

$$\Phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \quad \to \quad \phi_1 = \frac{1}{\sqrt{2}}\left(\Phi(x) + \Phi^\dagger(x)\right), \tag{11.2.7}$$
$$\phi_2 = \frac{-i}{\sqrt{2}}\left(\Phi(x) - \Phi^\dagger(x)\right) \tag{11.2.8}$$
$$\Rightarrow \quad \left(\phi_1^2 + \phi_2^2\right) = 2|\Phi|^2 \tag{11.2.9}$$
$$\text{and } (\partial^\mu\phi_i)(\partial_\mu\phi_i) = 2(\partial^\mu\Phi)^\dagger(\partial^\mu\Phi) \tag{11.2.10}$$

and so $\mathcal{L}_{O(2)}[\phi_1, \phi_2] = \mathcal{L}_{U(1)}[\Phi, \Phi^\dagger]$.

(iv) Remembering that we have a global symmetry, so that $\partial_\mu\theta = 0$, then

$$\Phi(x) \to \Phi'(x) = e^{i\theta}\Phi(x) \quad \Rightarrow \quad |\Phi(x)| \to |\Phi'(x)| = |\Phi(x)| \tag{11.2.11}$$
$$\partial_\mu\Phi(x) \to \partial_\mu\Phi'(x) = e^{i\theta}\partial_\mu\Phi(x) \qquad \text{since } \partial_\mu\theta = 0. \tag{11.2.12}$$
$$\Rightarrow \quad |\partial_\mu\Phi(x)| \to |\partial_\mu\Phi'(x)| = |\partial_\mu\Phi(x)| \tag{11.2.13}$$

[1] Up to an overall constant if we choose a Hermitian form.

(v) Just as in the previous question, the group elements U are the transformation matrices $\Phi \to \Phi' = \mathsf{U}\Phi$ so here they are just the phase factors (one-dimensional unitary matrices)

$$\mathsf{U} = e^{i\theta} = \exp\{i\theta\mathsf{T}\} \tag{11.2.14}$$

and we see the generator in this complex irreducible representation for the field Φ is a one-dimensional representation is simple $\mathsf{T}_\Phi = 1$.

The point made in the previous question is that the irreducible representation when using real fields is two-dimensional, but by moving to complex fields we can reduce this to these one-dimensional representations.

(vi) What we need to do is find a representation where the $O(2)$ generator of the doublet of real scalar fields $\boldsymbol{\phi}$, the T of (11.2.6) is diagonal. This was found in the previous question. There by transforming to a new basis of fields using matrix B of (11.2.9), i.e. this is the matrix M in this question, the field vector $\bar{\phi}$ is then the complex valued doublet $\mathbf{\Phi}$ of (11.2.13) and the diagonal form of the generator is then simply $\mathbf{\Lambda}$ of (11.2.8). The current is then

$$\begin{aligned} J^{\mu a} &= i\phi_i \overrightarrow{\partial}^\mu T^a_{ij}\phi_j - i\phi_i \overleftarrow{\partial}^\mu T^a_{ij}\phi_j & (11.2.15)\\ &= i\bar{\phi}^\dagger_i \overrightarrow{\partial}^\mu \left(\mathsf{M}^\dagger \mathsf{T}^a \mathsf{M}\right)_{ij} \bar{\phi}_j - i\bar{\phi}^\dagger_i \overleftarrow{\partial}^\mu \left(\mathsf{M}^\dagger \mathsf{T}^a \mathsf{M}\right)_{ij} \bar{\phi}_j & (11.2.16)\\ &= i\bar{\phi}^\dagger_i \overrightarrow{\partial}^\mu \Lambda_{ij}\bar{\phi}_j - i\bar{\phi}^\dagger_i \overleftarrow{\partial}^\mu \Lambda_{ij}\bar{\phi}_j & (11.2.17)\end{aligned}$$

Thus we see that the charge eigenstates are Φ and $\Phi^\dagger$, the components of $\mathbf{\Phi}$ of (11.2.13). Note from the transformation (11.2.17) that the components of the complex field doublet transform with *opposite* phases. Put another way, the generator in this basis is $\mathbf{\Lambda}$ which has diagonal entries $+1$ and -1 Thus the relative charge for the Φ and $\Phi^\dagger$ fields is $+1$ to -1. This is as it should be as the two fields represent a particle and its anti-particle. Such particle-anti-particle pairs will always have a complex representation like this, though it is not always convenient to work with this representation (see discussion of real scalar triplets fields with $SO(3)$ symmetry).

Q??.

(i) The Lagrangian contains factors of $(\Phi^\dagger_j)\Phi_j$ and $(\partial_\mu\Phi^\dagger_j)(\partial^\mu\Phi_j)$ only. So

$$\begin{aligned} \Phi_j \to {\Phi'}_j = \exp\{i\theta_j\}\Phi_j\ ,\ \ \Phi^\dagger_j \to {\Phi^\dagger}'_j = \exp\{i\theta_j\}\Phi^\dagger_j \ \ &\Rightarrow\ \ \Phi^\dagger_j\Phi_j \to {\Phi^\dagger}'_j{\Phi'}_j = \Phi^\dagger_j\Phi_j\ , & (11.2.1)\\ \partial_\mu\Phi_j \to \partial_\mu{\Phi'}_j = \exp\{+i\theta_j\}\partial_\mu\Phi_j\ , \qquad & \ \ \ \partial_\mu\Phi^\dagger_j \to \partial_\mu{\Phi^\dagger}'_j = \exp\{-i\theta_j\}\partial_\mu\Phi^\dagger_j \\ \Rightarrow (\partial_\mu\Phi^\dagger_j)\partial^\mu\Phi_j) \to (\partial_\mu{\Phi^\dagger}'_j)(\partial^\mu{\Phi'}_j) \ \ &= \ \ (\partial_\mu\Phi^\dagger_j)\partial^\mu\Phi_j & (11.2.2)\end{aligned}$$

Hence each term of the Lagrangian is invariant without any restriction on the θ_j's being required.

(ii) $\exp\{i\theta\}$ is a representation of $U(1)$ but we have two independent θ parameters so we have a $U(1)_1 \times U(1)_2$ symmetry. We we add the subscripts $1, 2$ to differentiate the two parts of the symmetry (θ_i transformations are associated with $U(1)_i$, $i = 1, 2$). With two independent parameters θ_1, θ_2 there must be two Noether currents.

Note that to be complete we have that under $U(1)_1$ the Φ_2 field is a singlet i.e. it lies in the trivial representation of the group, the generators are zero and the charge of field two under this first $U(1)_1$ symmetry is zero $\Phi_2 \to {\Phi'}_2 = \Phi_2$. Likewise under $U(1)_2$ the Φ_1 field is a singlet. A general notation would be to have two generators, Y_1 and Y_2 where $\exp\{i\theta_j Y_j\} \in \mathrm{U}(1)$. The fields are in different representations of the whole group so that the two generators look different when comparing the representations used on the Φ_1 and Φ_2 fields i.e. the fields are in different representations of the group $U(1) \times U(1)$. We wil have one Y_1 for the first field and *another* Y_1 for the second field! See the next section.

(iii) Denote the two charges as Y_1 and Y_2 (the Y's can also be thought of as the generators as for $U(1)$'s as the generators are 1 dimensional matrices. The eigenvalues of the diagonal matrices [= charges] are trivially related). We have

Field	Y_1	Y_2
Φ_1	$+1$	0
Φ_2	0	$+1$

This can be seen by calculating the Noether current, or if you have the courage, just read it off from the form of the generators implicit in the transformations of the fields under each of the two $U(1)$'s.

(iv) The new term $\mathcal{L}_3$ is invariant under a transformation $\exp\{i\theta_3\}$. However, one then finds that with the $\mathcal{L}_{123}$ term we require that $\theta_1 + \theta_2 + \theta_3 = 0$ if one wants to make the total $\mathcal{L}_{\text{new}}$ invariant. Eliminating θ_3 so that we can keep the previous notation, we see that we still have only two free parameters and so still have just the $U(1)_1 \times U(1)_2$ symmetry. However, the transformation of Φ_3 is now $\Phi_3 \to \Phi'_3 = \exp\{i\theta_1 Y_1\}.\exp\{i\theta_2 Y_2\} = \exp\{-i\theta_1\}.\exp\{-i\theta_2\}$. Thus the two generators of the symmetry group are represented by $Y_1 = Y_2 = -1$ for this third field Φ_3 so it has -1 charge under both $U(1)$'s.

Q??.

(i) **Rule**: A complex scalar field is needed to represent a particle and its anti-particle, though this can be hidden as two real fields (e.g. see my SO(3) real scalar models).

A complex scalar field Φ and its hermitian conjugate field $\Phi^\dagger$ represent two degrees of freedom, two ways of transporting quantum numbers through the system, either as a scalar particle or as its distinct anti-particle. The free quantum fields contain two distinct type of annihilation and creation operators, say $\hat{a}$ and $bhat$. For instance the π^+ and π^- are a scalar particle/anti-particle pair.

All particles which have non-zero electric charge have distinct anti-particles, as anti-particles always have the same mass but opposite conserved charges of their partner particle. NOT all anti-particles have non-zero electric charge though e.g. K^0, ν (neutrinos).

A real scalar field ϕ represent a single scalar particle which is its own anti-particle, and so only one degree of freedom (hermitian conjugate field is identical to field $\phi = \phi^\dagger$). The free quantum fields contain only one type of annihilation and creation operator, say $\hat{a}$. For instance the π^0 is a scalar particle which is its own anti-particle - note it has zero electric charge. All particles which are their own anti-particles have zero electric charge but NOT vice-versa - think about it.

As we have two particles and two distinct anti-particles in this question, this suggests we need two complex scalar fields. Of course, these could be hiding as four real fields but

Rule: Choose the simplest option first, until the evidence forces you to do otherwise.

(ii) A simple U(1) (phase) symmetry ensures that particles have the same mass as their distinct anti-particles. For complex fields this means

$$m_1^2 \Phi_1^\dagger \Phi_1 + m_2^2 \Phi_2^\dagger \Phi_2 \tag{11.2.1}$$

would be a valid mass term fitting in with U(1) symmetries. However, the key evidence here is that *all four* the scalar particles and anti-particles all have the same mass.

Rule: Particles with equal masses or couplings are mixed together by some symmetry. Note that particles related by symmetries will have different charges though these will be in specific patterns related to the representation theory of the symmetry group. e.g. π^+, π^0, π^- have essentially the same mass, but their electric charges are unequal.

Here therefore, we need some sort of symmetry mixing the one and two Φ_j fields and forcing us to write a single mass term in the Lagrangian. The obvious way to do this would be

$$m^2 \Phi_j^\dagger \Phi_j, \quad j = 1, 2 \tag{11.2.2}$$

which has a $U(2)$ symmetry.

(iii) To summarize so far, two complex scalar fields are needed, and they should be in the same U(2) doublet because of the mass equality. A simple answer, with some interactions to give some sort of non-trivial physics, would be

$$\mathcal{L} = (\partial_\mu \Phi_j)^\dagger (\partial^\mu \Phi_j) - m^2 \Phi_j^\dagger \Phi_j - \lambda (\Phi_j^\dagger \Phi_j)^2, \quad j = 1, 2 \tag{11.2.3}$$

The charges are those for $U(2) \approx U(1) \times SU(2)$. $U(1)$ is generated by the 2-by-2 unit matrix (generates overall phase transformations with same phase for both components of the complex scalar doublet), and we can generate the SU(2) using Pauli matrices, where only the third one is diagonal. Reading all the diagonal generators tells us that the charges for the fields

	U(1)	$2\tau^3$
Φ_1	+1	+1
Φ_2	+1	-1

Note that the overall scale of charge doesn't matter, its the relative charge that matters. So I have cheekily rescaled the diagonal $SU(2)$ generator. The anti-particles (represented by the hermitian conjugate fields) have the the opposite charge of their associated particle field.

Q??.

