For linear P.D.E. s we can solve inhomogeneous P.D.E. s (where \( c \psi(x) \) not a solution if \( \psi(x) \) is) using Green functions \( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \) (convention could absorb here \( c \).

\( \text{ie. here} \)

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) G(x, y) = -i \delta(x - y)
\]

Solution for \( \psi \) at \((x_0, 0)\) given \((y_0, y)\) driving kick at \(y\).

\( \text{This describes how disturbance from one } \)

\( \text{propagates through system} \)

\( \text{ie. free propagation} \)

\( \text{Green functions } G \)

\( \text{depends on B.C. (= Boundary Conditions)} \)

\( \text{eg.: nothing before kick } \Rightarrow \text{ Retarded Propagator} \)

\( \text{waves absorbed by kick } \Rightarrow \text{ Advanced Propagator} \)
Classically, we study propagation by solving the e.o.m. here the K-G equ.
As a P.D.E. we solve KG using GREEN FUNCTIONS $G(x, y) = G(x - y)$.

\[
\left( \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) G(x, y) = -i \delta^4(x - y)
\]

Let $G(x, y) = \int d^4p \ G(p) \ e^{-i p(x - y)}$

\[
\Rightarrow \Box = \left( \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) G(x, y)
\]

\[
= \int d^4p \ (p^2 + p^2 + m^2) G(p) \ e^{-i p(x - y)}
\]

so if $G(p) = \frac{1}{p^2 - m^2}$ we get

\[
\Rightarrow \int d^4p e^{-i p(x - y)} = -i \delta^4(x - y) \quad \text{as required}
\]

so $G(x, y) = \int d^4p \ \frac{i}{p^2 - m^2} \ e^{-i p(x - y)}$

Too quick! (a) P.D.E. soln / Green functions depend on boundary conditions

(b) what do we do about poles at $p_0 = \pm \omega_p \Rightarrow p^2 = m^2$

\[
G(x, y) = \int d^4p \ \frac{i}{(p_0 - \omega_p)(p_0 + \omega_p)} \ e^{-i \omega_p (x - y)}
\]

\[
p^2 - m^2 = p_0^2 - \omega^2 - m^2
\]

28/04/16 2.0.10

Answers: Push poles off Re axis $\Rightarrow$ Different B.C.
Cauchy’s Theorem (for simple poles)

\[ \oint f(z) = \sum_{j} 2\pi i \cdot R_j \]

Where near \( z = z_j \), \( f(z) \) diverges as

\( f(z) = \frac{R_j}{z-z_j} \) (single pole)

(F. Prop. Handbook, eqn 6)
Retarded Green Function $G_R$

Boundary condition: $G_R(x, y) = 0$ if $x_0 < y_0$

Only reacts to delta function impulse at $y_m$

Solution

$$G_R(x, y) = G_R(x - y) = \int dp \frac{i}{(p_0^2 - \omega^2)} e^{-ip(x-y)}$$

Note: $0 < \epsilon \ll 1$

$\epsilon \rightarrow 0^+$ at end.

$$(p_0^2 - \omega^2) = (p_0 - \omega_p)^2$$

To do $\int dp_0$, we note

a) $(x_0 - y_0) > 0$

$$\int dp_0 e^{-ip_0(x_0 - y_0)} = 0$$

b) $(x_0 - y_0) < 0$

$$\int dp_0 e^{-ip_0(x_0 - y_0)} = 0$$

\[\Rightarrow \int dp_0 = \theta(x_0 - y_0) \int dp_0^I + \theta(x_0 + y_0) \int dp_0^I\]
\begin{align*}
\oint C \rho_0 L &= \Theta(x_0 - y_0) \int_{C_R+\epsilon} \rho_0 L + \Theta(y_0 - x_0) \int_{C_R+\epsilon} \rho_0 L \\
&\text{\textit{No poles inside here}}
\end{align*}

Two poles \#1

\begin{align*}
\rho_0 &= \pm \alpha_p \\
\text{clockwise round pole} &\quad \int \frac{d\rho_0}{2\pi i} \left[ \frac{ie^{-i\beta(x-y)}}{2\omega_p + i\epsilon} \right] \\
&\text{\textit{Reversal}} \\
&\text{\textit{Change } } \rho \Rightarrow -\rho
\end{align*}
\[ G_R(x-y) = \theta(x_0-y_0) \sqrt{\text{det} \mathbf{P}} e^{+ip \cdot (x-y)} \times \]
\[ \frac{1}{2w} \left( e^{-iw(x_0-y_0)} - \frac{1}{2w} \right) \]

\[ G(x,y) = \theta(x_0-y_0) (D(x-y) - D(y-x)) \]

where \( D(x-y) = \frac{1}{2w} e^{-ip \cdot x} \quad p_0 = w > 0 \)

\[ \text{NOTES} \]
1. Clearly solves K.G. equ. as \( G(p) \frac{e^{-ip \cdot x}}{w} \)

2. B.C. as that \( G_R(x-y) = 0 \) if \( x < y_0 \)
   - NO perturbations at \( x^+ \) before \( y^+ \) impacts
   - ONLY \( y^- \) after \( y^+ \)
   - retarded

3. Clearly Lorentz invariant from last form
   \( \Rightarrow \) must be Lorentz in last form but \( \theta(x_0-y) \) is less obvious (EPS)

4. Clearly function of \( (x-y)^m \) not just \( x^m y^n \)
   \( \Rightarrow \) translation invariance
   Anticipated as wrote \( G(x-y) \)
Causal Properties of $G_R$

For physical propagation, perturbation at $y^\mu$ cannot produce any change at $x^\mu$ if $x$ and $y$ space-like separated, i.e. $(x-y)^2 < 0$

Consider $x_0 = y_0$

$G_R(x-y) = \frac{\sqrt{\beta}}{2\omega} \left( e^{i\sqrt{\gamma}(x-y)} - e^{-i\sqrt{\gamma}(x-y)} \right)$

Change $\gamma = -\beta$

$= 0$

This is true for any $x-y$ when $\omega = 0$

True for $(x-y)^2 < 0$ values.

