MO

5.2 Matrix Element - Simple Illustration

From now on

(i) ALWAYS in Interaction Picture
drop I subscript

(ii) DROP HATS on operators

Simple Example - SYTh

Tong §3.2 p53+

Consider a theory with

(i) ONE spin zero particle (ϕ) "phi" = own anti-particle, mass m with $\omega_k = \sqrt{\underline{k}^2 + m^2}$

(ii) ONE particle (ψ) "psi" & One anti-particle ($\bar{\psi}$) "psi-bar" (distinct), charge +1 & -1, mass M, spin 0, $\Omega_k = \sqrt{\underline{k}^2 + M^2}$

eoL14 6/11/17.

Use

(i) Real scalar field $\phi(x) = \phi^{+}(x)$

(ii) Complex scalar field $\psi(x) \neq \psi^{+}(x)$

Usual free Hamiltonian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + |\partial_\mu \psi|^2 - M^2 |\psi|^2$$

EFS check $\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ gives usual form

Interactions

$$H_{int} = g \int d^3x \; \underbrace{\phi(x)\, \psi^{+}(x)\, \psi(x)}$$

c.c. = COUPLING CONSTANT
- strength of interaction
- real

- Conserves $\#\psi - \#\bar{\psi}$
- $\#\phi$ not conserved

Theory called

eoL14 7/11/15

SCALAR YUKAWA THEORY

= SYTh

eoL16 13/11/15 afternoon

(Yukawa theory describes $\frac{n}{\bar{\psi}} / \frac{p}{\psi}$ interaction via 'MESON' ϕ)

Can we describe the decay of one ϕ into a $\psi/\bar{\psi}$ pair?

Need $\mathcal{M} = \langle f, t_f = +\infty | S | i, t_i = -\infty \rangle$
$= \langle f, t_f \quad | i, t = t_f \rangle \Rightarrow \text{Prob} \propto |\mathcal{M}|^2$

where
S Matrix $S := U(t_f = +\infty, t_i = -\infty)$
= (Scattering)

$$\simeq 1 - ig \int_{-\infty}^{+\infty} d^4x \; \phi(x) \psi^\dagger(x) \psi(x) + O(g^2)$$

Free = NO Scattering Term

STATES

- Our measurements are conducted at $t_f = +\infty$, $t_i = -\infty$ since interactions happen on very short time scale
- At t_i, t_f NOTHING is happening

Tong p54

- particles widely separated
  - $\Rightarrow$ behave like free fields/particles
  - $\Rightarrow$ Use FREE field eigenstates to represent $|i, t_i = -\infty\rangle$ & $|f, t_f = +\infty\rangle$

STATES (cont)

Tong §2.4 p29+

We use the free fields to define a useful basis in our Hilbert space of states ie $H_{int} \simeq 0$ at $t = \pm\infty$.

(i) Vacuum $|0\rangle$ vacuum $\hat{a}_p |0\rangle = 0$ etc.

(ii) One-Particle State: $|\underline{p}\rangle = \hat{a}^+_{\underline{p}} |0\rangle$

NOTN

(ii) Multi-particle States $|\underline{p}, \underline{q}\rangle = \hat{a}^+_{\underline{p}} \hat{a}^+_{\underline{q}} |0\rangle$ 2 particle

etc.

Normalisation

(i) Vacuum $\langle 0|0\rangle = 1$ by definition

(ii) 1 particle states $\langle \underline{p}|\underline{q}\rangle = \langle 0| \hat{a}_{\underline{p}} \hat{a}^+_{\underline{q}} |0\rangle$

$$= \langle 0| \left(\delta(\underline{p}-\underline{q}) + \hat{a}^+ \hat{a} \right) |0\rangle$$

$$= \delta^3(\underline{p}-\underline{q})$$

etc.

Relativistic Normalisation

The normalisation above is NOT Lorentz invariant (L.inv)
which makes relativistic calculations complicated

We know that

$$1 = \int d^3p \, |\underset{\sim}{p}\rangle\langle\underset{\sim}{p}|$$

e.g. $1|\underset{\sim}{q}\rangle = \int d^3p \, |\underset{\sim}{p}\rangle \underbrace{\delta^3(\underset{\sim}{p}-\underset{\sim}{q})}_{\langle p|q\rangle}$

LHS L.inv $\Rightarrow$ RHS L.inv.

