L_∞ Algebras and Double Field Theory

Supergravity, Strings and Dualities

A Meeting in Celebration of 60 years of Chris Hull

Imperial College London, April 29 2017

Barton Zwiebach, MIT

Enjoying a beer in Singapore's Chinatown (November 2014)

While we had known each other for a long time and had a friendly relation, in 2008 Chris and I began an intense collaboration that lasted for three years, until 2010.

We ended up writing four papers together in a subject that we called **double field theory**.

Chris had the conviction that we could use String Field Theory to improve the target-space understanding of T-duality.

We had fun and made good progress, rediscovering some earlier work of Siegel and developing the story in some new directions.

The impetus for our original work –and a clear conviction of Chris– has been that **double field theory should be more than a duality-covariant description of supergravity and its α' extensions.**

String Field Theory uses the **weak constraint**: $L_0 - \bar{L}_0$ kills fields and gauge parameters.

Most versions of Double Field Theory use the **strong constraint**: $L_0 - \bar{L}_0$ also kills products of fields and/or gauge parameters.

The strong constraint implies that in some duality frame the theory contains only momentum or only winding modes.

- Introduction
- Axioms of L_∞.
- L_∞ for field theory
- Yang Mills
- Courant brackets.
- Double field theory.
- Outlook.

O. Hohm, BZ, arXiv:1701.08824

INTRODUCTION

Open String Field Theory (OSFT) is based on A_∞ algebras.

Closed String Field Theory (CSFT) is based on L_∞ algebras.

A_∞ Stasheff

L_∞ Schlessinger & Stasheff 1985, Zwiebach 1993.

How are L_∞ algebras relevant to general field theories?

Use L_∞ to help organize the construction of new field theories.

Usually we think of Lie algebras and their extensions as governing the *gauge structure* of a field theory.

Generally the gauge principle does not completely determine the Lagrangian.

We will see that L_∞ algebras capture the information about the gauge structure **as well as** the interactions in the Lagrangian.

Motivation for this work:

1. If L_∞ applies to the Closed String Field Theory, which is a field theory for infinite number of component fields, it should also be valid for rather general field theories.

2. Effective actions for a large class of truncations of string field theory (Ashoke Sen, 1609.00459, Wilsonian Effective Action of Superstring Theory)

   Consider a projector P to a given set of number operator eigenstates such that P commutes with $b_0^\pm, L_0^\pm, Q$. The projected field theory has the algebraic structure of the full CSFT.

   The effective field theories will have an L_∞ structure.

   A. Sen: *"As an example, consider toroidal compactification of type II string theory and take P to be the projection operator on to states for which the left and right-moving oscillators are forced to be in their GSO invariant ground state. By integrating out all other fields following the procedure described in this paper, one would arrive at the double field theory action envisaged in [Hull and Zwiebach]"*

3. Earlier work on L_∞ for field theory by Barnich *et.al*, Berends *et.al.*, and Fulp *et.al.*

4. L_∞ algebra for Yang-Mills by Anton Zeitlin (arXiv:0708.1773)

5. Roytenberg and Weinstein "Courant Algebroids and Strongly Homotopy Lie Algebras" [arXiv:math/9802118]. Underlies the gauge principle of Double Field Theory. Followups by Deser and Stasheff (2014), and Deser and Saemann (2016).

L_∞ Axioms

Vector space: $X \;=\; \oplus X_n$

$$n \in \mathbb{Z} \;\; \text{is called the degree}$$

$$x \in X_n\,, \quad \deg(x) = n$$

$$X_3 \;\to\; X_2 \;\to\; X_1 \;\to\; X_0 \;\to\; X_{-1} \;\to X_{-2}$$

$x_1, x_2, x_3, \ldots$ will denote elements of fixed degree

Products: $\{\ell_1\,, \ell_2\,, \ell_3\,, \ldots\}$

$$\ell_k : \; X^{\otimes n} \;\to\; X \qquad \ell_k(x_1, \ldots x_k) \in X$$

$$\deg\, \ell_k \;=\; k - 2$$

$$\text{meaning} \quad \deg\, \ell_k(x_1, \ldots, x_k) \;=\; k - 2 + \sum_{i=1}^{k} \deg\, x_i$$

$$\deg \ell_1 = -1\,, \quad \deg \ell_2 = 0\,, \quad \deg \ell_3 = 1\,,$$

Graded commutativity

$$\ell_2(x_2, x_1) \;=\; (-1)^{1+x_1 x_2}\, \ell_2(x_1, x_2)$$

$$\sigma :\; (1, \ldots, k) \;\to\; (\sigma(1), \ldots, \sigma(k))$$

$$\ell_k(x_{\sigma(1)}, \cdots x_{\sigma(k)}) \;=\; (-1)^{\sigma}\, \epsilon(\sigma, x)\, \ell_k(x_1, \cdots, x_k)$$

