Susy and $\mathbb{RCHO}$ Redux

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- Why 60? Type "Chris Hull Obituary" into duckduckgo. Ten entries. They made it to $25(2), 30(2), 36, 53(2), 56, 58, 60$.

- What happened to the over 60s? It's all explained (naturally) by the theory of everything, which is not only U-dual (naturally) but intimately connected (some say supernaturally) to $\mathbb{RCHO}$.

Hull@60, 29 April 2017

Overview

1. $\mathbb{RCHO}$ and $d = 3, 4, 6, 10$ susy
2. $\mathbb{RCHO}$ and Superparticles
3. $\mathbb{RCHO}$ and $D = 4, 5, 7, 11$ sugra

$\mathbb{RCHO}$

Composition algebras

Algebra $\mathbb{K}$ is a composition algebra if

1. $\forall x \in \mathbb{K}\,, \quad \exists$ a non-degenerate quadratic form $|x|^2 \in \mathbb{R}$.
2. $\forall x, y \in \mathbb{K}\,, \quad |xy|^2 = |x|^2|y|^2$.

$\mathbb{K}$ is a "normed division algebra" if $|x|^2$ is positive definite. A theorem of Hurwitz says that there are just four possibilities:

- $\mathbb{R}$. Real numbers (ordered, commutative, associative)
- $\mathbb{C}$. Complex numbers (commutative, associative)
- $\mathbb{H}$. Quaternions (associative)
- $\mathbb{O}$. Octonions

Lorentz groups and $Sl(2;\mathbb{K})$ [Kugo & PKT, Sudbery]

- $d = 3$. $Sl(2;\mathbb{R}) \cong Spin(1,2)$
- $d = 4$. $Sl(2;\mathbb{C}) := Sl_1(2;\mathbb{C}) \cong Spin(1,3) \times U(1)$
- $d = 6$. $Sl(2;\mathbb{H}) \cong Spin(1,5)$
- $d = 10$. $Sl(2;\mathbb{O}) \cong Spin(1,9)$

Check: $Sl(2;\mathbb{K})$ has $4\dim\mathbb{K} - 1$ generators for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, but

$$\dim sl(2;\mathbb{O}) \neq 4 \times 8 - 1 = 31$$

because of failure of Jacobi identity [Sudbery]:

$$[A,[B,X]] - [B,[A,X]] = [[A,B],X] + E(A,B)X\,,$$

where $E(A,B) \in G_2$. Since $\dim G_2 = 14$, the correct count is

$$\dim[sl(2;\mathbb{O})] = 31 + 14 = 45 \qquad \checkmark$$

The conformal group and $Sp(4; \mathbb{K})$ [Sudbery]

Define $Sp(4; \mathbb{K})$ as group preserving skew-hermitian quadratic form on $\mathbb{K}^4$

- $d = 3$. $Sp(4; \mathbb{R}) \equiv Spin(2, 3)$
- $d = 4$. $Sp(4; \mathbb{C}) \equiv Spin(2, 4) \times U(1)$
- $d = 6$. $Sp(4; \mathbb{H}) \equiv Spin(2, 6)$
- $d = 10$ $Sp(4; \mathbb{O}) \equiv Spin(2, 10)$

These are conformal isometries of Mink_d except extra $U(1)$ for $\mathbb{K} = \mathbb{C}$
For $\mathbb{K} = \mathbb{RCH}$, we have $\dim[sp(4; \mathbb{K})] = 6 \dim \mathbb{K} + 4$

For $\mathbb{K} = \mathbb{O}$ the add 14 rule again applies [Chung & Sudbery]

$\dim sp(4; \mathbb{O}) = 6 \times 8 + 4 + 14 = 66$ ✓

Rotation subgroups and $SO(2; \mathbb{K})$

Define $SO(k; \mathbb{K})$ as group preserving Hermitian quadratic form on $\mathbb{K}^k$

Then:

- $d = 3$. $SO(2; \mathbb{R}) \equiv U(1) \cong Spin(2)$
- $d = 4$. $SO(2; \mathbb{C}) \equiv U(2) \cong Spin(3) \times U(1)$
- $d = 6$. $SO(2; \mathbb{H}) \cong SU^*(4) \cong Spin(5)$
- $d = 10$. $SO(2; \mathbb{O}) \cong Spin(9)$