(i)

$$\widehat{\phi}_i := \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega(k)}\left(\hat{a}_i(\boldsymbol{k})e^{-ik.x} + \hat{a}_i^\dagger(\boldsymbol{k})e^{+ik.x}\right), \quad k_0 = \omega(k) \tag{11.2.1}$$

$$[\hat{a}_i(k), \hat{a}_j^\dagger(k')] = 2\omega(k).(2\pi)^3\delta(\boldsymbol{k} - \boldsymbol{k}')\delta_{ij}. \tag{11.2.2}$$

For the charge we need the zero-th component of the four-current

$$J^0 = \partial^0\widehat{\phi}_i = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega(k)}\left(\hat{a}_i(\boldsymbol{k})k_0e^{-ik.x} - k_0\hat{a}_i^\dagger(\boldsymbol{k})e^{+ik.x}\right) \tag{11.2.3}$$

$$= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2}\left(\hat{a}_i(\boldsymbol{k})e^{-ik.x} - \hat{a}_i^\dagger(\boldsymbol{k})e^{+ik.x}\right) \tag{11.2.4}$$

Now we find that

$$(\partial^0\widehat{\phi}_1)\widehat{\phi}_2 = \frac{1}{(2\pi)^6}\int d^3\boldsymbol{k}d^3\boldsymbol{k}'\frac{1}{4\omega(k')}\left(\hat{a}_1(\boldsymbol{k})e^{-ik.x} - \hat{a}_1^\dagger(\boldsymbol{k})e^{+ik.x}\right)$$

$$\left(\hat{a}_2(\boldsymbol{k}')e^{-ik'.x} + \hat{a}_2^\dagger(\boldsymbol{k}')e^{+ik'.x}\right) \tag{11.2.5}$$

$$= \frac{1}{(2\pi)^6}\int d^3\boldsymbol{k}d^3\boldsymbol{k}'\frac{1}{4\omega(k')}\left[\left(\hat{a}_1(\boldsymbol{k})\hat{a}_2(\boldsymbol{k}')e^{-i(k+k').x} - \hat{a}_1^\dagger(\boldsymbol{k})\hat{a}_2^\dagger(\boldsymbol{k}')e^{+i(k+k').x}\right)\right. \tag{11.2.6}$$

$$\left. + \left(-\hat{a}_1^\dagger(\boldsymbol{k})\hat{a}_2(\boldsymbol{k}')e^{+i(k-k').x} + \hat{a}_1(\boldsymbol{k})\hat{a}_2^\dagger(\boldsymbol{k}')e^{-i(k-k').x}\right)\right] \tag{11.2.7}$$

Likewise

$$(\partial^0\widehat{\phi}_2)\widehat{\phi}_1 = \frac{1}{(2\pi)^6}\int d^3\boldsymbol{k}d^3\boldsymbol{k}'\frac{1}{4\omega(k')}\left[\left(\hat{a}_2(\boldsymbol{k})\hat{a}_1(\boldsymbol{k}')e^{-i(k+k').x} - \hat{a}_2^\dagger(\boldsymbol{k})\hat{a}_1^\dagger(\boldsymbol{k}')e^{+i(k+k').x}\right)\right. \tag{11.2.8}$$

$$\left. + \left(-\hat{a}_2^\dagger(\boldsymbol{k})\hat{a}_1(\boldsymbol{k}')e^{+i(k-k').x} + \hat{a}_2(\boldsymbol{k})\hat{a}_1^\dagger(\boldsymbol{k}')e^{-i(k-k').x}\right)\right] \tag{11.2.9}$$

Now when we subtract these two to make J^0 the first two terms in each case cancel since the operators are in different orders but these operators commute $[\hat{a}_1(\boldsymbol{k}), \hat{a}_2(\boldsymbol{k}')] = 0$. The operators in the third and fourth terms also commute but the signs mean these simply add to give

$$J^0 = \frac{1}{(2\pi)^6}\int d^3\boldsymbol{k}d^3\boldsymbol{k}'\frac{1}{2\omega(k')}\left(-\hat{a}_1^\dagger(\boldsymbol{k})\hat{a}_2(\boldsymbol{k}')e^{+i(k-k').x} + \hat{a}_1(\boldsymbol{k})\hat{a}_2^\dagger(\boldsymbol{k}')e^{-i(k-k').x}\right) \tag{11.2.10}$$

Integrating over all space means that we get factors of $(2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')$ from the exponentials. This serves to kill off one of the two momentum integrations to give

$$\hat{Q} = \int d^3\boldsymbol{x}J^0 = \frac{1}{(2\pi)^3}\int d^3\boldsymbol{k}\frac{1}{2\omega(k)}\left(\hat{a}_1^\dagger(\boldsymbol{k})\hat{a}_2(\boldsymbol{k}) - \hat{a}_1(\boldsymbol{k})\hat{a}_2^\dagger(\boldsymbol{k})\right) \tag{11.2.11}$$

Q??.

Q??.

Q??.

Unless a differential operator, ∂_μ, D_μ, is specifically bracketed off, it acts on everything to its right.

Q??. Group Theory of $U(1) \times SU(2)$

(i) Need to consider four group axioms for the set of $d \times d$ unitary matrices $G = \{\mathsf{U}\}$ ($d = 2$ in the EW theory).

Closure. The product of two unitary matrices A, B is also unitary,

$$(\mathsf{AB})^\dagger.(\mathsf{AB}) = \mathsf{B}^\dagger \mathsf{A}^\dagger.\mathsf{AB} = \mathsf{B}^\dagger \mathbb{1}\mathsf{B} = \mathbb{1}. \tag{11.2.1}$$

Thus products of unitary matrices also lie in G.

Associativity. A property of matrix multiplication.

Identity. Always the unit matrix for matrix representations, $\mathbb{1}$. It is unitary and so in the set G.

Inverse. The inverse group element of $\mathsf{A} \in G$ is the inverse matrix A^{-1}. This is also unitary and so is in the set G.

$$\mathsf{A}^\dagger \mathsf{A} = \mathbb{1} \quad \Rightarrow \quad \mathsf{A}^{-1} = \mathsf{A}^\dagger, \quad (\mathsf{A}^{-1})^\dagger.\mathsf{A}^{-1} = \mathsf{A}.\mathsf{A}^\dagger = \mathbb{1} \tag{11.2.2}$$

(ii)

$$1 = \det(\mathbb{1}) = \det\left(\mathsf{U}^\dagger.\mathsf{U}\right) = \det\left(\mathsf{U}^\dagger\right)\det(\mathsf{U}) = [\det(\mathsf{U})]^* \det(\mathsf{U}) = |\det(\mathsf{U})|^2 \tag{11.2.3}$$

Thus

$$\det(\mathsf{U}) = e^{i\theta}, \quad \theta \in \mathbb{R} \tag{11.2.4}$$

So for every unitary matrix $\mathsf{U} \in U(N)$, factor out this phase as an overall factor

$$\mathsf{U}_Y := e^{i\theta}\mathbb{1} \quad \Rightarrow \quad \mathsf{U}_S := \mathsf{U}_Y^{-1}\mathsf{U} \tag{11.2.5}$$

$$\Rightarrow \det(\mathsf{U}_S) = \det\left(\mathsf{U}_Y^{-1}\right).\det(\mathsf{U}) \quad = \quad e^{-i\theta}.e^{i\theta} = +1 \tag{11.2.6}$$

$$\mathsf{U}_S^\dagger.\mathsf{U}_S = \left(e^{-i\theta}\mathbb{1}.\mathsf{U}\right)^\dagger.\left(e^{-i\theta}\mathbb{1}.\mathsf{U}\right) = e^{i\theta}.\mathsf{U}^\dagger.e^{-i\theta}.\mathsf{U} \quad = \quad \mathbb{1} \tag{11.2.7}$$

Thus U_S is indeed a special unitary matrix. Now need to consider the four group axioms for the set of $d \times d$ special unitary matrices $SU(N) = \{\mathsf{U}\}$ ($N = d$).

Closure. The product of two unitary matrices is unitary ,as shown above. The determinant of the product of any two matrices, A, B is just the product of their determinants, so if both were special so is the product of the two

$$\det(\mathsf{AB}) = \det(\mathsf{A}).\det(\mathsf{B}) = 1.1 = 1 \tag{11.2.8}$$

Associativity. A property of matrix multiplication.

Identity. Always the unit matrix for matrix representations, $\mathbb{1}$. It is unitary, has determinant one and so in the set $SU(N)$.

Inverse. The inverse group element of $\mathsf{A} \in G$ is the inverse matrix A^{-1}. This is also unitary as proved above. It also has determinant one so is in $SU(N)$ as required

$$1 = \det(\mathbb{1}) = \det\left(\mathsf{A}^{-1}\mathsf{A}\right) = \det(\mathsf{A}^{-1}).\det(\mathsf{A}) = \det(\mathsf{A}^{-1}).1 = \det(\mathsf{A}^{-1}) \tag{11.2.9}$$

(iii) The set of phases $\{e^{i\theta}\}$ forms the representation of the group U(1). Multiplying by the unit matrix does not effect this except that it gives us a reducible representation of U(1).

(iv) The unit matrix commutes with every other matrix, $[\mathbb{1}, \mathsf{U}] = 0$. Likewise, multiplication by a constant such as $e^{i\theta}$ commutes. Thus U_Y commutes with all matrices including the special unitary ones $[\mathsf{U}_Y, \mathsf{U}_S] = 0$.

(v) For $d = N = 2$ we have simply that

$$U_Y = e^{i\theta}\mathbb{1} \approx \left(1 + i\theta + O(\theta^2)\right)\mathbb{1} = \mathbb{1} + i\theta\mathbb{1} + O(\theta^2) \tag{11.2.10}$$

and since $\mathsf{U} = \mathbb{1} + i\epsilon^a T^a + \dots$ we see that $\mathsf{T}_Y = \frac{1}{2}\mathsf{Y}, \mathsf{Y} := \mathbb{1}$ if we choose $\theta = \epsilon^Y$

For the $SU(2)$ part we find that there are three generators which in the two-dimensional representation and the standard normalisation are half the Pauli matrices, τ^a $(a = 1, 2, 3)$

$$\mathsf{T}^a = \frac{1}{2}\boldsymbol{\tau}^a, \quad \mathrm{Tr}\{\mathsf{T}^a\mathsf{T}^b\} = \frac{1}{2}\delta^{ab} \tag{11.2.11}$$

The Pauli matrices have the usual commutation relations $[\boldsymbol{\tau}^a, \boldsymbol{\tau}^b] = i\epsilon^{abc}\boldsymbol{\tau}^c$ where this ϵ^{abc} is the standard totally anti-symmetric tensor. The unit matrix, and any constant times the unit matrix, commutes with every other matrix. So the Lie algebra of $U(2)$ is defined through the commutation relations

$$[\mathsf{T}^a, \mathsf{T}^b] = \frac{i}{2}\epsilon^{abc}\mathsf{T}^c, \quad [\mathsf{T}_Y, \mathsf{T}^a] = 0 \tag{11.2.12}$$

Q??. $U(1) \times SU(2)$ and Goldstone's theorem

(i)

$$\begin{aligned}
\boldsymbol{\Phi} \to \boldsymbol{\Phi}' = \mathsf{U}\boldsymbol{\Phi} \quad &\Rightarrow \quad \boldsymbol{\Phi}^\dagger \to \boldsymbol{\Phi}'^\dagger = \boldsymbol{\Phi}^\dagger\mathsf{U}^\dagger && (11.2.1)\\
\Rightarrow \quad \boldsymbol{\Phi}^\dagger.\boldsymbol{\Phi} \quad &\to \quad \boldsymbol{\Phi}'^\dagger.\boldsymbol{\Phi}' = \boldsymbol{\Phi}^\dagger\mathsf{U}^\dagger.\mathsf{U}\boldsymbol{\Phi} && (11.2.2)\\
&= \quad \boldsymbol{\Phi}^\dagger.\boldsymbol{\Phi} \quad \text{iff} \quad \mathsf{U}^\dagger.\mathsf{U} = \mathbb{1} && (11.2.3)
\end{aligned}$$

Thus the mass term $m^2|\boldsymbol{\Phi}|^2$ of the potential V is invariant if U is a unitary matrix. As $\boldsymbol{\Phi}$ is two-dimensional,U must be a two-=by-two unitary matrix.

The interaction term, $\lambda(\boldsymbol{\Phi}^\dagger.\boldsymbol{\Phi})^2 = \lambda|\boldsymbol{\Phi}|^4$, also depends only on the modulus of the field which we have just shown to be invariant if U is a unitary matrix.

The derivative terms work the same only if U is also space-time independent

$$\begin{aligned}
\partial_\mu\boldsymbol{\Phi} \to \partial_\mu\boldsymbol{\Phi}' \quad &= \quad \partial_\mu(\mathsf{U}\boldsymbol{\Phi}) && (11.2.4)\\
&= \quad (\partial_\mu\mathsf{U})\boldsymbol{\Phi} + \mathsf{U}(\partial_\mu\boldsymbol{\Phi}) && (11.2.5)\\
(\partial^\mu\boldsymbol{\Phi})^\dagger \quad &\to \quad \boldsymbol{\Phi}^\dagger\left(\partial^\mu\mathsf{U}^\dagger\right) + \left(\partial_\mu\boldsymbol{\Phi}^\dagger\right)\mathsf{U}^\dagger && (11.2.6)\\
\boldsymbol{\Phi}^\dagger.\boldsymbol{\Phi} \quad &\to \quad \boldsymbol{\Phi}^\dagger\mathsf{U}^\dagger.\mathsf{U}\boldsymbol{\Phi} + O(\partial_\mu\mathsf{U}) && (11.2.7)\\
&= \quad \left(\partial^\mu\boldsymbol{\Phi}^\dagger\right).(\partial_\mu\boldsymbol{\Phi}) \quad \text{iff}\partial_\mu\mathsf{U} = 0 && (11.2.8)
\end{aligned}$$

i.e. we have a theory invariant under *global* U(2) symmetry transformations.