L. Inv. True for any $(x-y)^2 < 0$.

Whichever $x_0, y_0$ are.

$G_R$ is zero for unphysical separations.

N.B. For $(x-y)^2 > 0$ find $G_R > 0$ allowed.

E.g., $x = y \Rightarrow G_R(x-y) = \frac{\sqrt{\beta}}{2\omega} \text{ is real}$
Green Functions G & QFT Two-Point Function Δ, \( \rho \)

Classical Green function of \( H \) \( G \) are linked to Two-Point operators & expectation values of Free field QFT

**Wightman Function \( D(x,y) \)**

Define \( D(x,y) = \langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 1 \rangle \)

\[ D(x,y) = \frac{\delta^{(3)}(\mathbf{p}) \delta^{(3)}(\mathbf{q})}{\sqrt{2 \omega p} \sqrt{2 \omega q}} e^{-i p \cdot x + q \cdot y} \langle 0 | \hat{a}_p^\dagger \hat{a}_q^\dagger | 1 \rangle \]

as \( \hat{a}_p | 0 \rangle = \langle 0 | \hat{a}_q^\dagger = 0 \)

i.e. this encodes creation of quanta at \( y \) and annihilation removal at \( x \)\( p \)

i.e. propagation of influence from \( y \) to \( x \)

\( \Rightarrow \) hence term

\[ \Rightarrow D(x,y) = \frac{\delta^{(3)}(\mathbf{p}) e^{-i p \cdot (x-y)}}{2 \omega} \]

This is not physical in itself as \( D(x,y) \neq 0 \) for \( (x-y)^2 < 0 \)
Replied Two-Point Function $\Delta_R$

Since $G_R = \Theta(x-y) \left( \Delta(x-y) - \Delta(y-x) \right)$

we have that

$$G_R(x-y) = \Theta(x_0-y_0) \langle 01 | \langle \phi(x), \phi(y) | 10 \rangle$$

$$= \langle 01 | \Theta(x_0-y_0) \sum \phi(x), \phi(y) | 10 \rangle$$

$$G_R(x-y) = \Delta_R(x-y) \quad \text{Two-point vacuum expectation value}$$

N.B.

$$\Theta(x_0-y_0) \sum \phi(x), \phi(y) | 1\rangle = \Delta_R(x-y) \uparrow$$

unit operator
Feynman Propagator $\Delta_T$

The most important for QFT, only see this later.

$\Delta_T(x-y) = \langle 0 | T \phi(x) \bar{\phi}(y) | 0 \rangle$

where $\not{T}(\hat{A}(t) \hat{B}(t')) = \begin{cases} \hat{A} \hat{B} & \text{if } t > t' \\ \hat{B} \hat{A} & \text{if } t < t' \end{cases}$

BIGGEST SMALLEST

$T =$ Time Ordering, puts operators in order of their time, largest (smallest) times on left (right).

$\Rightarrow \Delta_T(x-y) = \Theta(x_0 - y_0) \Delta(x-y) + \Theta(y_0 - x_0) \Delta(y-x)$

1st Term

$\Theta(x_0 - y_0) \Delta(x-y)$

$= \Theta(x_0 - y_0) \int d^3p \frac{e^{-i p \cdot (x_0 - y)}}{(2\pi)^3} \frac{1}{2\omega_P} \left( \frac{\gamma}{\gamma^2 - 1} \right)\Theta(p_0 - \omega_P) e^{-i p \cdot x} e^{i p \cdot y}$

\[\Downarrow\]

\[\Theta(x_0 - y_0) \int d^3p \frac{1}{G_0} \frac{e^{-i p \cdot x}}{p_0 - \omega_P} e^{i p \cdot x} e^{i p \cdot y}\]

e.o.LH 3/10/4
Reverse and Distort contour to

$$\Gamma_0$$

Only pole at $$p = -y_0$$ inside

$$\left( e^{-i\phi(x-x_0)} \right) \rightarrow 0$$

$$\Rightarrow \int L p_0 \frac{1}{(p_0 - w)(p_0 + w)}$$

$$\Rightarrow 0 = \Theta(x_0 - x_0) \int L p_0 \frac{1}{(p_0 - w)(p_0 + w)}$$

OR

push poles above/below by $$\epsilon$$

local leave $$L p_0$$ along real axis

$$0 = e^{i(x_0 - x_0)} \int L p_0 \frac{+\infty}{-\infty} \frac{1}{(p_0 - w - i\epsilon)(p_0 + w + i\epsilon)} e^{-i\phi(x)}$$

(Take $$\epsilon \rightarrow 0^+$$ at end of calculation)

$$\theta(x_0 < x < 1)$$
\[ \Theta = \Theta(x_0 - y_0) \int \frac{d^4 p}{p^2 - (p^2 - i\epsilon)^2} e^{-ip(x-y)} \]

Integrate \( p^2 = m^2 + 2iM \) for real \( p^0 \).

Likewise, the 2nd term comes when we complete to upper half \( \Theta \) valid when \( x_0 < y_0 \).

\[ \Delta \Phi(x - y) = \int \frac{d^4 p}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \]

\[ (\partial_t - \nabla^2 + m^2) \Delta \Phi(x - y) = -i \delta(x-y) \]

with B.C. \( e^{-i\omega t} \) only as \( t \to +\infty \), \( e^{+i\omega t} \) " " \( t \to -\infty \).