Symmetry and Classical Field Theory

- The ACTION \( S = \int dt L = \int df^a \times L \) describes the whole of classical dynamics.

- Any solution to classical e.o.m. is a turning point in \( S \) e.g. local maximum.

- Want \( S \) to be invariant under physical symmetry.

  \( \Rightarrow \) symmetries map soln. to e.o.m. to other related (same energy) soln.
Symmetry only mixes objects (field, particles, ...) which lie in same irreducible representation of the group.
(and vice-versa irrep $\Rightarrow$ group of related objects)

Lorentz/Poincaré group labels irreps by:

1. MASS, $m \in \mathbb{R}^+$
2. SPIN, $s, 0, \frac{1}{2}, 1, \ldots$

E.g. Two degrees of freedom of photon (photonization) linked in one "particle" because space-time symmetry mixes them ($m=0, S=1$)

Three degrees of $f^0$ (Long, + 2x Transverse)
(since $m > 0$, $s=1$)
Relativistic Space-Time Symmetry

(i) Translations \( x^\mu \Rightarrow x'^\mu = x^\mu + a^\mu \) for constant \( a^\mu \)

(ii) Rotations/Boosts \( x^\mu \Rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \)

No detailed study here, only results quoted
Ex. Relativistic Scalar Field Theory \( x^\mu = x^\mu' \) Lorentz subgroup

\[ S = \int d^4 x \mathcal{L} = \int d^4 x \phi \]

**EFS**

\[ S d^4 x = S d^4 x' \text{ invariant (EFS)} \]

\[ \Rightarrow \text{Work with Lagrangian density} \]

**KE.** \( \dot{\phi}^2 = (\partial_0 \phi)^2 \) \text{ NOT invariant}

But we have \( \left( \frac{\partial \phi}{\partial t} \right)^2 \left( \frac{\partial \phi}{\partial x} \right)^2 \) \text{ in 1D example}

\[ c=1 \]

**EFS**

\[ \Rightarrow \left( \partial_0 \phi \right) \left( \partial_0 \phi \right) = \left( \frac{\partial \phi}{\partial t} \right)^2 \left( \frac{\partial \phi}{\partial x} \right)^2 \ldots \text{(with)} \]

\[ \text{This is invariant IFF } \phi(x) \mapsto \phi'(x') = \phi \left( \frac{x'}{c} \right) \]

i.e. coordinates change but function is UNCHANGED/INVARIANT

This is CALLED a (Lorentz) SCALAR FIELD \( c^I, A^\mu \to x^I, \phi \)

\[ = \text{Lagrangian} \]

Equivalent to having ZERO SPIN

\[ \text{Potential Terms (} = \text{No derivatives)} \]

\[ \int d^4 x \phi(x) \to \int d^4 x \left( \phi'(x') \right)^2 \text{ (} \phi \text{ unchanged)} \]

\[ = \int d^4 x' \phi(x') \cdot \right)^2 \text{ change label} \]

\[ = \int d^4 x \left( \phi(x) \right)^2 \Rightarrow \text{Action invariant} \]
Our 1D top model has given us a generic form of a relativistic scalar (spin 0) field theory.

Action
\[ S = S_{tr} + S_{L} \]

Lagrangian Density
\[ L = \frac{1}{2} \left( \partial_{\mu} \phi \right) \left( \partial^{\mu} \phi \right) - m^{2} \phi^{2} + \text{mass}^{2} \text{of particle} \]

assert: Mass^2 of particle
\( c = t = 1 \) (now)

\( \phi(x) \in \mathbb{R} \) = One degree of freedom

See p. S. ch 2, p. 16-47

For later use, we find the Hamiltonian Density H:

Momentum \( \Pi(x) := \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \), \( L = S_{tr} + S_{L} \)

Density

\( \Rightarrow H = \Pi \dot{\phi} - L \) where \( H = S_{tr} + H_{L} = \text{Hamiltonian} \)

\[ H = \Pi^{2} - \frac{1}{2} \phi^{2} - \frac{1}{2} (\nabla \phi)^{2} - m^{2} \phi^{2} + \text{mass}^{2} \phi^{2} \]

\( \phi \rightarrow H = \sum_{i=1}^{\infty} \frac{1}{2} \Pi^{2} + \frac{1}{2} (\nabla \phi)^{2} + m^{2} \phi^{2} \)

\[ \Rightarrow \Pi^{2} + \frac{1}{2} (\nabla \phi)^{2} + m^{2} \phi^{2} \]

\( \frac{1}{2m} \) x^2

\[ \text{cp.} \quad \frac{1}{2} (\pi^{2} + \frac{1}{2} \cos^{2} x^{2}) \]

(See p. S. ch 2, p. 16-47)
Other Limitations on Field Theories

1. Locality

Symmetry often allows terms such as $\int d^4x \int d^4x' (\phi(x))^2 V(x-x') (\phi(x'))^2$.

However, this is action at a distance for general $V$:
- NOT OK for fundamental physics
  - restrict $V$
- OK for low energy/velocity approximation
  - e.g., condensed matter physics

2. Renormalisation

We can only do calculations with polynomials of fields (usually)
- up to limited power
  - $\phi^2$, $\phi^4$, $\phi^6$ terms may be
    - depending on $\max_{\text{space-time dimension}}$ $\min_{\text{ok in 3d}}$ $\max_{\text{2d}}$
  - $\sin(\phi^2)$ NOT helpful

3. Hermitian

To ensure real probabilities need real action & $I=1$:

Here simple as $\phi(x) \in \mathbb{R} \Rightarrow I \in \mathbb{R}$

Complex action $\Rightarrow$ instabilities, non-equilibrium

11/10/16  Complex action $\Rightarrow$ instabilities, non-equilibrium