(ii) First, the lowest energy configurations are *constant* in space-time, $\partial_\mu\boldsymbol{\Phi} = 0$. To see this we need to look at the Hamiltonian, since this is the total of kinetic and potential energy terms. In the Hamiltonian we have fields $\boldsymbol{\Phi}, \boldsymbol{\Phi}^\dagger$ and their conjugate momenta $\boldsymbol{\Pi}, \boldsymbol{\Pi}^\dagger$, and the usual kinetic terms of the Lagrangian in the Hamiltonian become

$$\boldsymbol{\Pi}^\dagger.\boldsymbol{\Pi} + \left(\underline{\nabla}\boldsymbol{\Phi}^\dagger\right).\left(\underline{\nabla}\boldsymbol{\Phi}\right) = (\partial_0\Phi_1)^2 + (\partial_0\Phi_2)^2 + \left(\vec{\nabla}\Phi_1\right)^2 + \left(\vec{\nabla}\Phi_2\right)^2 \tag{11.2.9}$$

which is positive semi-definite reaching the minimum value zero only for constant field configurations.

The remaining terms in the Hamiltonian are just the potential with a positive overall coefficient, $+V$, so all we have to do is minimise the potential for constant space-time fields.

$$\frac{\partial V}{\partial \Phi_i} = m^2\Phi_i^\dagger + 2\lambda\Phi_i^\dagger\left(\Phi_j^\dagger\Phi_j\right) = 0 \tag{11.2.10}$$

$$\Rightarrow \qquad \Phi_i = 0 \text{ or } |\mathbf{\Phi}|^2 = -\frac{m^2}{2\lambda} \tag{11.2.11}$$

The latter only exists for $m^2 < 0$ but then is the lower of the two solutions.

(iii) For $m^2 < 0$, the minimum energy solutions have space-time constant fields of a given modulus. A space-time constant unitary transformation gives another space-time vector of the same modulus, thus there is a whole set of minimum energy solutions related by space-time constant unitary transformations. This should not be surprising as in general, symmetry transformations of a solution of the classical equations of motion (and the same happens in the quantum theory) *must*, by symmetry, give another solution of the same energy - the action (Hamiltonian etc.) are invariant after all. Sometimes, some symmetry transformations do not generate distinct solutions — these are unbroken symmetry transformations as the solution is invariant, unchanged by these symmetries. The $m^2 > 0$ solution is invariant under all symmetry transformations. A broken symmetry vacuum for some parts of the symmetries (the broken parts) does change.

In any case, we can take any constant vector of the right size $|\mathbf{\Phi}_0|$, and multiply by general symmetry transformations. This *always* be generates all possible lowest energy solutions - its just not all of them have to be distinct (multiplying by unbroken symmetries leaves the vacuum unchanged, one way of defining the little or stability group H)).

Here we define

$$v = \sqrt{\frac{-m^2}{\lambda}}, \tag{11.2.12}$$

and choose one arbitrary direction in complex two-dimensional space, $\boldsymbol{e}_0$, to form an exemplary lowest energy solution $(v/\sqrt{2})\boldsymbol{e}_0$. Then we multiply it by a symmetry transformation to get other vectors of the same modulus and the same (lowest) energy and so the general lowest energy solution for $m^2 < 0$ is $\mathbf{\Phi}_0$ where

$$\mathbf{\Phi}_0 = \frac{v}{\sqrt{2}}\mathsf{U}_0\boldsymbol{e}_0 \tag{11.2.13}$$

where U_0 is any two-dimensional unitary matrix.

(iv) No, a general space-time dependent unitary transformation is *not* a symmetry of the Lagrangian, as we showed above in (11.2.8). However, the potential terms only required U to be unitary, there was no limitation in (11.2.3) on space-time dependence. Thus a space-time dependent unitary matrix $\mathsf{U}(x)$ *is* a symmetry of the potential terms V.

As a result fields built using these space-time dependent $U(2)$ group elements on top of a minimum energy solution have the same potential energy V but will have non-zero derivative (kinetic) terms in the Hamiltonian and so they are not lowest energy states. Thus, to excite such states, one only need supply kinetic energy, i.e. they are acting like free massless modes.

(v) The definition of an unbroken symmetry, in terms of group transformations is that

$$\mathsf{U}_{\text{unbroken}}\mathbf{\Phi}_0 = \mathbf{\Phi}_0 \tag{11.2.14}$$

where $\mathbf{\Phi}_0$ is *any* vacuum solution. A small perturbation then gives a result for Lie algebra elements that

$$\mathsf{A}_{\text{unbroken}}\mathbf{\Phi}_0 = 0, \quad \mathsf{U}_{\text{unbroken}} = \exp\{i\mathsf{A}_{\text{unbroken}}\} \tag{11.2.15}$$

with broken parts of the symmetry being those which do not satisfy these equation. Without loss of generality, we can always choose a basis in the Lie algebra where the generators split into two sets, one spanning the unbroken directions which I call T'^A, and the other spanning the broken directions T''^Z. The general Lie algebra element is then written as $\mathsf{A} = \epsilon'^A \mathsf{T}'^A + \epsilon''^Z \mathsf{T}''^Z$ and the new basis of generators satisfies

$$\mathsf{T}'^A \mathbf{\Phi}_0 = 0, \quad \mathsf{T}''^Z \mathbf{\Phi}_0 \neq 0 \tag{11.2.16}$$

For small perturbations we can write the matrix $\mathsf{U}(x)$ in this basis as follows

$$\mathsf{U}(x) \approx \mathbb{1} + \epsilon'^A(x)\mathsf{T}'^A + \epsilon''^Z(x)\mathsf{T}''^Z + \ldots \tag{11.2.17}$$

The properties of the broken/unbroken generators are not effected by the space-time dependence of $\epsilon'^A(x), \epsilon''^Z(x)$ so

$$\mathsf{U}(x)\mathbf{\Phi}_0 \approx \mathbf{\Phi}_0 + \epsilon''^Z(x)\left(\mathsf{T}''^Z \mathbf{\Phi}_0\right) + \ldots \tag{11.2.18}$$

Fluctuations $\epsilon'^A(x)$ in the unbroken directions do *not* contribute to small perturbations about minimum energy solutions. Only fluctuations in the unbroken directions, $\epsilon''^Z(x)$ can describe some non-trivial perturbations (and then never all possible perturbations, for instance radial field fluctuations are not described).

(vi) Since the group representations are unitary, $\mathsf{U}(x)^\dagger \mathsf{U}(x) = \mathbb{1}$ and we find

$$|\mathbf{\Phi}_{\text{somepert}}(x)|^2 = (\mathbf{\Phi}_{\text{somepert}}(x))^\dagger \mathbf{\Phi}_{\text{somepert}}(x) \tag{11.2.19}$$

$$= (\mathsf{U}(x)\mathbf{\Phi}_0)^\dagger \mathsf{U}(x)\mathbf{\Phi}_0 = \mathbf{\Phi}_0^\dagger \mathsf{U}(x)^\dagger \mathsf{U}(x)\mathbf{\Phi}_0 = \mathbf{\Phi}_0^\dagger \mathbf{\Phi}_0 = |\mathbf{\Phi}_0|^2 \tag{11.2.20}$$

The unitary matrix $\mathsf{U}(x)$ keeps the modulus constant but allows all other possible variations to take place. Thus the only other variation in the field $\mathbf{\Phi}(x)$ to be accounted for is a variation in the modulus. If we add a real perturbation to vev, v replaced by $v + \sigma(x)$, this allows the modulus to vary and will give a complete description of all $\mathbf{\Phi}(x)$ variations. Thus most general field fluctuation can be written as

$$\mathbf{\Phi}(x) = \frac{1}{\sqrt{2}}\mathsf{U}(x)\mathsf{U}_0 \boldsymbol{e}_0 \left(v + \sigma(x)\right) \tag{11.2.21}$$

where $\sigma(x) \in \mathbb{R}$ is a *single, real* scalar field.

(vii) A $d \times d$ complex matrix has d^2 entries and so requires $2d^2$ real parameters to describe it. The unitary condition means that

$$\mathsf{U}^\dagger \mathsf{U} = \mathbb{1} \quad \Rightarrow \quad (\mathsf{U}^\dagger \mathsf{U})_{ij} = (U_k i)^* U_{kj} = \delta_{ij} \tag{11.2.22}$$

In principle this appears to give us $2d^2$ equations, one for each value of i and j indices but the equations with $i \leftrightarrow j$ are the same as the complex conjugate equations, i.e. only half of these equations are independent. Thus with d^2 equations for $2d^2$ real parameters we have d^2 independent real parameters for $d \times d$ unitary matrices leaving only d^2 independent real parameters. The dimension of the Lie Group $U(d)$ is d^2.

For the $\mathsf{U} \in U(2)$ the right-hand side of (11.2.21) has four real-functions in $\mathsf{U}(x)$ and one in the modulus $\sigma(x)$. The LHS requires only four real functions to describe the two complex components. Thus functions on the RHS are over complete, we should be able to find some relationship between them and still be able to describe the full $\mathbf{\Phi}(x)$. Physically, this means the five fields on the RHS can not all describe physical particles.

The resolution is that the $\mathsf{U}(x)$ must only have three independent functions giving non-zero fluctuations about the vacuum. This is because one generator will be unbroken and fluctuations based on this generator will leave the vacuum invariant and contribute nothing.

(viii) We discussed above why the general form of the field can be written as

$$\mathbf{\Phi}(x) = \frac{1}{\sqrt{2}}\mathsf{U}(x)\boldsymbol{e}_0\,(v + \sigma(x)) \tag{11.2.23}$$

(ix) The unbroken generators are given by combinations

$$(c^1T^1 + c^2T^2 + c^3T^3 + c^4Y)e_0 \quad = \quad 0 \tag{11.2.24}$$

The quickest way is to calculate $T^a e_0$ for each term and then to add the vectors to show that

$$\begin{pmatrix} c^1 - ic^2 \\ c^3 - c^4 \end{pmatrix} e_0 \quad = \quad 0 \tag{11.2.25}$$

$$\Rightarrow \quad c^1 = c^2 = 0, \qquad c^3 = c^4 \tag{11.2.26}$$

It is crucial to remember that the coefficients are real but the equation is complex, which is why we get both c^1 and c^2 zero. In this case, after suitable normalisation we see that there is only one unbroken generator

$$\mathsf{T}'^1 = \frac{1}{\sqrt{2}}\left(\mathsf{T}^3 + \mathsf{Y}\right) \tag{11.2.27}$$

To find the broken generators we just construct three basis vectors orthogonal to this one, using the trace of pairs of generators as the inner product on this vector space. The easiest choice is

$$\mathsf{T}''^1 = \frac{1}{\sqrt{2}}\left(\mathsf{T}^3 - \mathsf{Y}\right), \quad \mathsf{T}''^2 = \mathsf{T}^2, \quad \mathsf{T}''^3 = \mathsf{T}^3 \tag{11.2.28}$$

With one unbroken generator, the stability group must be $U(1)$.

(x) The unbroken generator has no effect on the small perturbations, $\mathsf{T}'^1 e_0 = 0$ by definition. Thus we can write

$$\mathbf{\Phi}(x) = \frac{1}{\sqrt{2}}\mathsf{U}''(x)\boldsymbol{e}_0\,(v + \sigma(x)), \quad \mathsf{U}''(x) := \exp\{i\theta^Z(x)\mathsf{T}''\} \tag{11.2.29}$$

Now there are only four real fields, three $\theta^Z(x)$ and the $\sigma(x)$.

(xi) The potential is

$$V(\Phi_i^\dagger, \Phi_i) = m^2\Phi_i^\dagger\Phi_i + \lambda(\Phi_i^\dagger\Phi_i)^2 \tag{11.2.30}$$

with $m^2 < 0$, but in terms of the parameterisation (11.2.29) we see that the result is simple

$$V(\Phi_i^\dagger, \Phi_i) \quad = \quad V((v + \sigma(x)^2/2) \tag{11.2.31}$$

$$= \quad \text{constants} + (m^2 + \lambda v^2)v\sigma + \frac{1}{2}(m^2 + 3\lambda v^2)\sigma^2 + O(\sigma^3, \sigma^4) \tag{11.2.32}$$

Note no Goldstone modes, θ^Z, appear at all. Removing the linear term we see that

$$v = \sqrt{\frac{-m^2}{\lambda}}, \quad m_\sigma^2 = -2m^2 = 2\lambda v^2 \tag{11.2.33}$$

while the mass of the θ^Z fields must be zero.

The potential terms for this parameterisation (11.2.29) of the field contain no Goldstone modes, θ^Z and certainly no Goldstone interactions, they have been removed totally from the potential. However, the kinetic terms are non-trivial in this parameterisation (11.2.29) and they are

$$\frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}v^2(\partial^\mu\theta^Z)(\partial_\mu\theta^Z) + \frac{1}{2}v\sigma(\partial^\mu\theta^Z)(\partial_\mu\theta^Z) + \frac{1}{2}\sigma^2(\partial^\mu\theta^Z)(\partial_\mu\theta^Z) \tag{11.2.34}$$

The third and fourth term are cubic and quartic in the fields so these are the Goldstone-Higgs interactions in this parameterisation. The derivatives in these interactions are not a problem, the cubic interactions between scalars and gauge fields contain derivative of fields too.