These are rotation subgroups except extra $U(1)$ for $\mathbb{K} = \mathbb{C}$.
For $\mathbb{K} = \mathbb{RCH}$, we have $\dim[so(2; \mathbb{K})] = 3 \dim \mathbb{K} - 2$

For $\mathbb{K} = \mathbb{O}$ apply add 14 rule

$\dim[so(2; \mathbb{O})] = (3 \times 8 - 2) + 14 = 36$ ✓

Super-Yang-Mills and $\mathbb{RCHO}$

Construction of SYM requires a Dirac Matrix Identity valid only for $d = 3, 4, 6, 10$ [Brink,Scherk & Schwarz] and DMI converts "transverse" $\mathbb{R}^{d-2}$ into $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [Evans, Schray, Baez & Huerta]

Same DMI needed for GS superstring [Green & Schwarz]

N.B. DMI is equivalent to existence of Jordan algebra of 3×3 Hermitian matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [Sierra, Fairlie & Manogue]

Super-Maxwell equations and $\mathbb{K} = \mathbb{RCHO}$ [Galperin, Howe & PKT]

For $d = 3, 4, 6, 10$, super-Maxwell equations are equivalent (via "twistor-type" transform) to $\mathbb{K}$-chirality constraint on a $\mathbb{K}$-valued worldline scalar superfield

Lorentz vectors and $\mathbb{RCHO}$

For spacetime dimension $d = 3, 4, 6, 10$ we can represent position in Minkowski spactime by a 2×2 Hermitian matrix $\mathbb{X}$ over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$:

$$\mathbb{X} = \begin{pmatrix} -X^0 + X^1 & \mathbf{X} \\ \bar{\mathbf{X}} & -X^0 - X^1 \end{pmatrix} \qquad (\mathbf{X} \in \mathbb{K}) \ .$$

For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, Lorentz transformation is

$$\mathbb{X} \to \mathbb{L}\mathbb{X}\,\mathbb{L}^\dagger\,, \quad \det(\mathbb{L}\mathbb{L}^\dagger) = 1 \quad \Rightarrow \quad \boxed{\mathbb{L} \in Sl(2;\mathbb{K})}$$

Hermitian $n \times n$ matrices over $\mathbb{H}$ have well-defined (real) determinant, as do hermitian matrices over $\mathbb{O}$ if $n \leq 3$.

$\mathbb{K} = \mathbb{O}$: $\mathbb{X} \to \mathbb{L}\mathbb{X}\mathbb{L}^\dagger$ for $\mathbb{L} \approx \mathbb{I}$ [Sudbery] but finite Lorentz transformation is more complicated [Manogue & Schray]

The relativistic particle and $\mathbb{RCHO}$

A particle has position *d*-vector $\mathbb{X}$ and momentum *d*-covector $\mathbb{P}$

$$\mathbb{P} \to (\mathbb{L}^\dagger)^{-1}\mathbb{P}\mathbb{L}^{-1}\,, \qquad \det \mathbb{P} = m^2.$$

Lorentz invariant action is

$$S = \int dt \left\{ \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{X}}\mathbb{P}) + \frac{1}{2} e \left(\det \mathbb{P} - m^2\right) \right\} .$$

- Real-trace satisfies $\mathrm{tr}_{\mathbb{R}}(\mathbb{A}\mathbb{B}) = \mathrm{tr}_{\mathbb{R}}(\mathbb{B}\mathbb{A})$.

Trace reversal [Schray]

If hermitian $\mathbb{V}$ is d-vector then $\tilde{\mathbb{V}} = \mathbb{V} - \mathrm{tr}\mathbb{V}$ is *d*-covector, and

$$\tilde{\tilde{\mathbb{V}}} = \mathbb{V}\,, \qquad \mathbb{V}\tilde{\mathbb{V}} = -(\det \mathbb{V})\mathbb{I}_2$$