EFS
Tony p261
e.o.L16 24/11/14

But $\int \frac{d^3p}{2\omega_p} = \int d^4p \, \delta(p^2-m^2)\,\theta(p_0)$ IS LORENTZ INV
NOT $\int d^3p$

(i) $\Rightarrow 2\omega_p \, \delta^3(\underset{\sim}{p}-\underset{\sim}{q})$ NOT $\delta^3(\underset{\sim}{p}-\underset{\sim}{q})$ is L.inv.

as $1 = \int \frac{d^3p}{2\omega_p} \cdot 2\omega_p \, \delta^3(\underset{\sim}{p}-\underset{\sim}{q})$

(ii) $1 = \int \frac{d^3p}{2\omega_p} \underbrace{\left(\sqrt{2\omega_p}\,|\underset{\sim}{p}\rangle\right)}_{|p\rangle} \cdot \underbrace{\left(\sqrt{2\omega_p}\,\langle\underset{\sim}{p}|\right)}_{\langle p|}$

Define new states $|p\rangle = \sqrt{2\omega_p}\,|\underset{\sim}{p}\rangle = |\phi(p)\rangle$ (If >1 particle)

*** NOTATION: No ~ under p for relativistic normalised states
with normalisation $\langle p|q\rangle = 2\omega_p \, \delta^3(\underset{\sim}{p}-\underset{\sim}{q})$

$$1 = \int \frac{d^3p}{2\omega_p} \, |p\rangle\langle p|$$

[N.B. $\hat{A}_p = \sqrt{2\omega_p}\,\hat{a}_{\underset{\sim}{p}}$ is rel. choice too!]

$\underline{\phi \to \psi\bar{\Psi} \text{ Decay}}$ (again) $\quad \phi(\omega_p, \underline{p}) \Rightarrow \psi(\Omega_{q_1}, \underline{q}_1)\,\bar{\Psi}(\Omega_{q_2}, \underline{q}_2)$

$$\mathcal{M} \approx -ig \langle f | \int d^4x\, \psi^+(x)\psi(x)\,\phi(x) | i \rangle + O(g^2)$$

$$= -ig \int d^4x \,\langle \text{left} | \text{right} \rangle$$

where

$$|\text{right}\rangle = \phi(x)|i\rangle$$

$$= \int \frac{d^3\underline{k}}{\sqrt{2\omega_k}} \left(a_{\underline{k}} e^{-ikx} + a^+_k e^{+ikx}\right) \sqrt{2\omega_p}\, \hat{a}^+_p |0\rangle$$

$(k_0 = \omega_k)$

$a^+a^+|0\rangle$ = Two ϕ term while $\langle f | \psi^+\psi$ terms contain zero ϕ

$$\sim \langle 0 | (b\text{'s} \,\&\, c\text{'s})\, a^+a^+|0\rangle = 0$$

commute

$\Rightarrow$ Overlap zero

$$\Rightarrow |\text{right}\rangle = \int d^3\underline{k} \sqrt{\frac{\omega_p}{\omega_k}}\, e^{-ikx}\, \underbrace{\hat{a}_k \hat{a}^+_p |0\rangle}_{(a^+_p a_k + \delta^3(\underline{k}-\underline{p}))|0\rangle}$$

$\underbrace{a^+_p a_k}$: $\; a^+a|0\rangle = 0$

e.e. 4.15 11/11/16 $\quad \Rightarrow |\text{right}\rangle = e^{-ipx}|0\rangle$

$$\langle \text{left} | = \langle f | \psi^+(x)\, \psi(x)$$

$$\Omega_k = k^2 + M^2$$

$$= \langle 0 | b_{q_1} c_{q_2} \sqrt{2\Omega_{q_1} 2\Omega_{q_2}}\,.$$

$k_1^{\mu=0} = +\Omega_{k_1}$ etc

(Sketch in)

$$\cdot \int \frac{d^3\underset{\sim}{k}}{\sqrt{2\Omega_{k_1}}} \left(b^+_{k_1} e^{+ik_1x} + c_{k_1} e^{-ik_1x}\right)$$

$$\cdot \int \frac{d^3\underset{\sim}{k}_2}{\sqrt{2\Omega_{k_2}}} \left(b_{k_2} e^{-ik_2x} + c^+_{k_2} e^{+ik_2x}\right)$$

Again only bb^+cc^+ term has zero b & c #
to match $|\text{left}\rangle$'s: ψ & ψ^+ #

$\vdots$

EPS $\Rightarrow \langle \text{left}| = \langle 0| \int d^3\underset{\sim}{k_1} \int d^3\underset{\sim}{k_2} \sqrt{\frac{\Omega_{q_1}\Omega_{q_2}}{\Omega_{k_1}\Omega_{k_2}}}\, e^{ik_1x + ik_2x}$

$\times\, \delta^3(k_1 - q_1)\,\delta^3(k_2 - q_2).$

$\langle\ = \langle 0| e^{iq_1x} e^{iq_2x}$

$\Rightarrow \mathcal{M} = -ig \int d^4x\, e^{-i(p - q_1 - q_2)x}$

Tong (3.30) P56 $\mathcal{M} = -ig\, \delta^4(p - q_1 - q_2)$

- A first scattering amplitude :) $\rightarrow$ Probability of sc $\propto g^2$
- Its zero or ∞ :(

(But this is not directly measurable.)