The products are also linear in all arguments (multilinear)

$$\ell_k(x_1 + \tilde{x}_1\,, \cdots) \;=\; \ell_k(x_1\,, \cdots) + \ell_k(\tilde{x}_1\,, \cdots)$$

$$\ell_k(a x_1\,, \cdots) \;=\; a\, \ell_k(x_1\,, \cdots)\,, \quad a \in \mathbb{C}$$

What makes this interesting is a series of identities that must be satisfied by the products. The main identity is an extension of the Jacobi identity of Lie algebras to the full set of products.

MAIN IDENTITY

$n \geq 1$ is the number of inputs, $\quad i, j \geq 1, \quad i + j = n + 1$

One identity for each value of n. Schematically:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \, \ell_j \, \ell_i \quad (x_1 \, , \cdots , x_n) \; = \; 0$$

In detail

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma;\text{unshuffle}} (-1)^{\sigma} \epsilon(\sigma, x) \, \ell_j \Big(\ell_i (x_{\sigma(1)} \cdots x_{\sigma(i)}) \, , \, x_{\sigma(i+1)} \, , \, \cdots \, , \, x_{\sigma(n)} \Big) = 0 \, .$$

A permutation of n inputs is an i-unshuffle if

$$\sigma(1) < \cdots < \sigma(i) \, , \quad \text{and} \quad \sigma(i+1) < \cdots < \sigma(n)$$

$\underline{n=1}$:

$$\ell_1\big(\ell_1(x)\big) \;=\; 0$$

$\underline{n=2}$:

$$\ell_1(\ell_2(x_1,x_2)) \;=\; \ell_2(\ell_1(x_1),x_2) + (-1)^{x_1}\ell_2(x_1,\ell_1(x_2)) \;.$$

$\underline{n=3}$:

$$\begin{aligned} 0 = \;& \ell_1(\ell_3(x_1,x_2,x_3)) \\ & + \ell_3(\ell_1(x_1),x_2,x_3) \\ & + (-1)^{x_1}\ell_3(x_1,\ell_1(x_2),x_3) + (-1)^{x_1+x_2}\ell_3(x_1,x_2,\ell_1(x_3)) \\ & + \ell_2(\ell_2(x_1,x_2),x_3) \\ & + (-1)^{(x_1+x_2)x_3}\ell_2(\ell_2(x_3,x_1),x_2) + (-1)^{(x_2+x_3)x_1}\ell_2(\ell_2(x_2,x_3),x_1) \;. \end{aligned}$$

First four terms: failure of ℓ_1 to be a derivation of ℓ_3.

Last three terms: Jacobiator or failure of ℓ_2 to be Lie.

FIELD THEORY: Gauge parameters Λ, fields Ψ , field equations E.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\ell_1} & X_0 & \xrightarrow{\ell_1} & X_{-1} & \xrightarrow{\ell_1} & X_{-2}\,. \\ & & \Lambda & & \Psi & & E \end{array}$$

Field equation:

$$\mathcal{F} \equiv \ell_1(\Psi) - \tfrac{1}{2!}\ell_2(\Psi,\Psi) - \tfrac{1}{3!}\ell_2(\Psi,\Psi,\Psi) + \tfrac{1}{4!}\ell_4(\Psi,\Psi,\Psi,\Psi) + \dots \;=\; 0\,.$$

Gauge transformations:

$$\delta_\Lambda \Psi \;=\; \ell_1(\Lambda) + \ell_2(\Lambda,\Psi) - \tfrac{1}{2}\ell_3(\Lambda,\Psi,\Psi) - \tfrac{1}{3!}\ell_4(\Lambda,\Psi,\Psi,\Psi) + \dots\,.$$

Covariance of the field equations:

$$\delta_\Lambda \mathcal{F}(\Psi) \;=\; \ell_2(\Lambda,\mathcal{F}) \,+\, \ell_3(\Lambda,\mathcal{F}(\Psi),\Psi) \,-\, \tfrac{1}{2}\ell_4(\Lambda,\mathcal{F}(\Psi),\Psi^2) + \cdots$$

Action:

$$S \;=\; \tfrac{1}{2}\langle\Psi,\ell_1(\Psi)\rangle - \tfrac{1}{3!}\langle\Psi,\ell_2(\Psi^2)\rangle - \tfrac{1}{4!}\langle\Psi,\ell_3(\Psi^3)\rangle + \tfrac{1}{5!}\langle\Psi,\ell_4(\Psi^4)\rangle + \dots$$

With a suitable inner product.