There is one warning though. Just as mentioned for local SSB, such polar coordinates can be misleading. Generally fine for this qualitative type of discussion or for small perturbations — low orders of perturbation theory. However, at some point we must deal with the fact that the fields σ and θ^Z do not take any real values and when quantising scalar fields this is usually implicitly assumed. In the path integral one enforces restrictions on fields (e.g. $|\theta^Z| < \pi$) via various tricks similar to those used for gauge fixing. Equivalently, there is a non-trivial Jacobian required in the path integral associated with the transformation to polar field parameterisation, and we do not see this term in the classical analysis.

(xii) We considered the minimum at $\mathsf{U}_0 = \mathbb{1}$. Why can we choose this with out loss of generality because the general U_0 is a *constant* and we can always use a global symmetry transformation to transform the vacuum and fields to any one value such as this.

Thus if we have solutions $\mathbf{\Phi}_1$ for some vacuum with $\mathsf{U}_0 = \mathbb{1}$, then the solutions for some general vacuum $\mathbf{\Phi}_0$ with some general U_0 factor are simply given by a global symmetry transformation $\mathsf{U}_0\mathbf{\Phi}_1$

Q??. Complex fields and Goldstone's theorem

(i) Taylor expansion of $V(\Phi)$ about the the stationary point $\Phi_x = v_i$ gives

$$V(\Phi) = V(v) + \psi_i A_{ij}\psi_j + \psi_i C_{ij}\psi_j^* + \psi_i^* B_{ij}\psi_j^* + O(\Phi^3) \tag{11.2.1}$$

$$A_{ij} = \frac{1}{2}\frac{\partial^2 V}{\partial\Phi_i\Phi_j}, \quad B_{ij} = \frac{1}{2}\frac{\partial^2 V}{\partial\Phi_i^\dagger\Phi_j^\dagger}, \quad C_{ij} = \frac{\partial^2 V}{\partial\Phi_i\Phi_j^\dagger}; \quad \psi_i := \Phi_i - v_i \tag{11.2.2}$$

where because v_i is a stationary point defined through the equations

$$0 = \frac{\partial V}{\partial\Phi_i} = \frac{\partial V}{\partial\Phi_i^\dagger} \tag{11.2.3}$$

Note that it is implicit that all partial derivatives are evaluated at $\Phi_i = v_i$. Also the * notation on the fields Φ is usually written as † but in this *classical* analysis the index notation takes care of any transposition in the i, j indices.

Real V means $V(\Phi) = [V(\Phi)]^*$ and hence

$$(A_{ij}\Phi_i\Phi_j)^* = A_{ij}^*\Phi_i^*\Phi_j^*, \ \text{etc.} \qquad \Rightarrow A = B^*, \ \ C = C^\dagger \tag{11.2.4}$$

Note that, without loss of generality, we may *choose*

$$A = A^T, \ \ B = B^T, \ \Rightarrow \ \ A = B^\dagger \tag{11.2.5}$$

but I will not use this if I can avoid it.

$$\psi(x) = \frac{1}{\sqrt{2}}(\eta(x) + i\zeta(x)), \quad \eta, \zeta \in \mathbb{R} \tag{11.2.6}$$

then substituting into V we find

$$\begin{aligned} V(\Phi) &= V(v) + \eta_i[A+B+C]_{ij}\eta_j + \eta_i[i(A-B-C)]_{ij}\zeta_j \\ &\quad +\zeta_i[i(A-B+C)]_{ij}\eta_j + \zeta_i[C-A-B]_{ij}\zeta_j + O(\Phi^3) \end{aligned} \tag{11.2.7}$$

Now define a *real* $2d$ entry scalar field ϕ_I, $I, J = 1, 2, \ldots, 2d$

$$\phi_i = \frac{1}{\sqrt{2}}(\psi + \psi^*)_i = \eta_i, \quad \phi_{i+d} = \frac{-i}{\sqrt{2}}(\psi - \psi^*)_i = \zeta_i \tag{11.2.8}$$

and we then have that

$$V = \frac{1}{2}\phi_I M^2_{IJ}\phi_J + \text{ constants } + O(\phi^3) \tag{11.2.9}$$

Hence diagonalising the $2d \times 2d$ M^2 matrix gives the classical masses for the scalar field modes where

$$M^2 = \begin{pmatrix} A+B+C & i(A-B-C) \\ i(A-B+C) & C-A-B \end{pmatrix} \tag{11.2.10}$$

This is Hermitian so it does have real eigenvalues as it must if they are to be masses. In (11.2.7) the $\eta\zeta$ cross terms can be mixed up and then rewritten in a more symmetric way

$$\begin{aligned} &\eta_i[i(A-B+C)]_{ij}\zeta_j^* + \zeta_i[i(A-B-C)]_{ij}\eta_j^* \\ &\quad = \eta_i[i(A-B+C) + i(A-B-C)^T]_{ij}\zeta_j^* \end{aligned} \tag{11.2.11}$$

$$= \quad \eta_i[i(A-B+\frac{1}{2}(C-C^T))]_{ij}\zeta_j^* + \zeta_i[i(A-B-\frac{1}{2}(C-C^T))]_{ij}\eta_j^* \tag{11.2.12}$$

Likewise

$$\eta_i C_{ij}\eta_j = \eta_j C_{ji}\eta_i = \eta_i\frac{1}{2}(C+C^T)_{ij}\eta_j \tag{11.2.13}$$

and also for the $\zeta_i\zeta_j$ term so that

$$M^2 = \begin{pmatrix} A+B+\frac{1}{2}(C+C^T) & i(A-B+\frac{1}{2}(C-C^T)) \\ i(A-B-\frac{1}{2}(C-C^T)) & \frac{1}{2}(C+C^T)-A-B \end{pmatrix} \tag{11.2.14}$$

From the properties (11.2.4) known for A, B, C only this second example is a pure real matrix yet encodes the same information as before. It is also hermitian.

(ii)

$$V(\Phi) = \frac{1}{2}m^2\Phi^*\Phi + \frac{1}{4}\lambda(\Phi^*\Phi)^2 \tag{11.2.15}$$

On expanding Φ about v we find that

$$A = \frac{1}{4}\lambda(v^*)^2, \quad B = \frac{1}{4}\lambda(v)^2, \quad C = \frac{1}{2}m^2 + \lambda|v|^2 = \frac{1}{2}\lambda|v|^2, \quad |v|^2 = -\frac{m^2}{\lambda} \tag{11.2.16}$$

Note that the properties for A, B, C derived in (ii), (11.2.4) are satisfied. If $v = |v|\exp\{i\alpha\}$ then the real M^2 matrix (11.2.14) is then given by

$$M^2 = \begin{pmatrix} \lambda|v|^2\cos^2(\alpha) & \frac{1}{2}\lambda|v|^2\sin(2\alpha) \\ \frac{1}{2}\lambda|v|^2\sin(2\alpha) & \lambda|v|^2\sin^2(\alpha) \end{pmatrix} \tag{11.2.17}$$

$$= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} \lambda|v|^2 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{11.2.18}$$

which has eigenvalues/eignvectors of $\lambda|v^2|, (\cos(\alpha), \sin(\alpha))$ and $0, (-\sin(\alpha), \cos(\alpha))$. When $\alpha = 0, v \in \mathbb{R}e$ then the results from the lecture are recovered.

Q??. SO(3) breaking schemes in 3d representations

(i) To solve these problems consider a general element of the Lie Algebra,

$$A = c_1 T^1 + c_2 T^2 + c_3 T^3, \quad \{c_i\} \in \mathbb{R} \tag{11.2.1}$$

and for a given vacuum solution $\phi = (x, y, z)$ solve

$$A. \begin{pmatrix} x \\ y \\ z \end{pmatrix} \propto c_1 \begin{pmatrix} 0 \\ -z \\ +y \end{pmatrix} + c_2 \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix} + c_3 \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = 0 \tag{11.2.2}$$

In general these will be *complex* equations and you must look at real and imaginary parts, but it is essential to remember that the c's **MUST** be real. The solutions here are

Question Number	Unbroken Generators	Broken Generators	H	Massive Scalars	Massless Scalars
a	T^1, T^2, T^2	(none)	SO(3)	3	0
b	T^1	T^2, T^2	SO(2)	1	2
c	$N(T^1 + T^2)$	$N(T^1 - T^2), T^2$	SO(2)	1	2
d	$N(T^1 - T^2)$	$N(T^1 + T^2), T^2$	SO(2)	1	2

where $N = 1/\sqrt{2}$ is only needed if you want to stick to the standard normalisation.

(ii) Suppose vector $\boldsymbol{\phi}_0$ is a valid vacuum pointing (in field space) in direction $\boldsymbol{e}_0$. Since we are talking 3D rotations (rotations of a three-dimensional real vector), we know that any non-zero vector has a direction and that rotations around that direction leave it unchanged. There are then two remaining independent rotations (say about axes orthogonal to the first axis) which must change the vector $\boldsymbol{\phi}_0$. Thus we have two broken and one unbroken direction. The *only* way to alter this is if the vector $\boldsymbol{\phi}_0 = 0$. This is invariant under all rotations so all three generators are unbroken.

You may also always argue that all non-zero vacua, $\bar{\boldsymbol{\phi}}_0$, are related by $U.\boldsymbol{\phi}_0$ to some standard vacuum $\boldsymbol{\phi}_0$, where U is some group element. Do this by transforming to new fields using $\boldsymbol{\phi}'(x) = \mathsf{U}^{-1}\boldsymbol{\phi}(x)$ as this is a global symmetry transformation and hence the Lagrangian is invariant. As a result, the vacuum solution for the $\boldsymbol{\phi}'$ fields is always $\boldsymbol{\phi}_0$ yet we have merely relabelled the fields so the physics will be identical. Since we are dealing with rotations, one can convince yourself that $\boldsymbol{\phi}_0$ may always be chosen to be of type b if non-zero, otherwise we have type a.

To prove this algebraically, expanding out (11.2.2) we find that $c_1 = cx, c_2 = cy, c_3 = cz$, where c is some normalisation constant, is the only solution (unless $x = y = z = 0$). Hence there is only one unbroken generator and therefore there must be two remaining independent remaining generators.

Q??. The O(N) group

(i) The defining representation of the $O(N)$ group is a set of real $N \times N$ matrices, M which satisfy $\mathsf{M}.\mathsf{M}^T = 1$ i.e.

$$M_{ij} M_{kj} = \delta_{ik}. \tag{11.2.1}$$

These matrices have N^2 real entries. For each i, k value when $i \neq k$, the defining equation, (11.2.1), is the same as the one when the values of i and k are the other way round. Thus only half of these equations are independent. This, with the n equations where $i = k$, give a total of $\frac{1}{2}N(N+1)$ equations. This means that only $N^2 - \frac{1}{2}N(N+1) = \frac{1}{2}N(N-1)$ of the entries are independent. There are no other limitations on the entries so this means that $\frac{1}{2}N(N-1)$ real parameters are needed to represent all elements of the O(N) group in this representations, and thus in any representation.

(ii) Looking at the N diagonal equations, $(i = k)$, $\sum_j (M_{ij})^2 = 1 \ \forall i$. Thus $|M_{ij}| \leq 1 \ \forall i, j$ so $O(N)$ is a compact group.

(iii) Consider the $N \times N$ matrix

$$\mathsf{M} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & & \\ \vdots & \vdots & & \mathsf{H} \end{pmatrix} \tag{11.2.2}$$

where H is an $h \times h$ matrix. If $\mathsf{M}.\mathsf{M}^T = 1$, so that M is in the defining representation of $O(N)$, then $\mathsf{H}.\mathsf{H}^T = 1$. It is easy to show that if we consider two such matrices $\mathsf{M}_1, \mathsf{M}_2$ with corresponding sub-matrices $\mathsf{H}_1, \mathsf{H}_2$, then the product $\mathsf{M}_1.\mathsf{M}_2$ contains the sub-matrix $\mathsf{H}_1.\mathsf{H}_2$, hence group closure. The remaining group properties are easy to demonstrate. Hence we have found an explicit representation for $O(N)$ with an explicit sub-group $O(h)$. If it is true in one representation, it is true in all representations.

(iv) Suppose that the original symmetry has g generators and that after spontaneous symmetry breaking there are f generators left in the unbroken symmetry. Let there be d real scalar particles. Goldstone's theorem states that the number of broken generators = the number of Goldstone bosons = $(g - f)$. There can not be more Goldstone bosons than real scalar modes so the key relation is

$$d \geq g - f \tag{11.2.3}$$

For $O(N) \to O(h)$ symmetry breaking we have

$$d \geq \frac{1}{2}N(N-1) - \frac{1}{2}h(h-1) = \frac{1}{2}x(2N - 1 - x) \text{ where } x = N - h. \tag{11.2.4}$$

For fields in the defining representation, $d = N$, we see that the only values allowed by (11.2.4) are $h = N, N-1$, see table (11.1).