Have to integrate over range of momenta, flavours etc – a probability density.

eo L16
16/11/15

Physical States & Vacuum Expectation Values

- We won't calculate matrix elements directly, but will consider $\langle 0|T\ldots|0\rangle$

Consider our initial state

$$|i\rangle = |\phi(p)\rangle \left(= \text{"}|p\rangle\text{"} \text{ for } \phi \text{ field}\right)$$

$$= \sqrt{2\omega_p}\, \hat{a}^\dagger_p |0\rangle$$

We may rewrite this in terms of $\hat{\phi}|0\rangle$

First note

$$\hat{\phi}(y)|0\rangle = \int \frac{d^3k}{\sqrt{2\omega_k}} \left(\hat{a}_k e^{-iky} + \hat{a}^\dagger_k e^{+iky}\right)|0\rangle$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}}\, e^{+iky}\, |\underset{\sim}{k}\rangle$$

To project out, just $k=p$ mode, use

$$\left(\int d^3y\, e^{-ipy}\, 2\omega_p\right) \times \left(\hat{\phi}(y)|0\rangle\right)$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}}\, 2\omega_p\, \underbrace{\int d^3y\, e^{-i(p-k)y}}_{\delta^3(\underline{p}-\underline{k})\, e^{-i(\omega_p-\omega_k)t}}\, |\underset{\sim}{k}\rangle$$

$$= \sqrt{2\omega_p}\, |\underset{\sim}{p}\rangle = \underbrace{|p\rangle}_{\text{Relativistically Normalised State}}$$

This allows us to link

MATRIX ELEMENTS $\mathcal{M}$

$$\mathcal{M} = \langle f | S | \underbrace{i \rangle}_{a^\dagger, b^\dagger, c^\dagger |0\rangle}$$

to

GREEN FUNCTIONS G

= vacuum expectation values of time ordered products of fields.

$$G = \langle 0 | T \; \hat{\phi}(x_1) \hat{\phi}^\dagger(x_2) \hat{\phi}(x_3) \ldots | 0 \rangle$$

Rule

For every particle of momentum p in initial state ($p_0 = \omega_p > 0$) as follows

skip [replace $|p\rangle = \sqrt{2\omega_p}\, \hat{a}^\dagger |0\rangle$ by y]

SKIP
$$\left[\left(\int d^3 y \, e^{-ipy} 2\omega_p \right) \underbrace{\hat{\phi}^\dagger(y) |0\rangle}_{\text{Choose } y_0 \to -\infty} \right]$$
SKIP

$$\Rightarrow \boxed{\begin{aligned} \mathcal{M} = & \prod_f \left(\int d^3 z_f \, e^{+iq_f z_f} 2\omega_f \right) \quad \& \; z_f^{\mu=0} \to +\infty, \; y_i^{\mu=0} \to -\infty \\ & \prod_i \left(\int d^3 y_i \, e^{-ip_i y_i} 2\omega_i \right) G(\{z\}, \{y\}) \\ G(\{z\}, \{y\}) = & \langle 0 | T \,(\text{final})\, S \,(\text{initial}) | 0 \rangle \end{aligned}}$$

e.g. for our $\phi \to \psi\psi$ decay we need

$$\mathcal{M} = \langle \bar{\psi}(q_2), \psi(q_1) | S | \phi(p) \rangle$$

$$= \prod_{f=1,2} \left(\int d^3 z_f \, e^{+iq_f z_f} \, 2\Omega_{p_f} \right)$$

$$\left(\int d^3_y \, e^{-ipy} \, 2\omega_p \right)$$

$$G(z_1, z_2, y) \qquad \begin{pmatrix} z_1^0, z_2^0 \to +\infty \\ y^0 \to -\infty \end{pmatrix}$$

where

$$G(z_1, z_2, y)$$

$$= \langle 0 | T \hat{\psi}(z_1) \hat{\psi}^+(z_2) \hat{S} \hat{\phi}(y) | 0 \rangle$$

For perturbation theory we expand S in powers of coupling constants so

$$G = \sum \underbrace{\langle 0 | T \text{ (fields) } | 0 \rangle}$$

Each term represented as part of one Feynman diagram

Lets see how ...

e.o. L16
14/11/16