Field equation determines

$$\ell_n(\Psi, \ldots, \Psi) \in X_{-2}\,, \quad n \geq 1\,,$$

Gauge transformations determine:

$$\ell_{n+1}(\Lambda, \Psi_1, \ldots \Psi_n) \in X_{-1}\,, \qquad n \geq 0\,,$$

Gauge algebra: $\left[\delta_{\Lambda_2}\,,\, \delta_{\Lambda_1}\right]$ is a gauge transformation with parameter

$$\Lambda_{12} \;=\; \ell_2(\Lambda_1, \Lambda_2) + \ell_3(\Lambda_1, \Lambda_2, \Psi) - \tfrac{1}{2}\ell_4(\Lambda_1, \Lambda_2, \Psi, \Psi) - \ldots\,,$$

YANG MILLS

$$X_0 \xrightarrow{\ell_1} X_{-1} \xrightarrow{\ell_1} X_{-2}$$

$$\lambda^\alpha \qquad A_\mu{}^\alpha \qquad E_\mu{}^\alpha$$

$$\delta_\lambda A_\mu{}^\alpha = \partial_\mu \lambda^\alpha + \big[A_\mu, \lambda\big]^\alpha ,$$

$$0 = D^\mu F_{\mu\nu} = \Box A_\nu - \partial_\nu \, \partial \cdot A + \partial^\mu [A_\mu, A_\nu] + [A^\mu, \partial_\mu A_\nu - \partial_\nu A_\mu] + [A^\mu, [A_\mu, A_\nu]]$$

Yang-Mills:

$$\begin{aligned}
\ell_1(\lambda) &= \partial\lambda \in X_{-1} \\
\ell_1(A) &= \Box A - \partial(\partial \cdot A) \in X_{-2} \\
\ell_2(\lambda_1, \lambda_2) &= -\big[\lambda_1, \lambda_2\big] \in X_0 \\
\ell_2(A, \lambda) &= -\big[A, \lambda\big] \in X_{-1} \\
\ell_2(A_1, A_2)_* &= -\partial[A_1, A_{2*}] - [\partial_* A_1 - \partial A_{1*} , A_2] + (1 \leftrightarrow 2) \in X_{-2} \\
\ell_2(E, \lambda) &= -\big[E, \lambda\big] \in X_{-2} \\
\ell_3(A_1, A_2, A_3)_* &= -[A_1, [A_2, A_{3*}]] + \text{sym.} \in X_{-2}
\end{aligned}$$

Courant algebroid ($O(D,D)$ covariantized)

This is the **gauge structure** of Double Field Theory.

$O(D,D)$ indices by $M,N = 1,\ldots,2D$

$$\langle V_1, V_2\rangle \;=\; \eta_{MN}\, V_1^M V_2^N\,, \qquad \eta_{MN} \;\equiv\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

C-bracket

$$[\xi_1, \xi_2]^M \;\equiv\; \xi_1^K \partial_K \xi_2^M - \tfrac{1}{2}\xi_1^K \partial^M \xi_{2K} - (1 \leftrightarrow 2)\,,$$

The C-bracket satisfies a Jacobiator identity

$$J(\xi_1,\xi_2,\xi_3) \;\equiv\; \big[\,[\xi_1,\xi_2]\,,\xi_3\big] + \text{c.p.} \;=\; \partial\, T(\xi_1,\xi_2,\xi_3)\,,$$

$$T(\xi_1,\xi_2,\xi_3) \;\equiv\; \tfrac{1}{6}\Big(\big\langle\,[\xi_1,\xi_2]\,,\xi_3\big\rangle + \text{c.p.}\Big)\,.$$

$$
\begin{array}{ccccccc}
0 \longrightarrow & X_2 & \xrightarrow{\iota} & X_1 & \xrightarrow{\partial} & X_0 \\
 & c & & \chi & & \xi^M
\end{array}
$$

$$
\boxed{\begin{aligned}
\ell_1(\chi) &= \partial\chi \ \in X_0\,, \\
\ell_1(c) &= \iota\, c \ \in X_1\,, \\
\ell_2(\xi_1,\xi_2) &= [\xi_1,\xi_2] \ \in X_0\,, \\
\ell_2\,(\xi,\,\chi) &= \tfrac{1}{2}\,\langle \xi\,, \partial\chi\rangle \ = \ \tfrac{1}{2}\xi^K\partial_K\chi \ \in X_1\,, \\
\ell_3(\xi_1,\xi_2,\xi_3) &= -\,T(\xi_1,\xi_2,\xi_3) \ \in X_1\,.
\end{aligned}}
$$

1. Begin with X_0 only. Set $\ell_2(\xi_1,\xi_2)$ equal to the C-bracket.

2. With just $\ell_2 \neq 0$ must have $\ell_2\ell_2 = 0$. This fails because of the non-zero Jacobiator. Have to introduce both ℓ_3 and ℓ_1.