For the adjoint representation, there are $d = \frac{1}{2}N(N-1)$ scalar modes and this means all possible $O(h)$ sub-groups are possible stability groups, i.e. $N \geq h \geq 0$. Thus in the adjoint representation it is possible to have no massive scalars if all the continuous symmetry is broken ($h = 0$ for example). See table (11.1).

Q6.1. Trying to combine gauge fields and matter

Representation, d	h	SSB	Massive Scalars	Goldstone Bosons
Defining, N	N	No SSB.	N	0
Defining, N	$N-1$	$O(N) \to O(N-1)$	1	$N-1$
Adjoint, $\frac{1}{2}N(N-1)$	N	No SSB.	$\frac{1}{2}N(N-1)$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Adjoint, $\frac{1}{2}N(N-1)$	h	$O(N) \to O(h)$	$\frac{1}{2}h(h-1)$	$\frac{1}{2}(N-h)(N+h-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Adjoint, $\frac{1}{2}N(N-1)$	0	$O(N) \to 1$	0	$\frac{1}{2}N(N-1)$

Table 11.1: Some possible $O(N)$ symmetry breaking scenarios.

(i)

$$\begin{aligned} D^\mu &:= \partial_\mu - ieA_\mu, \quad F^{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu && (11.2.1)\\ [D_\mu, D_\nu] &= [\partial_\mu, \partial_\nu] + e^2[A_\mu, A_\nu] - ie[\partial_\mu, A_\nu] - ie[A_\mu, \partial_\nu] \\ &= -ie[\partial_\mu, A_\nu] - ie[A_\mu, \partial_\nu] \\ [\partial_\mu, A_\nu] &= (\partial_\mu A_\nu) + A_\nu\partial_\mu - A_\nu\partial_\mu = (\partial_\mu A_\nu) \\ \Rightarrow [D_\mu, D_\nu] &= -ie\{(\partial_\mu A_\nu) - (\partial_\nu A_\mu)\}, \quad \Rightarrow F^{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu] && (11.2.2) \end{aligned}$$

(ii)

$$\Phi \to \Phi' = g\Phi, \quad A_\mu \to A'_\mu = A_\mu - \frac{i}{e}g(\partial_\mu g^{-1}), \quad \text{where } g = e^{i\theta} \in U(1) \tag{11.2.3}$$

(iii)

$$\begin{aligned} D_\mu\Phi &= (\partial_\mu - ieA_\mu)\Phi \to (\partial_\mu - ieA'_\mu)\Phi' \\ D_\mu\Phi &\to (\partial_\mu g)\Phi + g(\partial_\mu\Phi) - ie(A_\mu - \frac{i}{e}(\partial_\mu g^{-1})).g\Phi = g(D_\mu\Phi) = gD_\mu g^{-1}\Phi' && (11.2.4)\\ \Rightarrow D_\mu &\to D'_\mu = gD_\mu g^{-1} && (11.2.5) \end{aligned}$$

(iv) Using (11.2.4) and $g.g^\dagger = 1$ we find

$$\begin{aligned} D_\mu\Phi &\to g.(D_\mu\Phi) \Rightarrow (D_\mu\Phi)^\dagger \to (D_\mu\Phi)^\dagger.g^{-1} \Rightarrow (D_\mu\Phi)^\dagger D_\mu\Phi \text{ is invariant.} && (11.2.6)\\ \Phi^\dagger\Phi &\to \Phi^\dagger g^\dagger.g\Phi = \Phi^\dagger\Phi \Rightarrow V(\Phi^\dagger\Phi) \text{ is invariant} && (11.2.7) \end{aligned}$$

$$\begin{aligned} F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu] &\to \frac{i}{e}[gD_\mu g^{-1}, gD_\nu g^{-1}] = gF_{\mu\nu}g^{-1} && (11.2.8)\\ \Rightarrow F_{\mu\nu}F^{\mu\nu} \to gF_{\mu\nu}F^{\mu\nu}g^{-1} &= gg^{-1}F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}F^{\mu\nu} \text{ using Abelian property} && (11.2.9) \end{aligned}$$

(v)

$$A_\mu A_\mu \to [A_\mu - \frac{i}{e}g(\partial_\mu g^{-1})].[A_\mu - \frac{i}{e}g(\partial_\mu g^{-1})] \neq A_\mu A_\mu \text{ for general } g. \tag{11.2.10}$$

(vi) The equation of motion for Φ has no contribution from the F^2 term so

$$\Box\Phi + m^2\Phi - ie(\partial_\mu A_\mu)\Phi - 2ieA_\mu(\partial_\mu\Phi) = -\frac{\partial}{\partial\Phi^\dagger}[V_{int} + e^2A^2\Phi^\dagger\Phi] \tag{11.2.11}$$

where $V = m^2|\Phi|^2 + V_{int}$. The equation of motion for A_μ is

$$\partial_\mu F_{\mu\nu} = -J_\nu = ie(\Phi^\dagger(\partial_\nu\Phi) - (\partial_\nu\Phi^\dagger)\Phi) + 2e^2 A_\nu|\Phi|^2 \tag{11.2.12}$$

where the LHS only came from the F^2 term. If the U(1) symmetry involved is the usual E/M symmetry then this is Maxwell's equation for classical E/M fields and their interactions with scalar particles (e.g. pions, Cooper pairs). Note how the E/M current, the RHS, contains a term which depends on the E/M vector potential. This is quite unlike the case for fermions where the current is independent of A_μ. Classical electrodynamics with scalar particles will contain some interesting new physics!

(vii) The Noether current J_N^ν is

$$J_N^\nu = -i[\Phi^\dagger(\partial_\nu\Phi) - (\partial_\nu\Phi^\dagger)\Phi] - 2eA_\nu|\Phi|^2 \tag{11.2.13}$$

(viii) From (11.2.12), we see that the physical current, J^ν, in Maxwell's equation for E/M with scalar particles is the Noether current, (11.2.13), (which is essentially counting the number of particles minus the number of anti-particles) is $J^\nu = e \times J_N^\nu$.

(ix) The usual Maxwell's equations are of the form $\partial_\mu F_{\mu\nu} = -J_\nu$. The equation of motion for A^μ from $\mathcal{L}_{Maxwell}$ gives just this. The $\partial_\mu F^{\mu\nu}$ comes from the $-\frac{1}{4}F^2$ term, and then

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)}(-J_L^\rho A_\rho) = 0, \quad \frac{\partial}{\partial A_\nu}(-J_L^\rho A_\rho) = -J_L^\nu \tag{11.2.14}$$

provided that

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} J_L^\rho = 0, \quad \frac{\partial}{\partial A_\nu} J_L^\rho = 0 \tag{11.2.15}$$

The equations (11.2.15) are the conditions needed if J_L^μ is to be identical with the physical current in Maxwell's equations, J^μ.

(x) The scalar QED can be rewritten in this form with

$$J_L^\mu = -ie[\Phi^\dagger(D_\mu\Phi) - (D_\mu\Phi^\dagger)\Phi] = -ie[\Phi^\dagger(\partial_\mu\Phi) - (\partial_\mu\Phi^\dagger)\Phi] - e^2 A^\mu|\Phi|^2 \neq eJ_N^\mu \tag{11.2.16}$$

It does not satisfy (11.2.15). One has to be more careful when deriving the Maxwell currents when there are scalar particles present.

Physically the current in Maxwell's equations must be conserved and the Noether current is the only such quantity we have to hand.

Q7.1. Unitary Gauge: Abelian case

(i) $F^2 = \frac{1}{e^2}(\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial^\mu B^\nu - \partial^n u B^\mu)$ so dropping the F^2 term is equivalent to working with $e \to \infty$, $A^\mu \to 0$, $eA^\mu = B^\mu$ finite.

(ii) The key point is that with no F^2 term there are no $\partial_\mu A_\nu$ terms so the equations of motion become very simple, allowing one to eliminate A^μ - such a field is known as an *auxiliary* field.

(iii) Just substitute.

(iv) A gauge transformation is $eA^\mu \to aA'^\mu = eA^\mu + \partial^\mu\theta$ so $eA^\mu = \partial^\mu\theta$ is gauge equivalent to $A'^\mu = 0$. Direct substitution shows $F_{\mu\nu} = \partial_\mu\partial_\nu\theta - \partial_\nu\partial_\mu\theta$ which is zero by symmetry of $\partial_\mu\partial_\nu$. Alternatively, you know $F_{\mu\nu}$ is gauge invariant, and that the solution we have here is gauge equivalent to a zero gauge field, hence can not contain any physics.

(v) Just substitute. Note that the potential must be independent of the phase if the Lagrangian was to be U(1) symmetric.

(vi) Difficult. It appears that in $\mathcal{L}$ we have two particles in $\Phi, \Phi^\dagger$, and a field eA^μ that is *not* independent but completely fixed by the other two and so can not represent any degree of freedom and no physical particle. However, $\mathcal{L}_{\text{new}}$ depends on only one real field η which suggests the theory represents just one scalar particle, contradicting our first guess. The problem lies in the fact that in moving from $\mathcal{L}$ to $\mathcal{L}_{\text{new}}$ we switch to a *polar* representation of the fields rather than a cartesian $\phi_1 + i\phi_2$ representation (I'm referring to coordinates in the field space, not space-time). The Jacobian of the polar transformation is not trivial and one can not ought not simply read off the particle content from $\mathcal{L}_{\text{new}}$ in the usual way, and in any case η is confined to be positive, unlike the scalar models you have quantised where the fields range over all values. When we try to quantise the theories, I expect that one finds all sorts of complications so the precise content of these models is not clear.

The polar representation breaks down when $\Phi = 0$ so I expect that $\mathcal{L}_{\text{new}}$ is only equivalent to our original at points where $\Phi \neq 0$. When $\Phi = 0$ I think the form given for $\mathcal{L}_{\text{new}}$ is incomplete and that you need to add some sort of delta function in eA^μ. This might be thought of as some infinitely thin string (c.f. cosmic strings or vortices in a superconductor), and these represent the missing degree of freedom in the $\mathcal{L}_{\text{new}}$ case.

The Unitary gauge is traditionally used in situations where local symmetry is broken so that the expectation value of the scalar field is not zero. However, as our analysis shows, we have not referred to the potential, so one could use this argument at any time, with or without symmetry breaking. In any case quantum fluctuations will ensure the scalar field is zero briefly at many points in space-time.

The bottom line is that the Unitary gauge is a very dangerous concept!

Q6.2. Gauge fields and matter: Non-Abelian case

(i) Proof that $V(\mathbf{\Phi}^\dagger\mathbf{\Phi})$ is invariant is as the abelian case in Q.3.(iii). For the kinetic term of $\mathcal{L}_{\Phi,\Phi W}$ to be invariant we require that

$$\begin{aligned}(\mathsf{D}_\mu\mathbf{\Phi})^\dagger(\mathsf{D}^\mu\mathbf{\Phi}) &= (\mathsf{D}'_\mu\mathbf{\Phi}')^\dagger(\mathsf{D}'^\mu\mathbf{\Phi}') = (\mathsf{U}^{-1}\mathsf{D}'_\mu\mathsf{U}\mathbf{\Phi})^\dagger\mathsf{U}^\dagger.\mathsf{U}(\mathsf{U}^{-1}\mathsf{D}'^\mu\mathsf{U}\mathbf{\Phi}') \\ \Rightarrow \mathsf{D} \to \mathsf{D}' &= \mathsf{U}.\mathsf{D}.\mathsf{U}^{-1} \end{aligned} \tag{11.2.1}$$

(ii) We have that coordinates and ∂_μ are invariant and with (i) we have that

$$\mathsf{D}' = \mathsf{U}.\partial_\mu.\mathsf{U}^{-1} - ie\mathsf{U}.\mathsf{W}_\mu.\mathsf{U}^{-1} = \mathsf{U}.\mathsf{U}^{-1}.\partial_\mu + \mathsf{U}.(\partial_\mu.\mathsf{U}^{-1}) - ie\mathsf{U}.\mathsf{W}_\mu.\mathsf{U}^{-1} \tag{11.2.2}$$

$$\Rightarrow \mathsf{W}_\mu(x) \to \frac{i}{e}\mathsf{U}.(\partial_\mu.\mathsf{U}^{-1}) + \mathsf{U}.\mathsf{W}_\mu.\mathsf{U}^{-1} \tag{11.2.3}$$

where the $\mathsf{U}.\mathsf{W}_\mu.\mathsf{U}^{-1}$ is clearly zero only for abelian groups.

(iii) In principle one can multiply each side of the W transformation by a generator, T^a_{ij}, and take the trace over the matrix indices $(i, j\ )$. A useful trick is that the basis set of generators for the Lie algebra (which is a vector space) is chosen to be orthogonal, i.e. $\mathrm{Tr}\{T_aT_b\} \propto \delta_{ab}$. However I did not find anything very simple when I did this.