Bi-spinor formulation and spin-shell constaints

Write $\mathbb{P} = \mp \mathbb{U}\mathbb{U}^\dagger$, $\mathbb{U} \to (\mathbb{L}^\dagger)^{-1}\mathbb{U}\mathbb{R}$ $\mathbb{R} \in SO(2;\mathbb{K})_{\text{local}}$

cf. vielbein formulation of GR; expect local $SO(2;\mathbb{K})$ invariance

Substitute: $\frac{1}{2}\text{tr}_{\mathbb{R}}(\dot{\mathbb{X}}\mathbb{P}) = \text{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger) + d_t(\cdots)$, where

$$\boxed{W = \pm \mathbb{X}\mathbb{U}} \quad \Rightarrow \quad 0 \equiv \mathbb{U}^\dagger\mathbb{W} - \mathbb{W}^\dagger\mathbb{U} := \mathbb{G}$$

Incidence relation

View $\mathbb{W}$ as independent by imposing $\mathbb{G} = 0$ as a "spin-shell" constraint

Why "spin-shell"? For $d = 3, 4, 6$ [Arvanitakis, Mezincescu, PKT]

Pauli Lubanski 3-form (self-dual for $d = 6$) is $\mathbb{U}\mathbb{G}\mathbb{U}^\dagger$.
So, $\mathbb{G} = 0 \;\Rightarrow\;$ zero spin.

Bi-twistor action [Arvanitakis, Barns-Graham & PKT]

Now have equivalent "bi-twistor" action

$$S = \int dt \left\{ \mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger) - \mathrm{tr}_{\mathbb{R}}(\mathbb{S}\mathbb{G}) - \tfrac{1}{2} e \left(\det(\mathbb{U}\mathbb{U}^\dagger) - m^2\right) \right\}$$

- $\mathbb{G}$ generates expected $SO(2;\mathbb{K})_{\mathrm{local}}$ gauge transformations

Why "bi-twistor"? Because

$$\left.\begin{array}{rcl} \mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger) & = & \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\mathbb{Z}^\dagger\Omega\dot{\mathbb{Z}}) \\ \mathbb{G} & = & -\mathrm{tr}_{\mathbb{R}}(\mathbb{Z}^\dagger\Omega\dot{\mathbb{Z}}) \end{array}\right\} \text{ for } \mathbb{Z} = \begin{pmatrix} \mathbb{U} \\ \mathbb{W} \end{pmatrix} \;\&\; \Omega = \begin{pmatrix} 0 & -\mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}$$

and these expressions are unchanged if $\mathbb{Z} \to \mathbb{M}\mathbb{Z}\mathbb{R}$ for $\mathbb{M}^\dagger\Omega\mathbb{M} = \Omega$, which defines the conformal group $Sp(4;\mathbb{K})$.

- Only the mass-shell constraint breaks conformal invariance

Massless particle in AdS_D, $D = 4, 5, 7$

Omit the mass-shell constraint to get $Sp(4; \mathbb{K})$-invariant action

$$S = \int dt \left\{ \frac{1}{2} \mathrm{tr}_{\mathbb{R}} (\mathbb{Z}^\dagger \Omega D_t \mathbb{Z}) \right\} , \quad D_t \mathbb{Z} = \dot{\mathbb{Z}} + \mathbb{S}\mathbb{Z}$$

- Phase-space dimension has increased by 2, so spacetime dimension is now $D = d + 1$. What is this spacetime?
- For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ it is AdS_D, and for zero mass $Sp(4; \mathbb{K})$ is its isometry group [Arvanitakis, Barns-Graham & PKT]

Non-zero mass

For particle of mass m in AdS_5 of radius R, a complex field redefinition yields action of Claus, Rahmfeld & Zunger with $\mathbb{G} = imR\mathbb{I}$.
For $\mathbb{K} \neq \mathbb{C}$ need quadri-twistor variables [Cederwall]

Superparticle and $\mathbb{K} = \mathbb{RCHO}$

N-extended superparticle in Mink$_d$: make replacement

$$\dot{\mathbb{X}} \to \dot{\mathbb{X}} + \Theta^\dagger \overset{\leftrightarrow}{d}_t \Theta\,, \qquad \Theta \to \mathbb{N}\Theta\mathbb{L}^\dagger\,, \quad \mathbb{N} \in SO\,(N;\mathbb{K})$$

for anticommuting $Sl(2;\mathbb{K})$ spinors Θ $\Rightarrow$ $\boxed{2N\,\mathrm{dim}\mathbb{K}}$ susy charges