3. $J = \partial T(\xi_1,\xi_2,\xi_3)$ suggests $\ell_3(\xi_1,\xi_2,\xi_3) = T(\xi_1,\xi_2,\xi_3) \in X_1$, a space of functions. $\ell_1 = \partial$ on X_1.

4. Reconsider $\ell_1\ell_2 = \ell_2\ell_1$. That identity determines $\ell_2(\xi,\chi)$.

L_∞ for the full Double Field Theory

Introduce a field to the gauge structure!

$$\begin{array}{ccccccccc} 0 \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X_{-1} \\ & c & & \chi & & \xi^M & & \mathcal{H}_{MN} \end{array}$$

Without X_{-2} no field equations and no dynamics.

The gauge transformation of $\mathcal{H}_{MN}$

$$\delta_\xi \mathcal{H}_{MN} \equiv \mathcal{L}_\xi \mathcal{H}_{MN}$$

$$\rightarrow \quad \ell_2(\xi, \mathcal{H}) \equiv \mathcal{L}_\xi \mathcal{H} \, .$$

With this addition to the list of products we have a consistent L_∞ algebra structure on the vector space above.

Adding the interactions

Must work in a perturbative expansion! $h_{\underline{M}\bar{N}}$ fluctuation of $\mathcal{H}$.

One is using projected $O(D,D)$ indices.

New complex

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X_{-1} & \longrightarrow & X_{-2} \\ & & c & & \chi & & \xi^M & & (h_{\underline{M}\bar{N}}, \phi) & & (\mathcal{R}_{\underline{M}\bar{N}}, \mathcal{R}) \end{array}$$

$\Psi = (h_{\underline{M}\bar{N}}, \phi), \quad E = (\mathcal{R}_{\underline{M}\bar{N}}, \mathcal{R}),$

$$\begin{aligned} \delta_\xi h_{\underline{M}\bar{N}} &= 2(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}) \\ &\quad + \xi^P \partial_P h_{\underline{M}\bar{N}} + K_{\underline{M}}{}^{\underline{K}} h_{\underline{K}\bar{N}} + K_{\bar{N}}{}^{\bar{K}} h_{\underline{M}\bar{K}} \\ &\quad + \tfrac{1}{8} h_{\underline{M}\bar{K}} K^{\underline{L}\bar{K}} h_{\underline{L}\bar{N}} \\ \delta_\xi \phi &= \xi^N \partial_N \phi + \partial_N \xi^N \, . \end{aligned}$$

The non-vanishing products for the L_∞ algebra for the full DFT are :

i) the products for the gauge structure,

$$\ell_1(\chi)\,,\quad \ell_1(c)\,,\quad \ell_2(\xi_1,\xi_2)\,,\quad \ell_2(\xi,\chi)\,,\quad \ell_3(\xi_1,\xi_2,\xi_3)\,,$$

ii) the products required for gauge transformations

$$\ell_1(\xi)\,,\quad \ell_2(\xi,\Psi)\,,\quad \ell_3(\xi,\Psi_1,\Psi_2)\,,$$

iii) products ℓ_n for arbitrary $n \geq 2$ involving only fields Ψ,

$$\ell_n(\Psi_1,\ldots,\Psi_n) \qquad \text{for} \qquad \Psi_1,\ldots,\Psi_n \;\in\; X_{-1}\,, \tag{1}$$

iv) the product between gauge parameter and field equation,

$$\ell_2(\xi,E) \;=\; \mathcal{L}_\xi E\,. \tag{2}$$

CONCLUSIONS AND OPEN QUESTIONS

- L_∞ captures both the gauge structure and field dynamics.
- Does L_∞ classify all gauge-invariant perturbative field theories?
- Have discussed the L_∞ algebras of general field theories.
- Have set up the L_∞ algebra of Double Field Theory (DFT).
- Build a genuine DFT that uses momentum and winding?
- Reformulate Vasiliev higher-spin theories in terms of L_∞ algebras.