What I meant to ask was for the infinitesimal transformation law. Looking at this directly I find

$$\begin{aligned}\delta\mathsf{W}_\mu &= \delta W^a_\mu.\mathsf{T}^a = i\epsilon^b\mathsf{T}^b.W^c_\mu\mathsf{T}^c - W^c_\mu\mathsf{T}^c.i\epsilon^b\mathsf{T}^b + \frac{i}{e}.(-i\partial_\mu\epsilon^b)\mathsf{T}^b \\ \delta W^a_\mu.\mathsf{T}^a &= i\epsilon^bW^c_\mu[\mathsf{T}^c\mathsf{T}^b. - \mathsf{T}^c\mathsf{T}^b] + \frac{1}{e}.(\partial_\mu\epsilon^a).\mathsf{T}^a, \\ \Rightarrow \delta W^a_\mu &= -\epsilon^bW^c_\mu f^{cba} + \frac{1}{e}.(\partial_\mu\epsilon^a)\end{aligned} \tag{11.2.4}$$

(iv) For infinitesimal changes

$$\delta(W^a_\mu W^{\mu,a}) = 2W^a_\mu \delta W^{\mu,a} = \epsilon^c W^{\mu,a} W^b_\mu f^{cba} + \frac{1}{e}.W^{\mu,a}(\partial_\mu \epsilon^a) \neq 0 \tag{11.2.5}$$

True even for the abelian case when the second term remains non-zero.

(v)

$$\begin{aligned} \mathsf{F}^{\mu\nu} &\rightarrow = \frac{i}{e}[\mathsf{U}^{-1}.\mathsf{D}^\mu.\mathsf{U}, \mathsf{U}^{-1}.\mathsf{D}^\nu.\mathsf{U}] = \mathsf{U}^{-1}.\mathsf{F}^{\mu\nu}.\mathsf{U} & (11.2.6) \\ \mathrm{Tr}\{\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\} &\rightarrow \mathrm{Tr}\{\mathsf{U}^{-1}.\mathsf{F}^{\mu\nu}.\mathsf{F}_{\mu\nu}.\mathsf{U}\} = \mathrm{Tr}\{\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\} & (11.2.7) \end{aligned}$$

(vi) Just as in question 3.(i) we find

$$\begin{aligned} \mathsf{F}^{\mu\nu} &= \frac{i}{e}[\partial_\mu - ie\mathsf{W}_\mu, \partial_\nu - ie\mathsf{W}_\nu] = \frac{i}{e}[\partial_\mu, \partial_\nu] + [\partial_\mu, \mathsf{W}_\nu] + [\mathsf{W}_\mu, \partial_\nu] - ie[\mathsf{W}_\mu, \mathsf{W}_\nu] \\ &= 0 + (\partial_\mu \mathsf{W}_\nu) - (\partial_\nu \mathsf{W}_\mu) - ie[\mathsf{W}_\mu, \mathsf{W}_\nu] & (11.2.8) \end{aligned}$$

and this is not a differential operator.

(vii)

$$\mathsf{F}^{\mu\nu} = F^{\mu\nu,a}.\mathsf{T}^a \rightarrow \mathsf{U}^{-1}.F^{\mu\nu,a}.\mathsf{T}^a.\mathsf{U} \tag{11.2.9}$$

and the infinitesimal version is

$$\begin{aligned} \delta F^{\mu\nu,a}.\mathsf{T}^a &\rightarrow -F^{\mu\nu,a} i\epsilon^b [\mathsf{T}^b, \mathsf{T}^a] = -iF^{\mu\nu,a}\epsilon^b (if^{bac})\mathsf{T}^c \\ \delta F^{\mu\nu,b} &= \epsilon_a T^a_{bc} F^{\mu\nu,c} & (11.2.10) \end{aligned}$$

where the generators in the last line are $T^a_{bc} = if^{abc}$, which means they are in the adjoint representation, c.f. $\Phi_b \rightarrow \epsilon^a T^a_{bc}\Phi_c$.

(viii)

$$\begin{aligned} F^{\mu\nu,a} &= (\partial_\mu W^{\nu,a}) - (\partial_\nu W^{\mu,a}) - ef^{abc} W^{\mu,b} W^{\nu,c} & (11.2.11) \\ \Rightarrow F^{\mu\nu,a} F^a_{\mu\nu} &= (\partial_\mu W^{\nu,a})^2 - (\partial_\nu W^{\mu,a})(\partial_\mu W^a_\nu) - (\partial_\nu W^a_\mu)^2 \\ &\quad -2ef^{abc}((\partial_\mu W^{\nu,a}) - (\partial_\nu W^{\mu,a}))W^b_\mu W^c_\nu \\ &\quad +e^2 f^{abc} f^{ade} W^{\mu,b} W^{\nu,c} W^d_\mu W^e_\nu & (11.2.12) \end{aligned}$$

The quadratic terms are as with QED and when quantised give the usual type of gauge boson propagator. The terms cubic and quartic in W are treated as interaction terms when quantising. These are represented by vertices in Feynman diagrams with three or four gauge boson legs only being connected to them. They are as messy as the result above suggests.

For abelian theories only, $f^{abc} = 0$, and only then are the cubic and quartic terms zero. This is as it should be as we know photons do not interact directly with each other, they carry no E/M charge. Conversely these extra interaction for non-abelian gauge bosons means that they *do* carry charges so that they interact with themselves.

(ix) I found that the equation for $\mathbf{\Phi}$ was just as in the abelian case except the order of terms must be preserved more carefully.

$$\Box\mathbf{\Phi} + m^2\mathbf{\Phi} - 2ie\mathsf{W}^\mu.(\partial_\mu\mathbf{\Phi}) - ie(\partial_\mu\mathsf{W}^\mu).\mathbf{\Phi} = e^2\mathsf{W}^\mu.\mathsf{W}_\mu.\mathbf{\Phi} + \frac{\partial V_{int}}{\partial\Phi^\dagger} \tag{11.2.13}$$

The equation of motion for W, the non-abelian version of Maxwell's equations, is more complicated than the abelian case

$$\partial_\mu F^{\mu\nu,a} = ef^{abc}W^{\nu,b}F^{\mu\nu,c} + \left[ie(\mathbf{\Phi}(\mathsf{D}^\nu\mathbf{\Phi}) - (\mathsf{D}^\nu\mathbf{\Phi})^\dagger\mathbf{\Phi})\right] \tag{11.2.14}$$

The kinetic terms for W, the F^2 terms, give the pure W terms. Note that this includes the first term on the LHS which is zero for the abelian case and it is the only difference between the non-abelian and abelian cases.

(x) Writing the above as $\partial_\mu F^{\mu\nu} = -J^\nu$, then J is a conserved current as is is meant to be a physical current. The fact that $J = J_W + J_{\Phi,\Phi W}$ and J_W is non-zero for non-abelian cases but zero for abelian theories, indicates that the W field itself is charged as J_W is a pure W term. This is a major departure from the abelian case where the gauge bosons (photons etc.) are always uncharged

(xi) The complication in calculating the Noether current for the non-abelian case is that non-abelian fields *change* under global transformations. From (ii) we have that

$$\mathsf{W}_\mu(x) \to -ie\mathsf{U}.\mathsf{W}_\mu(x).\mathsf{U}^{-1} \neq 0 \tag{11.2.15}$$

for global transformations $\partial_\mu \mathsf{U} = 0$. Thus in calculating the Noether current one must include global variation of W fields at the same time as adding in terms coming from $\mathbf{\Phi}$ and $\mathbf{\Phi}^\dagger$ variations. This is not surprising as again this is because the non-abelian gauge fields carry charge. You should find that the Noether current is the same as the LHS of (11.2.14) less a factor of e.

11.1.

(i) Precise rules:

(a) Two or more particles of the same non-zero mass are related by an unbroken symmetry.

(b) Each gauge boson is associated with one generator of a local symmetry.

(c) No particle is massless unless required by a symmetry.

(d) For every distinct massless scalar (spin 0) particle there is a broken generator of a global symmetry (Goldstone's theorem).

(e) A massless fermion has a chiral symmetry.

(f) For every massless Gauge Boson (spin one) there is an unbroken generator of a local symmetry.

(g) For every massive Gauge Boson (spin one) there is an broken generator of a local symmetry.

(h) If particles of different spin have equal non-zero mass then there is an unbroken supersymmetry present.

Some not very precise non-rules:

(a) Two particles of different masses may be related by a broken symmetry.

(b) SSB can change the masses of particles of any spin.

(ii) For each distinct observable charge there is one generator in the Cartan subalgebra of the unbroken symmetry group (the little or stability group), and the symmetry can be local or global.

(iii)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{2}m^2(\phi_1^2+\phi_2^2) - \lambda_1\phi^4 - \lambda_{12}\phi_1^2\phi_2^2 - \lambda_2\phi_1^4, \quad \phi_i(x) \in \mathbb{R} \tag{11.2.1}$$

Note that

$$\lambda_1 = \frac{1}{2}\lambda_{12} = \lambda_2 \Leftrightarrow O(2) \text{ symmetry} \tag{11.2.2}$$

and in this case rule 1 is illustrated. Rule 1 is also illustrated by the case where (11.2.2) is not true as there is then no O(2) symmetry. While in this case it still appears that the two scalar particles have the same mass m despite the lack of an O(2) symmetry and in contradiction to rule 1, the lack of O(2) symmetry in the interactions will ensure that the quantum corrections to the masses will generate a mass difference, the difference being of order the difference in the square of these λ's.

Rule 3 and 4 are illustrated by the case of (11.2.3) for $m^2 < 0$ when (11.2.2) is true. In that case the one generator of $O(2)$ is broken and we have one massless scalar mode and one massive one of mass $(-2m^2) > 0$.

Rule 2 is illustrated by any local theory as we have to replace partial derivatives by covariant derivatives and $D_\mu = \partial_\mu - igW_\mu^a(x)\mathsf{T}^a$, clearly one new gauge field for each generator T^a e.g. a locally symmetric $O(d)$ symmetric Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^aF^{\mu\nu a} + \frac{1}{2}(D_\mu\boldsymbol{\phi})^\dagger(D^\mu\boldsymbol{\phi}) - \frac{1}{2}m^2(\boldsymbol{\phi}.\boldsymbol{\phi}) - \lambda_1(\boldsymbol{\phi}.\boldsymbol{\phi})^2, \quad \phi_i(x)\in\mathbb{R}, \quad i=1,2,\ldots,d \tag{11.2.3}$$

Remember that the symmetry is that of rotations of d-coordinates and then suppose that $m^2 < 0$. Then the vacuum state is $\boldsymbol{\phi}_0 \neq 0$, which picks out one of d-directions as special. Rotations around this direction will leave the vacuum unchanged, but the $d-1$ remaining independent rotations will change the vacuum. The infinitesimal forms of these rotations tell us that one generator is unbroken and the rest are broken. Thus $G = SO(d) \to H = O(2)$. Rules 6 and 7 are illustrated by the observation that the gauge bosons, $W_\mu^a(x)$ acquire a mass matrix M^2 of the form

$$(\mathsf{M}^2)^{ab} = g^2\boldsymbol{\phi}_0^T\mathsf{T}^a\mathsf{T}^b\boldsymbol{\phi}_0 \tag{11.2.4}$$

so that the unbroken generator gives a row and column full of zeros, while the remaining broken generators give a non-zero entries in the rest of the matrix. Thus there is clearly one zero eigenvector.

11.3.

(i) New long range force $\Rightarrow$ one massless gauge boson $\Rightarrow$ one unbroken generator of a local symmetry.

One observed charge $\Rightarrow$ one generator in CSA of H, the little group.

Two scalar particles of different non-zero masses and zero charge $\Rightarrow$ must use two different real fields for these, not one complex field.

First attempt would be to make a local O(2) or U(1) theory which has one generator. The two real scalar fields would be in the trivial representation and not two parts of a complex field, which makes sense as then they would have zero charge.

Thus the simplest answer for (a) is

$$\begin{aligned}\mathcal{L} \quad &= \quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathsf{D}_{\mu(\phi)}\phi)^\dagger(\mathsf{D}^\mu_{(\phi)}\phi) + (\mathsf{D}_{\mu(\xi)}\xi)^\dagger(\mathsf{D}^\mu_{(\xi)}\xi) \\ &\quad -\frac{1}{2}m^2_{(\phi)}\phi^2 - \frac{1}{2}m^2_{(\xi)}\xi^2 - \lambda_1\phi^4 - \lambda_{12}\phi^2\xi^2 - \lambda_2\xi^4,\end{aligned} \tag{11.2.1}$$

where

$$\mathsf{D}^{\mu}_{(\phi)}(x) = \mathsf{D}^{\mu}_{(\xi)}(x) \quad := \quad \partial_\mu \mathbb{1} - igW^{\mu a}(x)\mathsf{T}^a_{(\phi)} \equiv \partial_\mu \quad \mathsf{T}^a_{(\phi)} = \mathsf{T}^a_{(\xi)} = 1 \tag{11.2.2}$$

where the $\phi(x), \xi(x) \in \mathbb{R}$. Note that the λ_{12} term mixes the two scalar particles but there is no direct scalar-gauge boson interaction as the scalars have no charge, they are in the trivial representation of $SO(2)$. No extra particles are then predicted.