Proceeding as before, for $\mathbb{K} = \mathbb{RCH}$ we get

$$S = \int dt \left\{ \tfrac{1}{2}\mathrm{tr}_{\mathbb{R}}(\mathbb{Z}^\dagger \Omega D_t \mathbb{Z} \mp \Xi^\dagger D_t \Xi) - \tfrac{1}{2} e \left(\det(\mathbb{U}\mathbb{U}^\dagger) - m^2\right) \right\}$$

where $\Xi = \Theta\mathbb{U}$ are anticommuting Lorentz scalars:

$$\Xi \to \mathbb{N}\Xi\mathbb{R}^\dagger \qquad \mathbb{R} \in SO(2;\mathbb{K})_{\rm gauge}$$

$\mathbb{K} = \mathbb{O}$: *massless* $Sl(2;\mathbb{O})$ superparticle known [Oda, Kimura & Nakamura]

Superparticle in $\mathrm{AdS}_{4,5,7}$

Omitting mass-shell constraint, we get bi-supertwistor action

$$S = \int dt \left\{ \tfrac{1}{2} \mathrm{tr}_{\mathbb{R}} (\mathscr{Z}^\dagger \boldsymbol{\Omega} D_t \mathscr{Z} \right\} \qquad \mathscr{Z} = \begin{pmatrix} \mathbb{Z} \\ \Xi \end{pmatrix}, \quad \boldsymbol{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & \pm 2\mathbb{I}_N \end{pmatrix}$$

This is $OSp(N|4;\mathbb{K})$-invariant.

It describes a massless superparticle in AdS_D for $D = 3 + \mathrm{dim}\mathbb{K}$ [Arvanitakis, Barns-Graham & PKT]

Quantum Theory: $\Xi \rightarrow N\mathrm{dim}\mathbb{K}$ fermi oscillators

$\Rightarrow$ Supermultiplet of $2^{N\,\mathrm{dim}\mathbb{K}}$ polarization states

M-theory supergravitons and $\mathbb{RCHO}$

The M2, D3 & M5 branes interpolate between the Minkowski vacuum and the maximally supersymmetric "$\mathrm{AdS} \times S$" vacuum [Gibbons & PKT]. The isometry supergroups of these near-horizon vacua are as follows:

$M2$	:	$AdS_4 \times S^7$	:	$\mathrm{OSp}(8\|4;\mathbb{R}) \supset \mathrm{Spin}(8) \times Sp(4;\mathbb{R})$
$D3$	:	$AdS_5 \times S^5$	:	$\mathrm{OSp}(4\|4;\mathbb{C}) \supset \mathrm{U}(4) \times Sp(4;\mathbb{C})$
$M5$	:	$AdS_7 \times S^4$	:	$\mathrm{OSp}(2\|4;\mathbb{H}) \supset \mathrm{USp}(4) \times Sp(4;\mathbb{H})$

Isometry supergroup is $OSp(N|4;\mathbb{K})$ with $N\dim\mathbb{K} = 8$

$\Rightarrow$ $2^8 = 128 + 128$ polarization states

$\Rightarrow$ Massless superparticle is a supergraviton

Speculations

According to Nahm, $\nexists$ supergroup for $D = 11$, but the $\mathbb{K} = \mathbb{O}$ case of $M2, D3, M5$ sequence yields the "soft" Lie supergroup $\boxed{OSp(1|4;\mathbb{O})}$ [Hasiewicz & Lukierski]

This should corresponds to some "M9-brane", but only candidate is a Horava-Witten Mink_{10} boundary of $D = 11$ spacetime.

- Do higher-deriv. corrections to $D = 11$ sugra allow AdS_{11} vacuum?
- If so, is there a M9-brane solution of the corrected equations?
- If so, is the M9-brane worldvolume action an E_8 SYM theory.
- If so, is this holographic dual of M-theory?

in short, $\boxed{\text{Is M} - \text{theory octonionic?}}$

Why 60? Redux

Question: Does the afterlife really begin at 60 for those individuals unfortunate enough to be called "Chris Hull"?

Answer: Let's investigate using $\mathbb{RCHO}$

- Sergeant Pepper's theorem [Beatles, 1967] states that

$$(\dim\mathbb{R})(\dim\mathbb{C})(\dim\mathbb{H})(\dim\mathbb{O}) = 64$$

- And let's not forget the add 14 rule: $64 + 14 = \boxed{78}$

So Chris, no need to panic! Welcome to the > 60 club.