(ii) A short range force suggest massive gauge bosons, and therefore we must now embed the $SO(2)$ theory as part of the unbroken sector of a larger symmetry. The simplest one from the point of view of the group theory would be to look at $G = SO(3)$ with its three generators broken to a little group $H = SO(2) \subset G$ used in (i). This means we need three gauge bosons.

Further, we have to break the symmetry so we need a Higgs field, a scalar field with a non-zero vev. The smallest one in an $SO(3)$ theory would be a real scalar triplet, i.e. a scalar field lying in the fundamental or defining representation of $SO(3)$. Remembering discussions of $SO(3)$ we see that the usual $m^2\boldsymbol{\phi}^2 + \lambda(\boldsymbol{\phi}^2)^2$ with $m^2 < 0$ easily generates a vev in one direction, all directions equivalent (the rotation symmetry). For instance if we choose $\boldsymbol{\phi}_0 \propto (1,0,0)$ the T^1 generator of rotations about axis one ((A.5) of the "Some SU(2) ≅ SO(3) Representations" handout) would be unbroken, the other two, generating rotations about axes two and three, would be broken. We'd therefore get two broken generators, two massive gauge bosons. If they were heavy enough, the experimentalists may well not have found them directly (they work only up to some maximum energy). To get two massive scalars they must "eat up" two would-be-Goldstone bosons and we lose two scalar modes of the three in the scalar Higgs triplet. One massive scalar mode would be left over from this Higgs field. This could then be identified with one of the two original scalar fields, say ϕ in (11.2.1).

It also implies that there is one unbroken generator so the little group is $SO(2)$, one massless gauge boson, exactly as we need for the evidence given .

This then leaves the second massive scalar, ξ. There is no need for this to be involved in the SSB, so its just a scalar field, not a Higgs field. We can leave this as before in (a), though now it is in the trivial representation of $SO(3)$.

Putting this together we see that

$$\begin{aligned}\mathcal{L} \quad = \quad & -\frac{1}{2}\mathrm{Tr}\{\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\} + \frac{1}{2}(\mathsf{D}_{\mu(\phi)}\boldsymbol{\phi})^\dagger.(\mathsf{D}^\mu_{(\phi)}\boldsymbol{\phi}) + \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) \\ & -\frac{1}{2}m^2_{(\phi)}\boldsymbol{\phi}.\boldsymbol{\phi} - \frac{1}{2}m^2_{(\xi)}\xi^2 - \lambda_1(\boldsymbol{\phi}.\boldsymbol{\phi})^2 - \lambda_{12}(\boldsymbol{\phi}.\boldsymbol{\phi})\xi^2 - \lambda_2\xi^4,\end{aligned} \tag{11.2.3}$$

where

$$\mathsf{D}_{\mu(\phi)}(x) \quad := \quad \partial_\mu\mathbb{1} - ig\mathsf{W}_\mu(x)\mathsf{T}^{\mu a}_{(\phi)}, \tag{11.2.4}$$

where the $[\boldsymbol{\phi}(x)]_i \in \mathbb{R}$ $(i = 1,2,3)$ lie in the three dimensional representation of $SO(3)$. Rather than using a basis where T^3 is diagonal, we choose the basis where $T^a_{bc} = -\frac{i}{2}\epsilon_{abc}$ $(a,b,c = 1,2,3$, ϵ^{abc} is the completely anti-symmetric tensor with $\epsilon^{123} = +1)$. This basis is suitable for real scalar triplets.

Then *one* of many equally good vacuum solutions with $m^2 = m^2_{(\phi)} < 0$ is

$$\boldsymbol{\phi}_0 \quad = \quad v\boldsymbol{e}_1, \quad \boldsymbol{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v = \sqrt{\left(\frac{-m^2}{4\lambda_1}\right)} \tag{11.2.5}$$

This gives

$$\text{Unbroken } \mathsf{T}'^{1} = \mathsf{T}^{1}; \quad \text{Broken } \mathsf{T}''^{2} = \mathsf{T}^{2}, \mathsf{T}''^{3} = \mathsf{T}^{3}; \tag{11.2.6}$$

using the generators of (A.5) on the "Some SU(2) $\cong$ SO(3) Representations" handout. Thus amongst the gauge bosons $W_\mu^{a=1}$ is massless while $W_\mu^{a=2}$ and $W_\mu^{a=3}$ have mass $gv/2$. The mass of the one remaining scalar ϕ particle, the Higgs particle, ϕ_1, is $\sqrt{-2m^2_{(\phi)}}$.

Note that the λ_{12} term guarantees mixing of the Higgs and the ξ particles. It also means that the mass of the ξ is $(m^2_{(\xi)} + \lambda_{12}v^2)^{1/2}$, so its mass is also changed by the SSB.

Q??. EW currents

(i) (a) The lepton sector contributes in the form $\bar{\boldsymbol{\psi}}\gamma^\mu \mathsf{T}^a \boldsymbol{\psi}$. The right-handed neutrino has no weak isospin, its in the trivial representation where $\mathsf{T}^a = 0$, so it does not contribute to the isospin current. The left-handed leptons are in the two-dimensional representation $\mathsf{T}^a = \frac{1}{2}\boldsymbol{\tau}$ so the contribute

$$J^{\mu a}_{\text{lepton}} = \frac{1}{2}\bar{\boldsymbol{l}}\gamma^\mu \boldsymbol{\tau}^a \boldsymbol{l} \tag{11.2.1}$$

The interesting case is the diagonal generator $a = 3$ where

$$J^{\mu 3}_{\text{lepton}} = \frac{1}{2}\bar{\nu}_L\gamma^\mu \nu_L - \frac{1}{2}\bar{e}_L\gamma^\mu e_L \tag{11.2.2}$$

The weak hyper charge contributions is of the usual U(1) form $q_Y\bar{\psi}\gamma^\mu\psi$ where q_Y is the charge, so

$$J^{\mu Y}_{\text{lepton}} = -\bar{\nu}_L\gamma^\mu\nu_L + \bar{e}_L\gamma^\mu e_L - 2\bar{e}_R\gamma^\mu e_R \tag{11.2.3}$$

We can read the charges from the covariant derivatives on the EW handout.

(b) Scalar contributions to Noether currents are of the form

$$i\left(\mathsf{D}^\mu\boldsymbol{\Phi}\right)^\dagger \mathsf{T}^a\boldsymbol{\Phi} - i\boldsymbol{\Phi}^\dagger\mathsf{T}^a\left(\mathsf{D}^\mu\boldsymbol{\Phi}\right) \tag{11.2.4}$$

Some gauge boson fields are present in the covariant derivatives. The Higgs doublet has $\mathsf{T}^a = \frac{1}{2}\boldsymbol{\tau}$ so

$$\begin{aligned} J^{\mu a}_{\Phi} &= i\left(\mathsf{D}^\mu\boldsymbol{\Phi}\right)^\dagger \mathsf{T}^a\boldsymbol{\Phi} + (\text{h.c.}) && (11.2.5)\\ &= i\boldsymbol{\Phi}^\dagger\left(\overleftarrow{\partial^\mu} - \overrightarrow{\partial^\mu}\right)\mathsf{T}^a\boldsymbol{\Phi} + gW^{\mu b}\boldsymbol{\Phi}^\dagger\{\mathsf{T}^b,\mathsf{T}^a\}\boldsymbol{\Phi} && (11.2.6)\\ &= i\boldsymbol{\Phi}^\dagger\left(\overleftarrow{\partial^\mu} - \overrightarrow{\partial^\mu}\right)\mathsf{T}^a\boldsymbol{\Phi} + \frac{g}{2}W^{\mu a}|\boldsymbol{\Phi}|^2 && (11.2.7) \end{aligned}$$

The weak hypercharge is of the same form but where the generator is now proportional to the unit matrix with a factor $q_Y = +1$

$$\begin{aligned} J^{\mu Y}_{\Phi} &= +i\left(\mathsf{D}^\mu\boldsymbol{\Phi}\right)^\dagger\boldsymbol{\Phi} + (\text{h.c.}) && (11.2.8)\\ &= i\boldsymbol{\Phi}^\dagger\left(\overleftarrow{\partial^\mu} - \overrightarrow{\partial^\mu}\right)\boldsymbol{\Phi} + g'B|\boldsymbol{\Phi}|^2 && (11.2.9) \end{aligned}$$

(c) (optional, very messy calculation) the pure gauge boson sector

(ii) Why does the Lagrangian contain gauge-boson/lepton interactions as a Noether Current multiplied by a gauge boson field term $W^{\mu a}J^a_{\mu\text{lepton}}$

(iii) Find the electromagnetic current contribution from the leptons and make sure that it contains the generator, $\mathsf{Q} = \frac{1}{2}\boldsymbol{\tau}^3 + \frac{1}{2}\mathbb{1}$. Check that the lepton currents calculated above satisfy combine in the same way to give the EM current.

(iv) Hence deduce that the W^1, W^2 are representing gauge bosons carrying EM charges, $W^{\pm} = (W^1 \pm iW^2)/(\sqrt{2})$

Appendix A

Summary of QFT

A.1 Equations of motion from Lagrangians

The equations of motion are given by

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f_i)} - \frac{\partial \mathcal{L}}{\partial f_i} = 0 \tag{A.1.1}$$

where the fields $f_i(x)$ represents any of the fields present in the theory.

The contribution to the electroweak Lagrangian coming from the lepton (first generation only) and scalar sectors can be written as $\mathcal{L}_{W,B} + \mathcal{L}_l + \mathcal{L}_\phi$ where

$$\begin{aligned}
\mathcal{L}_l &= \bar{\boldsymbol{L}}_L i \mathsf{D}_L^\mu \gamma_\mu \boldsymbol{L}_L + \bar{e}_R i \mathsf{D}_R^\mu \gamma_\mu e_R - g_m \bar{\boldsymbol{L}}_L e_R \boldsymbol{\Phi} - g_m \boldsymbol{\Phi}^\dagger \bar{e}_R \boldsymbol{L}_L, \\
\mathcal{L}_\phi &= (\mathsf{D}_\mu \boldsymbol{\Phi})^\dagger (\mathsf{D}^\mu \boldsymbol{\Phi}) - V(\boldsymbol{\Phi}^\dagger . \boldsymbol{\Phi}) && \text{(A.1.2)} \\
\mathcal{L}_{W,B} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}, && \text{(A.1.3)}
\end{aligned}$$

where

$$\begin{aligned}
F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, &\qquad G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu && \text{(A.1.4)} \\
\mathsf{D}_L^\mu = \partial^\mu + ig \frac{\boldsymbol{\tau}^a}{2} W^{a,\mu}(x) - \frac{i}{2} g' B^\mu(x), &\qquad \mathsf{D}_R^\mu = \partial^\mu - i g' B^\mu(x) \\
\mathsf{D}^\mu = \partial^\mu + ig \frac{\boldsymbol{\tau}^a}{2} W^{a,\mu}(x) + \frac{i}{2} g' B^\mu(x) & && \text{(A.1.5)}
\end{aligned}$$

and $\boldsymbol{L}_L$ is a left-handed fermion SU(2) doublet, e_R is a right handed SU(2) singlet, Φ is a scalar SU(2) doublet. The τ^a matrices are the Pauli matrices, which are given in (**??**).

Appendix B

Group Theory

There are numerous texts with much more detail than given here. Jones [24] especially Ch. 1-4,6,8 starts from the beginning and covers all the relevant applications without going into too much detail, e.g. good on Poincaré group ch. 9. Covers Lie groups in just sufficient depth for this book. Tung [27] also covers basics, and does simple particle physics cases, interesting advanced topics like Poincaré group of space-time symmetries too.

Many of the books on quantum field theory will contain brief summaries of the required group theory. I like the one in Cheng and Li [2], chapter 4. It is a very compact outline of key ideas about group theory required for particle physics and a brief but good introduction of main particle symmetry groups.

For more advanced treatments try the following. Hammermesh [25] is a standard classic text, mathematical and perhaps a bit old fashioned but comprehensive and accessible by physicists. Try chapters 1-3 and 8. Georgi [26], chapters Ch. 1-3,7, has advanced particle symmetry topics covered for physicists. I have also used Cornwall [28] (lots of physical examples) and Joshi (try chapters 1,2,4) in my time.

B.1 Summary of Group Theory

A **group** is a set G of elements $g \in G$ which can be combined under **group multiplication law**, $*$, such that it obeys four axioms:

(i) **Closure**: $\forall a, b \in G$, $a * b \in G$

(ii) **Associativity**: $\forall a, b, c \in G$, $a * (b * c) = (a * b) * c$

(iii) **Identity**: $\exists e \in G$ s.t. $g * e = e * g = g \;\forall\; g \in G$

(iv) **Inverse**: $\forall g \in G \;\exists g^{-1} \in G$ s.t. $g * g^{-1} = g^{-1} * g = e$

Only certain special groups, called **Abelian groups**, have the additional property:

$$\text{(Abelian Groups)} \quad a * b = b * a \;\forall\; a, b \in G. \tag{B.1.1}$$

There are many distinct groups, even sets of the same number of elements can often be given different group multiplications rules which lead to distinct groups. By distinct we mean that there is no one-to-one and onto map which preserves the group structure. Such maps are called **isomorphisms**.

In practice groups are not found as the elements of some abstract set with some abstract multiplication law, but are found in terms of other objects with known properties. This is called a **representation** of the group to distinguish it from the fundamental and pure abstract group.[1] In practice we will use only **matrix**

[1]Rather than define a group by labouriously specifying the result of every possible combination of these abstract group elements $a * b$ (usually in a **group multiplication table**, almost all groups are defined using **representations**.

representations where each abstract group element g is represented by a matrix, denoted as $\mathsf{D}(g)$. That is,

$$\text{(abstract group)}\;\; c = a * b \quad \Rightarrow \quad \mathsf{D}(c) = \mathsf{D}(a).\mathsf{D}(b) \;\; \text{(matrix representation)} \tag{B.1.2}$$

Normal matrix multiplication is playing the role here of group multiplication. If we have a representation made from $d \times d$ matrices, then we say that the **dimension of the representation** is d. Matrix multiplication always satisfies associativity, the identity is always represented by the unit matrix $\mathsf{D}(e) = \mathbb{1}$ and the inverse group element must be the inverse matrix $\mathsf{D}(g^{-1}) = [\mathsf{D}(g)]^{-1}$.[2] If every abstract group element is represented by a unique matrix, then we say we have a **faithful representation**:

$$\text{(Faithful Representation)}\;\; \mathsf{D}(a) \neq \mathsf{D}(b) \text{ if } a \neq b \,\forall a, b \in G, \tag{B.1.3}$$

Only faithful representations can be used to define groups.

Many representations of groups are not faithful. All groups have a representation where all elements are represented by the number 1 and this is called the **trivial representation**:

$$\text{(Trivial Representation)}\;\; \mathsf{D}(g) = 1 \;\forall\; g \in G. \tag{B.1.4}$$

Many representations are essentially the same as others. Similarity transformations. Unitary representations. Reducible representations.

One important type of group are **product groups**. These are constructed from two groups, G and H, and are denoted $G \times H$. Each element of the abstract group can be thought as as a pair of elements (g, h) where $g \in G, h \in H$ and the multiplication law is

$$(g_1, h_1) * (g_2, h_2) = (g_1 * g_2, h_1 * h_2) \in G \times H, \quad g_1, g_2 \in G,\; h_1, h_2 \in H \tag{B.1.5}$$

Note the key property of product groups is that all elements can be split into a product of two terms, each involving one of the identities $e_G \in G$ and $e_H \in H$, and these two terms commute

$$(g, h) = (g, e_H) * (e_G, h) = (e_G, h) * (g, e_H) \tag{B.1.6}$$

In terms of a matrix representation, a product group is formed by taking a **direct product** of the matrices making up representations of the two groups. Thus ...

The number of elements in a group is called the **order** of a group. Groups with a finite order called **finite groups**, and groups of infinite order are the **infinite groups**.

A **Lie Group** is a special type of infinite group. Elements of these groups[3], D, can be parameterised by a finite number of *continuous* and *real* parameters ϵ, so that $\mathsf{D} \equiv \mathsf{D}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$. We invariably choose the identity group elements to be $\mathbb{1} = \mathsf{D}(0, 0, \ldots, 0)$. The number of such parameters needed, n, is the **dimension of the Lie group**, not to be confused with the size of the matrices used, d — the dimension of the representation. For most representations, $d \neq n$. The **fundamental representation** is the representation of smallest dimension.

If the elements of a faithful representation of a Lie group are always finite $|D_{ij}| < \infty$ then we have a **compact Lie group**. These obey a useful theorem:

> All compact Lie groups have finite dimensional unitary representations,
> i.e. $\exists\{\mathsf{D}(g)\}$ s.t. $\mathsf{D}^\dagger(g)\mathsf{D}(g) = \mathbb{1}$.

[2]Thus closure is the only tricky group axiom. Thus matrices lend themselves to the representation of groups, though most sets of matrices do not form groups.

[3]We will only use matrix representations here

Such unitary finite dimensional representation matrices can then be written in terms of hermitian d dimensional matrices A

$$\mathsf{D}(g) = \exp\{i\mathsf{A}(g)\}, \quad \mathsf{A}^\dagger = \mathsf{A} \tag{B.1.7}$$

Note that $\mathsf{A} = \mathsf{0}$ generates the identity element. One can then show that the matrices $\mathsf{A}(g)$ are elements of another type of algebraic object called a **Lie algebra** $\mathcal{A}$ which is defined through the following axioms:

(i) The Lie Algebra is a *vector space*, so that laws of multiplication by a real number and addition of elements are defined

   (a) $c_1 A_1 + c_2 A_2 \in \mathcal{A}, \quad \forall\, A_1, A_2 \in \mathcal{A}, \quad c_1, c_2 \in \mathbb{R}$

(ii) Elements of the algebra can also be multiplied together to give another algebra element (closure under multiplication) and this product is denoted as
$[A_1, A_2] \in \mathcal{A}, \quad \forall\, A_1, A_2 \in \mathcal{A},$

(iii) Elements of a Lie algebra also satisfy the **Jacobi identity**
$[[A_1, A_2], A_3] + [[A_2, A_3], A_1] + [[A_3, A_1], A_2] = 0 \quad \forall\, A_1, A_2, A_3 \in \mathcal{A},$

Every abstract Lie group is linked with an abstract Lie algebra. However, a small number of Lie groups, differing in the group elements a long way from the identity, can share the same Lie algebra, since the relationship between Lie group and algebra is only a simple one-to-one map near the identity.

Like the group elements, there are many representations of a Lie Algebra, and we see that for every d-dimensional unitary matrix representation of the Lie group, we have a d-dimensional matrix hermitian representation of the algebra, the matrices $\{\mathsf{A}(g)\}$ of (**??**). Finite dimensional square matrices automatically satisfy the Jacobi identity and the Lie Algebra product law is just the usual commutator of matrices

$$\text{(abstract) } [A_1, A_2] \in \mathcal{A} \quad \Rightarrow \quad [\mathsf{A}_1, \mathsf{A}_2] = \mathsf{A}_1\mathsf{A}_2 - \mathsf{A}_2\mathsf{A}_1 \in \mathcal{A} \text{ (matrix representation)} \tag{B.1.8}$$

Note that the usual laws of matrix multiplication and subtraction are used in this definition of the algebra product. Thus closure of the algebra is the only problem for matrix representations.

As with all vector spaces, we can express all vectors in terms of sums of basis vectors. For a Lie algebra, the basis elements are called **generators** and are denoted as $\{\mathsf{T}^a\}$ in a matrix representation so

$$\mathsf{A} = \sum_a c^a \mathsf{T}^a, \ \forall \mathsf{A} \in \mathcal{A}, \ c^a \in \mathbb{R}, \quad \mathsf{T}^a = (\mathsf{T}^a)^\dagger. \tag{B.1.9}$$

Returning to the Lie group elements we see that we have[4]

$$\mathsf{D}(g) = \exp\{i \sum_{a=1}^{n} \epsilon_a \mathsf{T}^a\}, \quad \epsilon_a \equiv \epsilon_a(g) \in \mathbb{R}, \quad a = 1, 2, \ldots n = \dim(G) \tag{B.1.10}$$

There are as many generators as ϵ_a coefficients i.e. the Lie group dimension. For $\epsilon_a = 0 \forall a$ we get the identity element $\mathsf{D}(e) = \mathbb{1} \exp\{i\mathsf{0}\}$. Depending on the group and the representation, only for a limited range of real values do the ϵ_a's give unique group elements. At least for group elements 'near' the identity element, i.e. 'small' ϵ_a, we can choose a fixed set T^a matrices, and let different values of ϵ_a take us through the different group elements[5]

$$D(g) \approx \mathbb{1} + i \sum_{a=1}^{n} \epsilon_a \mathsf{T}^a + \frac{i}{2} \sum_{a,b=1}^{n} \epsilon_a \epsilon_b [\mathsf{T}^a, T^b] + \ldots \tag{B.1.11}$$

[4]There is no meaning attached to raised rather than lowered indices in the Lie group and algebra context.

[5]Technically, you may not be able to reach all group elements, in particular ones far from the identity. This depends on the *global* properties of the group.

The Lie algebra has a scalar product[6] which for matrix representations is given by the trace $\mathrm{Tr}\{\mathsf{A}_1\mathsf{A}_2\}$. This means we can define orthonormal generators, and in particle physics we *invariably* define

$$\mathrm{Tr}\{\mathsf{T}^a\mathsf{T}^b\} = \frac{1}{2}\delta_{ab} \quad a,b = 1,2,\ldots,n \tag{B.1.12}$$

The one exception is the generator of a U(1) group where normalisation is usually a matter of choice.

In terms of the generators, closure of multiplication in the Lie algebra has a simple form

$$[\mathsf{T}^a,\mathsf{T}^b] = if^{abc}\mathsf{T}^c \quad a,b,c = 1,2,\ldots,n, \qquad f^{abc} = -f^{bac} = -f^{acb} \in \mathbb{R} \tag{B.1.13}$$

where the f^{abc} are called the **structure constants**. They are real and completely antisymmetric and completely specify the algebra (and hence the Lie group close to the identity). However, many different values of these constants are possible for the same Lie algebra.

For a matrix representation of a product Lie group $G \times H$, we have that the associated Lie algebra is the *sum* of the algebras of the two parts where generators associated with each part commute, i.e.

$$\mathcal{A}_{G\times H} = \left\{ \sum_{a=1}^{\dim G+\dim H} c_a\mathsf{T}^a \right\} = \{\mathsf{A}_G + \mathsf{A}_H\}, \tag{B.1.14}$$

$$[\mathsf{T}^b,\mathsf{T}^c] = 0, \qquad \mathsf{A}_G = \sum_{b=1}^{\dim G} c_b\mathsf{T}^b \in \mathcal{A}_G, \quad \mathsf{A}_H = \sum_{c=1+dimG}^{\dim G+\dim H} c_c\mathsf{T}^c \in \mathcal{A}_H \tag{B.1.15}$$

A Lie group which can *not* be expressed as a product of two smaller Lie groups, i.e. its algebra can not be split into two mutually commuting parts, is called a **simple Lie Group**. If a Lie group is a product group but none of thee parts is a pure U(1) group then it is called a **semi-simple Lie group**.

There are as many coefficients ϵ_a and matrices T^a as the dimension of the compact Lie group, $a = 1,2,\ldots n$. The real coefficients, $\{\epsilon_a\}$ vary with the group element chosen.

B.2 Some SU(2) and SO(3) representations

The group SU(2) is three dimensional. Thus we work with generators T^a_{ij} where $a,b,c = 1,2,3$ while i,j range over the dimension of the representation. In the following, we work with representations which satisfy the usual orthogonality

$$\mathrm{Tr}\{T^aT^b\} = \frac{1}{2}\delta^{ab} \tag{B.2.1}$$

The Cartan sub-algebra has only one generator in it, i.e. only one of the three generators can be diagonal.

Two-dimensional representation (SU(2) only) The matrices satisfy

$$[T^a,T^b] = i\epsilon^{abc}T^c. \tag{B.2.2}$$

where ϵ^{abc} is the totally anti-symmetric tensor with $\epsilon^{123} = +1$. A suitable choice for the generators

$$\mathsf{T}^a = \frac{1}{2}\boldsymbol{\tau}^a, \tag{B.2.3}$$

[6] A vector space need not have a scalar product but most encountered in physics do.

half the **Pauli matrices** which are given by

$$\boldsymbol{\tau}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\tau}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\tau}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.2.4}$$

Three-dimensional representations SU(2) and SO(3) The adjoint representation is the three-dimensional case. There are two common forms found, related to each other by a unitary transformation. In both the following cases, the matrices satisfy

$$[\mathsf{T}^a, \mathsf{T}^b] = \frac{i}{2}\epsilon^{abc}\mathsf{T}^c. \tag{B.2.5}$$

where ϵ^{abc} is the totally anti-symmetric tensor with $\epsilon^{123} = +1$. *Note* that the structure constants differ from those used in the two-dimensional case, (B.2.2), by a factor of 2.

The first three dimensional representation has T^3 in diagonal form[7].

$$\mathsf{T}^1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{T}^2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathsf{T}^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{B.2.6}$$

A second way of writing the three dimensional representation has no generator in diagonal form[8]. It is also the natural way to write the three dimensional representation in the way which is common for the adjoint representation. The adjoint representation is present in *all* Lie Algebras, where one can write $T^a_{ij} = -if^{aij}$, $a, i, j = 1, \ldots, \dim(G)$. Here this gives

$$T^a_{bc} \quad = \quad -\frac{i}{2}\epsilon_{abc} \tag{B.2.7}$$

which can be written out as

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.2.8}$$

[7]See for example Cheng and Li (4.52). It can be used both for SU(2) and SO(3) but in the case of the latter the way to represent fields is less obvious.

[8]If we think in terms of rotations of real three-dimensional vectors, the definition of SO(3), this second representation is quickly found. It is therefore natural to use this when wanting to find the representation of SO(3) in terms of real fields.

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